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THE FUNDAMENTAL GROUP OF COMPLEX HYPERPLANES ARRANGEMENTS

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1. Introduction

Let V be a complex vector space of finite dimension. An arrangement is a finite family of affine hyperplanes in V. Given an arrangement $\mathcal{A} = \{H\}$. Let's consider the complement

$$M = V - N,$$

where $N = \bigcup_{H\in\mathcal{A}} H$. The purpose of this paper is to give a presentation for the fundamental group $\pi_1(M)$ of this complement. By standard arguments using Lefschetz Theorem (see [3]) we can always reduce to the case when V is a complex vector space of dimension 2. So, from now on we consider only the case when $\mathcal{A} = \{H_i | i = 1, \dots, n\}$ is an arrangement of complex lines in $V = \mathbf{C}^2$.

Such a presentation has been given in [2], [7], [9] for the case when $\mathcal{A}$ is the complexification of a real arrangement. In [1], W. Arvola has considered the case of an arbitrary complex arrangement and has given a presentation for the fundamental group of its complement using a certain planar graph defined by himself. In this paper we will follow a quite different method. Our computation is based on the so-called labyrinth of an arrangement of complex lines, which dues to L. Rudolph [8]. This construction of L.Rudolph has been used also by Y. Orevkov [5], in proving the Zariski's conjecture of the abelity of the fundamental group of the complement of a curve with nodes.

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2. Labyrinth

Suppose that $\mathbf{C}^2$ is given with the coordinates (x, y). Each hyperplane H_i of $\mathcal{A}$ can be then defined by an equation $y = \alpha_i(x)$, where α_i is a certain linear function. We denote by R_i and I_i the real and the imaginary parts of $\alpha_i(x)$ respectively. Then the subset $L_{i,j}$ of the x-axis $\mathbf{C}_x$, defined by

$$L_{i,j} = \{x \in \mathbf{C}_x \mid R_i = R_j\},$$

is a *(real)* line on the *(real)* plane $\mathbf{C}_x$. The union of these lines

$$\mathcal{L}(\mathcal{A}) = \bigcup_{1 \le i < j \le n} L_{i,j}$$

will be called the labyrinth of the arrangement $\mathcal{A}$. For each line $L \in \mathcal{L}(\mathcal{A})$ there may are $i_1, \dots, i_k, 1 \le i_1 < i_k \le n$, such that L is defined by $R_{i_s} = R_{i_t}$, $1 \le s < t \le k$. The number $k - 1$ will be called the multiplicity of L.

REMARK 2.1. After a suitable change of coordinates we can always assume that the multiplicity of any line $L \in \mathcal{L}(\mathcal{A})$ is 1.

Clearly, each line $L_{i,j}$ divides $\mathbf{C}_x$ into two parts $L^+_{i,j} = \{x \in \mathbf{C}_x | R_i(x) < R_j(x)\}$ and $L^-_{i,j} = \{x \in \mathbf{C}_x | R_i(x) > R_j(x)\}$. Each component U of $\mathbf{C}_x - \mathcal{L}(\mathcal{A})$ can be defined by $R_{s(1)} < \dots < R_{s(n)}$ for a certain permutation s of the set $\{1, \dots, n\}$.

Now we describe the group $\pi_1(M)$ in terms of the labyrinth $\mathcal{L}(\mathcal{A})$. We choose the base point to be the infinite point on the positive half of the axis $Im\, y$. Let U be a connected component of $\mathbf{C}_x - \mathcal{L}(\mathcal{A})$. Choose an arbitrary point $x_0 \in U$. Set $\tilde{\mathbf{C}}_{x_0} = \{(x, y) \in \mathbf{C}^2 | x = x_0\}$. It is the standard fiber of the projection $\pi : \mathbf{C} \to \mathbf{C}_x$. Then $\tilde{\mathbf{C}}_{x_0} - N$ is a punctured plane with n points removed. Suppose that these points $Q_1, \dots, Q_n$ are ordered in such a way that their real parts are increasing. For $1 \le i \le n$ let γ_i denote the path on $\tilde{\mathbf{C}}_{x_0}$ which comes from infinity straight down to a point, a little higher than Q_i, then goes counterclockwise along a small circle around Q_i and goes back to the infinity by the same line as illustrated in Figure 1.

