ACTA MATHEMATICA VIETNAMICA
Volume 20, Number 1, 1995, pp. 55-66

REMARKS ON KALTON'S PAPER: "COMPACT CONVEX SETS AND COMPLEX CONVEXITY"

TRAN VAN AN*, NGUYEN NHUY* AND LE HOANG TRI**

Abstract. Kalton [K] constructed compact convex sets which cannot be affinely embedded into the space L_0 of all measurable functions. This paper proves that compact convex sets constructed by Kalton's method have the fixed point property.

1. Introduction

In 1935 Schauder proved that every compact convex set in a locally convex space has the fixed point property. Moreover, he conjectured that his result holds for non-locally convex spaces as well. Schauder's conjecture is one of the most resistant and outstanding open problems in the fixed point theory. In fact Schauder posed this problem in the Scotish book in 1935 and despite great efforts by topologists for more than half a century his conjecture is still unproved. This problem is still open even in some very special cases: for instance, it is not known whether compact convex sets in the spaces $L_p, 0 \leq p < 1$, have the fixed point property. In [NT] , it was shown that all Roberts spaces have the fixed point property. The aim of this paper is to prove that all compact convex sets constructed by Kalton's method [K] have the fixed point property. Our result is a counter-example to the problem of whether every compact convex set having the fixed point property can be reduced to considering L_0. It is also similar to that of [NT] and could be through of as a positive step toward a solution of Schauder's conjecture.

Notation and conventions. Let X be a linear space over C. A *quasi-norm* on a linear space X is a real non-negative function $x \mapsto \|x\|_*$ such that

Received July 12, 1994; in revised form January 20, 1995.
This paper was financially supported by the National Basis Research Program in Natural Sciences

(i) $\|x\|_* > 0$ for every non-zero point $x \in X$;

(ii) $\|\alpha x\|_* = |\alpha|\|x\|_*$ for every $\alpha \in C$ and $x \in X$;

(iii) $\|x+y\|_* \leq k(\|x\|_* + \|y\|_*)$ for every $x, y \in X$,

where k is a constant independent of x, y.

The zero element of a linear space X is denoted by θ.

The sets $\{x \in X : \|x\|_* < \varepsilon\}$ form a base of neighbourhoods of θ for a metrizable linear topology on X. If this topology is complete then X is called a *complex quasi-Banach space*. We shall say that a quasi-norm $\|.\|$ is a *p-norm* $(0 < p \leq 1)$ if it satisfies

$$\|x+y\|_*^p \leq \|x\|_*^p + \|y\|_*^p \quad \text{for every } x, y \in X.$$

The space $(X, \|.\|_*)$ is called a *p-normed space.*

A well-known theorem of Aoki and Rolewicz [Ro] asserts that every quasi-norm is equivalent to a p-norm for a certain number p with $0 < p \leq 1$. Therefore, from now on we shall suppose that a complex quasi-Banach space X is p-normed for some $0 < p \leq 1$ and denote $\|x\| = \|x\|_*^p$ for every $x \in X$. Then the topology induced by the metric $\|.\|$ is equivalent to the original one.

Let Δ denote the open unit disc in the complex plane and T the unit circle. Let X be a complex quasi-Banach space. A function $f : \Delta \to X$ is called *analytic* iff for every $z \in \Delta$, f(z) can be represented as the sum of a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where the constant coefficients a_n belong to X.

By $A_0(X)$ we denote the space of functions $f : \overline{\Delta} \to X$ so that f is continuous on $\overline{\Delta}$ and analytic on Δ .

Let A, B be subsets of a complex quasi-Banach space X. By span A we denote the linear subspace of X spanned by A. By co A we denote the convex hull of A in X . We also use the following notation:

$$A^+ = \text{co } (A \cup \{\theta\}),$$

$$\hat{A} = \text{co } (A^+ \cup (-A^+) \cup (iA^+) \cup (-iA^+)),$$

and if $x, y \in X, \alpha \in C$ we write

$$\begin{aligned} \alpha A =& \{\alpha a;\ a \in A\}, \\ \|x - A\| =& \inf\{\|x - y\|;\ y \in A\}, \\ [x, y] =& \{tx + (1 - t)y;\ t \in [0, 1]\}. \end{aligned}$$