The fundamental group of $\tilde{\mathbf{C}}_{x_0} - N$ is then the free group with generators being the homotopy classes of these loops γ_i, $i = 1, \dots, n$. Identifying these

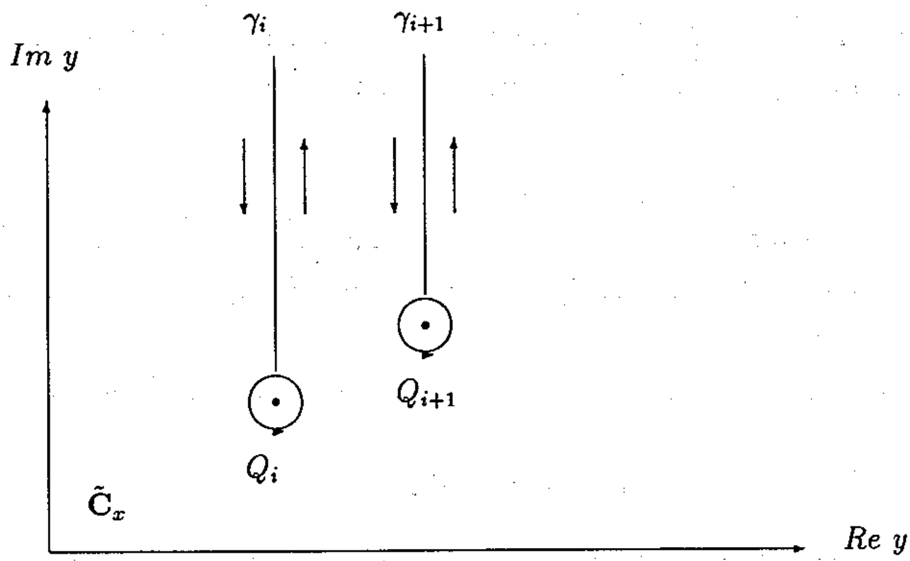

Figure 1

classes with their images in the group $\pi_1(C^2 - N)$, it is easy to show (see e.g [10] that for a chosen component U, the homotopy classes of $\gamma_1, \ldots, \gamma_n$; which will be denoted by the same notation $\gamma_1, \ldots, \gamma_n$, generate the group $\pi_1(M)$. Obviously, the ordered set $\{\gamma_1, \ldots, \gamma_n\}$ does not depend on the choice of x_0 in U. Therefore, from now on we will denote these paths by $\gamma_1(U), \ldots, \gamma_n(U)$ and the set $\{\gamma_1(U), \ldots, \gamma_n(U)\}$ by $\Gamma(U)$.

In order to determine defining relations of $\pi_1(M)$ we consider multiple points of the arrangement $\mathcal{A}$. By definition, a multiple point is the intersection of two or more hyperplanes of $\mathcal{A}$. After a suitable change of coordinates we can assume that multiple points are distinct by their x-coordinates. In other words, the images of multiple points on C_x are pairwise distinct. Denote these images by $\bar{P}_1, \dots, \bar{P}_m$. Take m simple loops $\eta_1, \dots, \eta_m$ based at x_0 on the x-asix $\mathbb{C}_x$, which surround $\bar{P}_1, \dots, \bar{P}_m$ respectively and exclusively and do not surround any point of $\mathbb{C}_x \bigcap N$. Moving the fiber $\tilde{\mathbb{C}}_{x_0}$ along the loop η_j we will get a deformation of $\tilde{\mathbb{C}}_{x_0} - N$ into itself. This deformation sends each loop γ_i to a new loop in $\tilde{\mathbb{C}}_{x_0} - N$. The homotopy class of the later can be expressed as a word $w_{i,j}$ of $\gamma_1, \dots, \gamma_n$. Then among generators of $\pi_1(\mathbb{C}^2 - N)$ we have the relation

$$\gamma_i = w_{i,j}(\gamma_1, \dots, \gamma_n). \qquad (R_{i,j})$$

The classical method (see [10], [4])shows that the group $\pi_1(\mathbb{C}^2 - N)$ has a presentation with generators $\{\gamma_1, \dots, \gamma_n\}$ and defining relations $R_{i,j} : 1 \le i \le n, 1 \le j \le m$.

In the rest of this paper we will write out $R_{i,j}$ from the labyrinth $\mathcal{L}(\mathcal{A})$.

3. Determining relations

Let $\mathcal{A}$ be an arrangement in $V = \mathbb{C}^2$ and $\mathcal{L}(\mathcal{A})$ its labyrinth. As in the previous section, for each component U of $\mathbb{C}_x - \mathcal{L}(\mathcal{A})$ we associate an ordered set $\Gamma(U) = \{\gamma_1(U), \dots, \gamma_n(U)\}$ of paths on the standard fiber of the projection π over a certain point $x_0 \in U$.

Suppose that U and U' are two adjacent components of $\mathbb{C}_x - \mathcal{L}(\mathcal{A})$, i.e. the affine support of the intersection $U \cap U'$ is a line $L \in \mathcal{L}(\mathcal{A})$. First we want to know how the set $\Gamma(U)$ is changed when the point x_0 moves from U to U'. Let $\Gamma(U) = \{\gamma_1(U), \dots, \gamma_n(U)\}$. By means of Remark 2.1 we can assume that the line $L \in \mathcal{L}(\mathcal{A})$ is defined by real parts of two hyperplanes, which determine $\gamma_i(U)$ and $\gamma_{i+1}(U)$, for some $1 \le i \le n-1$. Without loss of generality, we may also assume that these hyperplanes are H_i and H_{i+1}, i.e. L is defined by the equation $R_i = R_{i+1}$. Suppose that $U \subset L^+$ and $U' \subset L^-$. We have then $\Gamma(U') = \{\gamma_1(U'), \dots, \gamma_n(U')\}$, where

$$\gamma_j(U') = \gamma_j(U) \text{ with } j < i \text{ and } j > i+1.$$

In order to relate $\gamma_i(U), \gamma_{i+1}(U)$ and $\gamma_i(U'), \gamma_{i+1}(U')$ we consider the (*real*) line L' on $\mathbb{C}_x$, defined by the equation $I_i = I_{i+1}$. Also, the line L' divides $\mathbb{C}_x$ into two parts $L'^{+} = \{c \in \mathbf{C}_x; I_i(c) < I_{i+1}(c)\}$ and $L'^{-} = \{c \in \mathbf{C}_x; I_i(c) > I_{i+1}(c)\}$. There are two cases.

1) x_0 moves from U to U' alongs a path lying in L'^{+}. In this case we have

$$(A) \quad \begin{cases} \gamma_i(U') = \gamma_i(U)^{-1}.\gamma_{i+1}(U).\gamma_i(U), \\ \gamma_{i+1}(U') = \gamma_i(U). \end{cases}$$

2) x_0 moves from U to U' alongs a path lying in L'^{-}. In this case we have

$$(B) \quad \begin{cases} \gamma_i(U') = \gamma_{i+1}(U), \\ \gamma_{i+1}(U') = \gamma_{i+1}(U).\gamma_i(U).\gamma_{i+1}(U)^{-1} \end{cases}$$

The above relations are consequences of the definition of $\gamma_i(U)$ and of the following lemma.

LEMMA 3.1(see [6]). *Consider the three loops in* $\mathbf{R}^2 - \{2\ points\}$, *which are depicted as in the Figure 2. Among these we have two relations*

$$\gamma = \beta^{-1}.\alpha.\beta \text{ and } \alpha = \beta.\gamma.\beta^{-1} \quad .$$

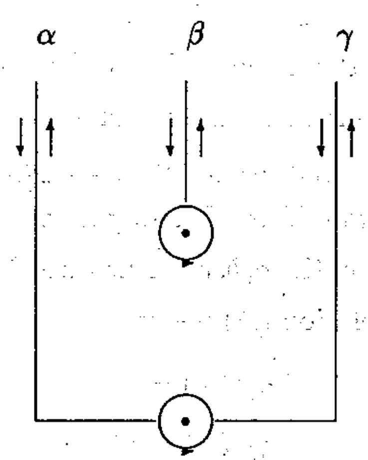

Figure 2

Next we want to know what relation will be given locally when x_0 moves around the image of a multiple point. Suppose that $\bar{P}$ is the image of a multiple point P, U is a component of $\mathbf{C}_x - \mathcal{L}(\mathcal{A})$ which is incident to $\bar{P}$, i.e. U has $\bar{P}$ on its boundary and $\Gamma(U) = \{\gamma_1(U), \dots, \gamma_n(U)\}$. Suppose that P is the intersection of the hyperplanes which define $\gamma_i(U), \gamma_{i+1}(U), \dots, \gamma_j(U)$ for certain $1 \leq i < j \leq n$. Without loss of generality we may assume that these hyperplanes are $H_i, H_{i+1}, \dots, H_j$. Then we have

Lemma 3.2. *Consider the fundamental group $\pi_1(M)$ with generators $\gamma_1(U), \dots, \gamma_n(U)$. We have*

$$\gamma_j(U)\gamma_{j-1}(U)...\gamma_i(U) = \gamma_{c(j)}(U)\gamma_{c(j-1)}(U)\dots\gamma_{c(i)}(U),$$

where c is any cyclic permutation of $\{j, j-1, \dots, i\}$.