Let L_0 denote the space of all measurable functions from $[0,1]$ into $I\!R$. Then L_0 is a linear metric space with the F-norm :

$$\|f\| = \int_0^1 \frac{|f(t)|}{1 + |f(t)|} dt \quad \text{for every} \quad f \in L_0.$$

Let X be a complex quasi-Banach space. Then we say that $x \in X$ is an *analytic needle point* of X iff for any $\varepsilon > 0$ there exists $g \in A_0(X)$ such that:

$$g(0) = x, \tag{1}$$

$$\|g(z)\|_* < \varepsilon \quad \text{for every } z \in T, \tag{2}$$

and

If $y \in$ co $g\,(\overline{\Delta})$ then there exists an $\alpha \in [0, 1]$ such that $\|y - \alpha x\|_* < \varepsilon$. (3)

A complex quasi-Banach space X is called an *analytic needle point space* iff every non-zero point of X is an analytic needle point.

For undefined notations, see [BP], [K] and [Ro].

LEMMA 1.1 ([K]). *Let x be an analytic needle point of X. Then given any $\varepsilon > 0$ there is a finite set $F = F(x, \varepsilon) \subset X$ and a polynomial $P \in A_0(X)$ so that:*

$$P(\overline{\Delta}) \subset \text{ co } F; \tag{4}$$

$$P(o) = x; \tag{5}$$

$$\|P(z)\|_* < \varepsilon \quad \textit{for every } z \in T; \tag{6}$$

$$\textit{If } y \in \text{ co } F \textit{ then there exists } \alpha \in [0, 1] \textit{ such that } \|y - \alpha x\|_* < \varepsilon; \tag{7}$$

$$\textit{If } y \in F \textit{ then } \quad \|y\|_* < \varepsilon. \tag{8}$$

Let X be an analytic needle point space. We shall describe compact convex sets constructed by Kalton's method. Let $\{\delta_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty}(\delta_n)^p < \infty$. Let $G_0 = \{x_0\}$, where x_0 is any non-zero point of X. Assume that $G_{n-1} = \{y_1, \dots, y_{m(n-1)}\}$ has been selected. Let $\varepsilon_n = m(n-1)^{\frac{-1}{p}}\delta_n$ and put $G_n = \bigcup_{j=1}^{m(n-1)} F(y_j, \varepsilon_n)$, where $F(y_j, \varepsilon_n)$ is given by Lemma 1.1 . Then we have

$$G_{n-1}^{+} \subset G_n^{+} \quad \text{for every } n \in N; \tag{9}$$

$$\|x - G_{n-1}^{+}\| \le m(n-1)\varepsilon_n^p \le (\delta_n)^p \ \text{ for every } \ x \in G_n^{+}. \tag{10}$$

Denote

$$K_0 = \overline{\bigcup_{n=0}^{\infty} G_n^{+}}; \qquad \text{and} \quad K = \hat{K}_0. \tag{11}$$

CLAIM 1.2. *K is a compact convex set.*

PROOF. Obviously, K is convex. Since G_n is finite, G_n^{+} is a finite dimensional compact convex set. Thus from (10) K_0 is totally bounded. Therefore by the completeness of X, K_0 is compact. It implies that K is compact.

CLAIM 1.3. *K has no extreme points.*

PROOF. Assume that a is an extreme point of K. Obviously, $\theta \in K$ is not an extreme point, so we infer that $a \neq \theta$. We have $a = \alpha_1 x_1 - \alpha_2 x_2 + i\alpha_3 x_3 - i\alpha_4 x_4$ for some $x_i \in K_0$, $\alpha_i \in [0,1]$, $i = 1, \dots, 4$ such that $\sum_{i=1}^{4} \alpha_i = 1$. Since a is an extreme point we have $\alpha_i = 1$ for some i. Without loss of generality we may say that $\alpha_1 = 1$. Therefore $a \in K_0$. Take $n \in N$ such that $\sum_{j=n}^{\infty}(\delta_j)^p < \frac{1}{2}\|a\|$.

Let $G_n = \{y_1^n, \dots, y_{m(n)}^n\}$ and choose $x_k \in \bigcup_{i=0}^{\infty} G_i^{+}$ such that $x_k \to a$.