Proof. By moving $x_0 \in U$ around $\bar{P}$ the paths $\gamma_1(U), \dots, \gamma(U)$ are deformed into paths $\alpha_1(U), \dots, \alpha_n(U)$. Denote the homotopy classes of these paths by the same notation $\alpha_1(U), \dots, \alpha_n(U)$. Obviously, we have

$$\alpha_l(U) = \gamma_l(U) \text{ for } l < i \text{ and } l > j.$$

We will try to relate $\gamma_i(U), \dots, \gamma_j(U)$ to $\gamma_i(U), \dots, \gamma_j(U)$. For the sake of simplicity we consider the case when $i = 1$ and $j = 3$, as illustrated in Figure 3. The general case is quite similar.

Let U be contained in the domain of $\mathbf{C}_x$ defined by $R_1 < R_2 < R_3$ and other components, which incident to $\bar{P}$, in corresponding domains as determined in Figure 3. Denote $\gamma_1(U), \gamma_2(U)$ and $\gamma_3(U)$ simply by a, b and c respectively. Suppose that x_0 moves from U to U_1 along a path lying in $L^+_{1,2}$. After x_0 has moved from U to U_1, the paths a, b, c are deformed into $\gamma_1(U_1), \gamma_2(U_1), \gamma_3(U_1)$, respectively. Using the relation (A) we have

$$\gamma_1(U_1) = a^{-1}ba$$
$$\gamma_2(U_1) = a$$
$$\gamma_3(U_1) = c.$$

Suppose that when coming back to the component U the paths a, b, c are deformed into the paths α, β, γ, respectively. Continuing the above procedure, the homotopy classes of these new paths, denoted by the same notations α, β, γ, can be expressed as

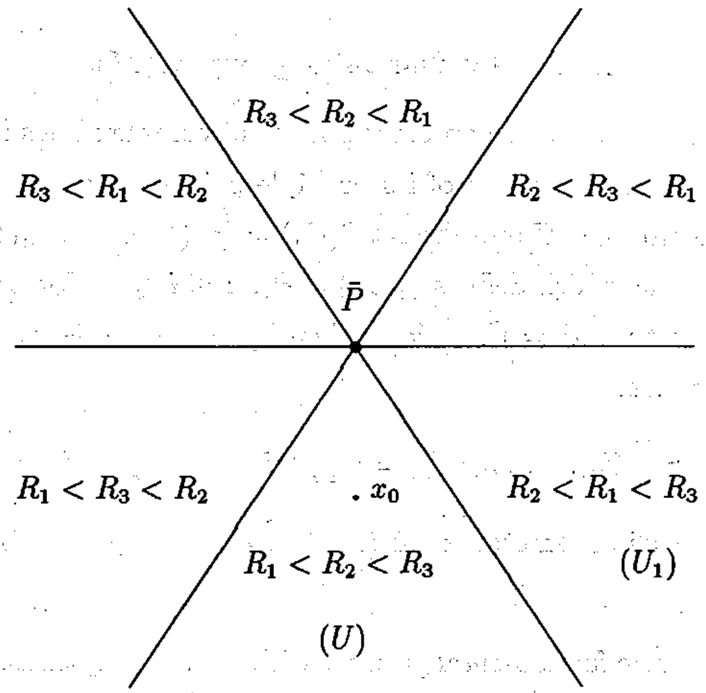

Figure 3

$$\alpha = a^{-1}b^{-1}c^{-1}acba$$
$$\beta = a^{-1}b^{-1}c^{-1}bcba$$
$$\gamma = a^{-1}b^{-1}cba\,.$$

Therefore, in the group $\pi_1(M)$ we have the following relations

$$a = a^{-1}b^{-1}c^{-1}acba$$
$$b = a^{-1}b^{-1}c^{-1}bcba$$
$$c = a^{-1}b^{-1}cba\,.$$

From these relations we easily get

$$cba = bac = acb.$$

The proof is complete.