For $k > n$ we define $G_k(y_i^n)$ by induction

$$G_k(y_i^n) = \bigcup \{F(b, \varepsilon_k) : b \in G_{k-1}(y_i^n)\}.$$

(Note that $G_{n+1}(y_i^n) = F(y_i^n, \varepsilon_{n+1})$). By (9) we may assume that for each $k \in N$, there exists an $n_k \in N$ such that $x_k \in G_{n+n_k}^{+}$. Since $\theta \in G_j^{+}$ and G_j^{+} is convex for every $j \in N$ we get

$$x_k = \sum_{i=1}^{m(n)} \alpha_i^k b_i^k, \ b_i^k \in G_{n+n_k}^{+}(y_i^n), \ \alpha_i^n \ge 0, \ i = 1, \dots, m(n) \text{ and } \sum_{i=1}^{m(n)} \alpha_i^k = 1.$$

By selecting subsequences we may assume that $\alpha_i^k \to \alpha_i$ and $b_i^k \to b_i$ for each $i = 1, \ldots, m(n)$. Then we get

$$a = \sum_{i=1}^{m(n)} \alpha_i b_i,\ b_i \in K_0,\ \alpha_i \geq 0,\ i = 1, \ldots, m(n) \text{ and } \sum_{i=1}^{m(n)} \alpha_i = 1. \tag{12}$$

Since $b_i^k \in G^+_{n+n_k}(y_i^n)$, from (10), we compute

$$\|b_i^k\| \leq \|a_i^n\| + \sum_{j=n}^{n+n_k} (\delta_{j+1})^p < (\varepsilon_n)^p + \sum_{j=n}^{n+n_k} (\delta_{j+1})^p$$
$$< \sum_{j=n}^{\infty} (\delta_j)^p < \frac{1}{2}\|a\|.$$

Hence $\|b_i\| < \frac{1}{2}\|a\|$ for every $i = 1, \ldots, m(n)$. Consequently, from (12) we get that a is not an extreme point of K.

The claim is proved.

By Kalton's method [K] we can prove that there is no affine embedding of K into L_0.

REMARK. Our construction of K (11) is slightly different from that of Kalton [K] . However it also has the property that K cannot be embedded into L_o. As pointed out by Kalton in his recent letter to the authors, there is no reason for the convexity of K in [K], so our definition of K will replace Kalton's compact set [K].

FACT 1.4.

$$K = \overline{\bigcup_{n=0}^{\infty} \hat{G}_n}$$

PROOF. Let $x \in K$. Because $\theta \in K_0$ and K_0 is convex, x has the form $x = \lambda_1 x_1 - \lambda_2 x_2 + i\lambda_3 x_3 - i\lambda_4 x_4$ with $x_i \in K_0$, $\lambda_i \geq 0$, $i = 1, \ldots, 4$ and $\sum_{i=1}^4 \lambda_i = 1$. We may assume that $x_i = \lim_{n\to\infty} x_n^i$ with $x_n^i \in G_n^+$. Put

$$x_n = \lambda_1 x_n^1 - \lambda_2 x_n^2 + i\lambda_3 x_n^3 - i\lambda_4 x_n^4.$$

Then we have $x_n \in \hat{G}_n$ and $x_n \to x$ as $n \to \infty$. Therefore $x \in \overline{\bigcup_{n=0}^{\infty} \hat{G}_n}$. Conversely, if $x \in \overline{\bigcup_{n=0}^{\infty} \hat{G}_n}$, then $x = \lim_{n\to\infty} x_n$ with $x_n \in \bigcup_{n=0}^{\infty} \hat{G}_n$. Without loss of generality we may assume that $x_n \in \hat{G}_n$, for every $n \in N$. Then there exist $x_n^i \in G_n^+, \lambda_n^i \geq 0, i = 1, \dots, 4$ with $\sum_{i=1}^4 \lambda_n^i = 1$ such that

$$x_n = \lambda_n^1 x_n^1 - \lambda_n^2 x_n^2 + i\lambda_n^3 x_n^3 - i\lambda_n^4 x_n^4.$$

Because of the compactness of the sets $J = \{(\lambda_1, \lambda_2, \lambda_3, \lambda_4);\ \lambda_i \in [0,1],\ i = 1, \dots, 4, \sum_{i=1}^4 \lambda_i = 1\}$ and K_0, by passing to an appropriate subsequence and renumbering it if necessary, one can assume that

$$(\lambda_n^1, \lambda_n^2, \lambda_n^3, \lambda_n^4) \to (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in J \text{ and } x_n^i \to x_i \in K_0, i = 1, \dots, 4.$$

Then we have $x = \lambda_1 x_1 - \lambda_2 x_2 + i\lambda_3 x_3 - i\lambda_4 x_4 \in K$.

2. The main result

THEOREM 2.1. *K has the fixed point property.*

To prove Theorem 2.1 we need the following facts

LEMMA 2.2([NT]). *Let X be a quasi-Banach space and let a be a non-zero point of X. Then there is a retraction $r_a : X \to [\theta, a]$ such that*

$$\|x - r_a(x)\| \leq 4\|x - [\theta, a]\| \text{ for every } x \in X.$$

Moreover, if $x \in co\, F(a, \varepsilon)$ then $\|x - r_a(x)\| < 4\varepsilon^p$.

Obviously, we have

FACT 2.3. *If $\tilde{B}_n = \{b_1, \dots, b_{2n}\}$ and $B_n = \{b_1, \dots, b_{2n}, ib_1, \dots, ib_{2n}\}$ then B_n^+ can be written in the form:*

$$B_n^+ = \{x = \alpha_1 \sum_{j=1}^n \lambda_j^1 b_j + \alpha_2 \sum_{j=1}^n \lambda_j^2 b_{n+j} + i\alpha_3 \sum_{j=1}^n \lambda_j^3 b_j + i\alpha_4 \sum_{j=1}^n \lambda_j^4 b_{n+j};$$

$$\alpha_i \geq 0,\ \lambda_j^i \geq 0,\ i = 1, \dots, 4;\ j = 1, \dots, n; \sum_{j=1}^n \lambda_j^i = 1 \text{ and } \sum_{i=1}^4 \alpha_i \leq 1\}.$$

Moreover, if $\tilde{B}_n$ is a linearly independent subset of X then every $x \in B_n^+$ can be written uniquely in the form :

$$x = \alpha_1 \sum_{j=1}^{n} \lambda_j^1 b_j + \alpha_2 \sum_{j=1}^{n} \lambda_j^2 b_{n+j} + i\alpha_3 \sum_{j=1}^{n} \lambda_j^3 b_j + i\alpha_4 \sum_{j=1}^{n} \lambda_j^4 b_{n+j}$$

with $\alpha_i \geq 0$, $\lambda_j^i \geq 0$, $\sum_{j=1}^{n} \lambda_j^i = 1$, $i = 1, \dots, 4$; $j = 1, \dots, n$ *and* $\sum_{i=1}^{4} \alpha_i \leq 1$ *(if* $\alpha_i = 0$ *we write, for convenience,* $\lambda_j^i = 0$ *for* $j = 1, \dots, n-1$, *and* $\lambda_j^i = 1$.)

The next lemma will play a crucial rolè in the sequel.

LEMMA 2.4. *Let* X *be an infinite dimensional quasi-Banach space,* $G_n = \{a_1, \dots, a_n\}$ *a finite subset of* X *and* $\varepsilon > 0$. *Then for every* $i = 1, \dots, 2n$ *there exists* $b_i = b(a_i) \in X$, *where* $a_{n+i} = -a_i$ *for* $i = 1, \dots, n$ *with the following properties:*

(i) $\|b_i - a_i\| < (4n)^{-1}\varepsilon^p$ *for every* $i = 1, \dots, 2n$,

(ii) $\tilde{B}_n = \{b_1, \dots, b_{2n}\}$ *is a linearly independent subset of* X,

(iii) Let $B_n = \{b_1, \dots, b_{2n}, ib_1, \dots, ib_{2n}\}$, *then there exists a continuous map* $p : B_n^+ \to \hat{G}_n$ *satisfying* $\|p(x) - x\| < \varepsilon^p$ *for every* $x \in B_n^+$,

(iv) $\|x - B_n^+\| < \varepsilon^p$ *for every* $x \in \hat{G}_n$.

PROOF. Obviously, we may assume that $a_i \neq 0$ for every $i = 1, \dots, n$. Since X is infinite dimensional we can define by induction $b_1, \dots, b_{2n}$ such that

$$\tilde{B}_n = \{b_1, \dots, b_{2n}\} \quad \text{is a linearly independent subset of } X, \tag{13}$$

and

$$\|b_i - a_i\| < (4n)^{-1}\varepsilon^p \quad \text{for every } i = 1, \dots, 2n, \tag{14}$$

where $a_{n+i} = -a_i$ for every $i = 1, \dots, n$.

Indeed, put $b_1 = a_1$ and assume that $\{b_1, \dots, b_k\}$ have been defined such that the conditions (13) and (14) hold. We take $b_{k+1} \in X \setminus \text{span}\,\{b_1, \dots, b_k\}$ so that the condition (14) is satisfied.

Since $\tilde{B}_n$ is linearly independent, by Lemma 2.3, every $x \in B_n^+$ can be written uniquely in the form:

$$x = \alpha_1 \sum_{j=1}^{n} \lambda_j^1 b_j + \alpha_2 \sum_{j=1}^{n} \lambda_j^2 b_{n+j} + i\alpha_3 \sum_{j=1}^{n} \lambda_j^3 b_j + i\alpha_4 \sum_{j=1}^{n} \lambda_j^4 b_{n+j},$$

with $\alpha_i \geq 0$, $\lambda_j^i \geq 0$, $\sum_{j=1}^{n} \lambda_j^i = 1$, $i = 1, \ldots, 4$, $j = 1, \ldots, n$ and $\sum_{i=1}^{4} \alpha_i \leq 1$.

Set

$$p(x) = \alpha_1 \sum_{j=1}^{n} \lambda_j^1 a_j - \alpha_2 \sum_{j=1}^{n} \lambda_j^2 a_j + i\alpha_3 \sum_{j=1}^{n} \lambda_j^3 a_j - i\alpha_4 \sum_{j=1}^{n} \lambda_j^4 a_j.$$

It is easy to see that $p(x) \in \hat{G}_n$ and the map $p : B_n^+ \to \hat{G}_n$ is well-defined, continuous. Thus the conditions (iii) and (iv) are satisfied.

From Lemma 2.4 we get the following corollary which is analogous to Corollary 1 in [NT].

COROLLARY 2.5. *Let $G_n = \{a_1^n, \ldots, a_{m(n)}^n\}$, where $m(n) =$ card G_n. Then there are finite subsets $\tilde{B}_n, B_n$ of X satisfying the following properties :*

(i) $\tilde{B}_n = \{b_1^n, \ldots, b_{2m(n)}^n\}$, where $b_i^n = b(a_i^n)$ for all $i = 1, \ldots, 2m(n)$ and $a_{m(n)+i}^n = -a_i$ for all $i = 1, \ldots, m(n)$;

$B_n = \{b_1^n, \ldots, b_{2m(n)}^n, ib_1^n, \ldots, ib_{2m(n)}^n\}$;

(ii) $\|a_i^n - b_i^n\| < [4m(n)]^{-1}(\varepsilon_n)^p$, where $\varepsilon_n = [m(n-1)]^{-1/p}\delta_n$;

(iii) $\tilde{B}_n = \{b_1^n, \ldots, b_{2m(n)}^n\}$ is a linearly independent subset of X;

(iv) $B_{n+1} = (\bigcup_{j=1}^{2m(n)} B_{n+1}(b_j^n)) \bigcup (\bigcup_{j=1}^{2m(n)} iB_{n+1}(b_j^n))$, where $b_i^n = b(a_i^n)$ and

(a) $B_{n+1}(b_j^n) = \{b \in B_{n+1} : b = b(a)$ for some $a \in F(a_j^n, \varepsilon_{n+1}\}$

$B_{n+1}(b_{m(n)+j}^n) = \{b \in B_{n+1} : b = b(-a)$ for some $a \in F(a_j^n, \varepsilon_{n+1})\}$,

where $j = 1, \ldots, m(n)$;

(b) $B_{n+1}(b_j^n) \cap B_{n+1}(b_k^n) = \emptyset$ for all $j \neq k$;

(v) For every $n \in N$ there exists a continuous map $p_n : B_n^+ \to \hat{G}_n$ such that $\|p_n(x) - x\| < (\varepsilon_n)^p$ for all $x \in B_n^+$;

(vi) $\|x - B_n^+\| < (\varepsilon_n)^p$ for all $x \in \hat{G}_n$.

LEMMA 2.6. *For every $j = 1, \ldots, 2m(n)$ there exists a continuous map f_j : co $B_{n+1}(b_j^n) \to [\theta, b_j^n]$ such that $\|x - f_j(x)\| < [4m(n)]^{-1}(\varepsilon_n)^p + 5(\varepsilon_{n+1})^p$ for all $x \in$ co $B_{n+1}(b_j^n)$.*

PROOF. From the proof of Lemma 2.4 it follows that

$$p_{n+1}(\text{ co } B_{n+1}(b_j^n)) \subset \text{ co } F(a_j^n, \varepsilon_{n+1}) \quad \text{for all } j = 1, \ldots, m(n).$$

$$p_{n+1}(\text{ co } B_{n+1}(b_{m(n)+j}^n)) \subset -\text{ co}F(a_j^n, \varepsilon_{n+1}) \text{ for all } j = 1, \ldots, m(n)$$

and

$$\|p_{n+1}(x) - x\| < (\varepsilon_{n+1})^p \quad \text{for all } x \in \text{ co } B_{n+1}(b_j^n). \tag{15}$$

Denote

$$p_{n+1}^j = p_{n+1}| \text{ co } B_{n+1}(b_j^n),\ j = 1, \dots, 2m(n).$$

By Lemma 2.2 we have $r_{a_j^n}(\text{ co } F(a_j^n, \varepsilon_{n+1})) \subset [\theta, a_j^n]$ and

$$\|x - r_{a_j^n}(x)\| < 4(\varepsilon_{n+1})^p \quad \text{for all } x \in \text{ co } F(a_j^n, \varepsilon_{n+1}),\ j = 1, \dots, m(n).$$

Now let $r_j :$ co $F(a_j^n, \varepsilon_{n+1}) \to [\theta, a_j^n]$ for $j = 1, \dots, m(n)$, be defined by

$$r_j(x) = r_{a_j^n}(x) \quad \text{for every } x \in \text{ co } F(a_j^n, \varepsilon_{n+1}),$$

and let $r_{m(n)+j} : -$ co $F(a_j^n, \varepsilon_{n+1}) \to [\theta, -a_j^n]$ for $j = 1, \dots, m(n)$, be defined by

$$r_{m(n)+j}(-x) = -r_{a_j^n}(x) \quad \text{for all } x \in \text{ co } F(a_j^n, \varepsilon_{n+1}).$$

From the above argument, it follows that r_j is continuous map such that

$$\|y - r_j(y)\| < 4(\varepsilon_{n+1})^p \quad \text{for all } j = 1, \dots, 2m(n). \tag{16}$$

Let $h_j : [\theta, a_j^n] \to [\theta, b_j^n]$ be the continuous map defined by $h_j(ta_j^n) = tb_j^n$ for all $t \in [0,1]$, $j = 1, \dots, 2m(n)$. By Corollary 2.5 we have

$$\|h_j(x) - x\| < [4m(n)]^{-1}(\varepsilon_n)^p \quad \text{for all } x \in [\theta, a_j^n] \text{ and } j = 1, \dots, 2m(n).$$

Put $f_j = h_j r_j p_{n+1}^j$. It is easy to see that f_j is continuous. Moreover, we have

$$\|f_j(x) - x\| = \|h_j r_j p_{n+1}^j(x) - x\| \le \|h_j r_j p_{n+1}^j(x) - r_j p_{n+1}^j(x)\| +$$

$$+ \|r_j p_{n+1}^j(x) - p_{n+1}^j(x)\| + \|p_{n+1}^j(x) - x\| \le$$

$$\le [4m(n)]^{-1}(\varepsilon_n)^p + 4(\varepsilon_{n+1})^p + (\varepsilon_{n+1})^p = [4m(n)]^{-1}(\varepsilon_n)^p + 5(\varepsilon_{n+1})^p,$$

for all $x \in$ co $B_{n+1}(b_j^n)$ and $j = 1, \dots, 2m(n)$.

The lemma is proved.

COROLLARY 2.7. *Let* $X^1 = \bigcup_{j=1}^{m(n)} B_{n+1}(b_j^n)$ *and* $X^2 = \bigcup_{j=1}^{m(n)} B_{n+1}(b_{m(n)+j}^n)$. *Then for each* $k = 1, 2$ *there exists a continuous map* $r^k : \text{co } X^k \to B_n^+$ *such that* $\|r^k(x) - x\| < [4m(n-1)]^{-1}(\delta_n)^p + 5(\delta_{n+1})^p$ *for all* $x \in \text{co } X^k$.

PROOF. Since X^1 and X^2 are linearly independent, for $x \in \text{co } X^1$ and $y \in \text{co } X^2$ there exist uniquely $\lambda_j \geq 0, z_j \in \text{co } B_{n+1}(b_j^n), \mu_j \geq 0, \omega_j \in \text{co } B_{n+1}(b_{m(n)+j}^n)$, $j = 1, \dots, m(n)$ with $\sum_{j=1}^{m(n)} \lambda_j = \sum_{j=1}^{m(n)} \mu_j = 1$ such that

$$x = \sum_{j=1}^{m(n)} \lambda_j z_j \text{ and } y = \sum_{j=1}^{m(n)} \mu_j \omega_j.$$

Put

$$r^1(x) = \sum_{j=1}^{m(n)} \lambda_j f_j(z_j) \text{ and } r^2(y) = \sum_{j=1}^{m(n)} \mu_j f_{m(n)+j}(\omega_j).$$

Obviously, $r^k(x) \in B_n^+$ for $k = 1, 2$. Because of the linear independence of X^1 and X^2, r^1, r^2 are continuous map, and we have

$$\begin{aligned}\|r^1(x) - x\| &\leq \sum_{j=1}^{m(n)} \|f_j(z_j) - z_j\| < m(n)[[4m(n)]^{-1}(\varepsilon_n)^p + 5(\varepsilon_{n+1})^p] \\ &= m(n)[[4m(n)]^{-1}[m(n-1)]^{-1}(\delta_n)^p + 5[m(n)]^{-1}(\delta_{n+1})^p] \\ &= [4m(n-1)]^{-1}(\delta_n)^p + 5(\delta_{n+1})^p \text{ for every } x \in \text{co } X^1,\end{aligned}$$

and

$$\begin{aligned}\|r^2(y) - y\| &\leq \sum_{j=1}^{m(n)} \|f_{m(n)+j}(\omega_j) - \omega_j\| \\ &< m(n)[[4m(n)]^{-1}(\varepsilon_n)^p + 5(\varepsilon_{n+1})^p] \\ &= m(n)[[4m(n)]^{-1}[m(n-1)]^{-1}(\delta_n)^p + 5[m(n)]^{-1}(\delta_{n+1})^p] \\ &= [4m(n-1)]^{-1}(\delta_n)^p + 5(\delta_{n+1})^p \quad \text{for all } y \in \text{co } X^2.\end{aligned}$$

The corollary is demonstrated.

COROLLARY 2.8. *For every* $n \in N$ *there exists a continuous map* $R_{n+1}: B_{n+1}^+ \to B_n^+$ *such that* $\|R_{n+1}(x) - x\| < (\delta_n)^p + 20(\delta_{n+1})^p$ *for all* $x \in B_{n+1}^+$.

PROOF. Since $\tilde{B}_{n+1}$ is linearly independent, every $x \in B^+_{n+1}$ is written uniquely in the form: $x = \alpha_1 x_1 + \alpha_2 x_2 + i\alpha_3 x_3 + i\alpha_4 x_4$, where $x_1, x_3 \in$ co $X^1; x_2, x_4 \in$ co $X^2, \alpha_i \geq 0, i = 1, \ldots, 4$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 1$. Put

$$R_{n+1}(x) = \alpha_1 r^1(x_1) + \alpha_2 r^2(x_2) + i\alpha_3 r^1(x_3) + i\alpha_4 r^2(x_4).$$

Obviously, the map $R_{n+1} : B^+_{n+1} \to B^+_n$ is continuous. Moreover, we get

$$\begin{aligned}\|R_{n+1}(x) - x\| &= \|\alpha_1 r^1(x_1) + \alpha_2 r^2(x_2) + i\alpha_3 r^1(x_3) + i\alpha_4 r^2(x_4) \\ &\quad - \alpha_1 x_1 - \alpha_2 x_2 - i\alpha_3 x_3 - i\alpha_4 x_4\| \\ &\leq \|r^1(x_1) - x_1\| + \|r^2(x_2) - x_2\| + \|r^1(x_3) - x_3\| + \|r^2(x_4) - x_4\| \\ &\leq 4[[4m(n-1)]^{-1}(\delta_n)^p + 5(\delta_{n+1})^p] < (\delta_n)^p + 20(\delta_{n+1})^p\end{aligned}$$

for all $x \in B^+_{n+1}$.

COROLLARY 2.9. *For each $n \in N$ and $k \in N$ there exists a continuous map $R_{k,n} : B^+_{n+k} \to B^+_n$ such that*

$$\|R_{k,n}(x) - x\| < 21[(\delta_n)^p + \cdots + (\delta_{n+k})^p] \quad \textit{for all} \;\; x \in B^+_{n+k}.$$

PROOF. Put $R_{k,n} = R_{n+1}R_{n+2}\ldots R_{n+k}$. From Corollary 2.8 it follows that

$$\begin{aligned}\|R_{k,n}(x) - x\| &\leq \|R_{k,n}(x) - R_{n+2}R_{n+3}\ldots R_{n+k}(x)\| + \cdots + \|R_{n+k}(x) - x\| \\ &< (\delta_n)^p + 20(\delta_{n+1})^p + \cdots + (\delta_{n+k-1})^p + 20(\delta_{n+k})^p \\ &< 21[(\delta_n)^p + \cdots + (\delta_{n+k})^p] \quad \text{for all} \;\; x \in B^+_{n+k}.\end{aligned}$$

The proof of the Corollary 2.9 is completed.

LEMMA 2.10 ([NT]). *Let P be a finite dimensional compact convex polyhedron in X and let $f : P \to K$ be a continuous map. Then for every $\varepsilon > 0$ there exist $k \in N$ and an affine map $g : P \to B^+_k$ such that $\|f(x) - g(x)\| < \varepsilon$ for all $x \in P$.*

PROOF OF THEOREM 2.1. Assume that there is a continuous map $f : K \to K$ such that $f(x) \neq x$ for every $x \in K$. By the compactness of K there exists an $\varepsilon > 0$ such that $\|f(x) - x\| \geq \varepsilon$ for every $x \in K$. Take $n \in N$ such that $88\sum_{j=n}^{\infty}(\delta_j)^p < \varepsilon$ and let $f_n = f|_{\hat{G}_n} : \hat{G}_n \to K$. By Lemma 2.10 there exists an affine map $g_n : \hat{G}_n \to B^+_{n+k}$ such that $\|g_n(x) - f_n(x)\| < 2^{-2}\varepsilon$ for all $x \in \hat{G}_n$.

We put

$$g = p_n R_{k,n} g_n : \hat{G}_n \to \hat{G}_n, \quad (17)$$

where p_n and $R_{k,n}$ were defined in Corollary 2.5 and Corollary 2.9. Then for all $x \in \hat{G}_n$, from (17) we have

$$\begin{aligned}\|f_n(x) - x\| \leq \|f_n(x) - g_n(x)\| + \|g_n(x) - R_{k,n}g_n(x)\| + \\ \|R_{k,n}g_n(x) - p_n R_{k,n}g_n(x)\| + \|p_n R_{k,n} g_n(x) - x\| \\ < 2^{-1}\varepsilon + \|g(x) - x\|.\end{aligned}$$

Therefore for all $x \in \hat{G}_n$ we have

$$\|g(x) - x\| \geq \|f_n(x) - x\| - 2^{-1}\varepsilon \geq 2^{-1}\varepsilon.$$

This contradicts the fact that a finite dimensional compact convex set $\hat{G}_n$ has the fixed point property.

The proof of Theorem is finished.

ACKNOWLEDGEMENTS. The authors would like to thank Nguyen To Nhu for his comments and N. Kalton for discussion about the convexity of his compact set [K].

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*DEPARTMENT OF MATHEMATICS
VINH UNIVERSITY OF EDUCATION
VINH CITY, VIETNAM

**DEPARTMENT OF MATHEMATICS
POLYTECHNICAL UNIVERSITY DA NANG