4. The fundamental group $\pi_1(M)$.

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of complex lines in $\mathbb{C}^2$ with the complement M and the set $\mathcal{M}$ of its multiple points. Fix a component U_0 of $\mathbb{C}_x - \mathcal{L}(\mathcal{A})$, a point $x_0 \in U$ and the set $\Gamma(U_0) = \{\gamma_1(U_0), \dots, \gamma_n(U_0)\}$. Suppose that the hyperplane which defines $\gamma_i(U_0)$ is H_i, $1 \leq i \leq n$. For a multiple point $P \in \mathcal{M}$ suppose that $P = \bigcap_{j=1}^{s} H_{i_j}$. Then by $R(P)$ we denote the set of all relations of the form

$$\gamma_{i_s}(U_0). \dots .\gamma_{i_1}(U_0) = \gamma_{i_{c(s)}}(U_0). \dots .\gamma_{i_{c(1)}}(U_0),$$

where c is any cyclic permutation of $\{s, \dots, 1\}$. Now we state the main result of this paper.

THEOREM 4.1. *The fundamental group $\pi_1(M)$ of the complement of the arrangement $\mathcal{A}$ has a presentation with generators $\gamma_1(U_0), \dots, \gamma_n(U)$ and defining relations $R(P)$, $P \in \mathcal{M}$.*

PROOF. Consider a multiple point $P = \bigcap_{j=1}^{s} H_{i_j}$ in $\mathcal{M}$. Suppose that the component U_0 is defined by $R_1 < \dots < R_{i_1} < \dots < R_{i_2} < \dots < R_{i_s} < \dots < R_n$. Without loss of generality we can assume that $i_1 = 1$ and $i_s = n$. Also, for the sake of simplicity we can assume that $s = 3$. Let fix a component U_P of $\mathbb{C}_x - \mathcal{L}(\mathcal{A})$, which is incident to the image $\bar{P}$ on $\mathbb{C}_x$ of P. Suppose that U_P is defined by $R_2 < R_3 < \dots < R_{j-1} < R_{j+1} < \dots < R_{n-1} < R_1 < R_j < R_n$.

Now let γ be the path going from $x_0 \in U_0$ to a point $x_P \in U_P$ defined as follows. It goes from one component to another component of $\mathbb{C}_x - \mathcal{L}(\mathcal{A})$, step by step, according to the following order

$$R_1 < R_2 < R_3 < \dots < R_{j-1} < R_j < R_{j+1} < \dots < R_{n-2} < R_{n-1} < R_n$$

$$R_1 < R_2 < R_3 < \dots < R_{j-1} < R_{j+1} < R_j < R_{j+2} < \dots < R_{n-1} < R_n$$

$$\dots$$

$$R_1 < R_2 < R_3 < \dots < R_{j-1} < R_{j+1} < R_{j+2} < \dots < R_{n-1} < R_j < R_n$$

$$R_2 < R_1 < R_3 < \dots < R_{j-1} < R_{j+1} < R_{j+2} < \dots < R_{n-1} < R_j < R_n$$

$$\dots$$

$$R_2 < R_3 < \dots < R_{j-1} < R_{j+1} < R_{j+2} < \dots < R_{n-1} < R_1 < R_j < R_n.$$

In each step, when γ crosses a line $L \in \mathcal{L}(\mathcal{A})$, we can choose γ so that it lies in the domain L'^{+}. We can then use the relations (A) to determine $\Gamma(U)$.

By moving x_0 along the path γ to $x_P \in U_P$, then surrounding it counterclockwise around $\bar{P}$ and moving it back to x_0 by the same path γ we obtain the relations of the group $\pi_1(M)$.

First, when x_0 moves to x_p along γ, the paths $\gamma_1(U_0), \dots, \gamma_n(U_0)$ are deformed into paths $\gamma_1(U_P), \dots, \gamma_n(U_P)$ and we have

$$\begin{aligned}
\gamma_i(U_P) &= \gamma_1(U_0)^{-1}\gamma_{i+1}(U_0)\gamma_1(U_0) \text{ for } i < j-1\\
\gamma_i(U_P) &= \gamma_1(U_0)^{-1}\gamma_j(U_0)^{-1}\gamma_{i+2}(U_0)\gamma_j(U_0)\gamma_1(U_0) \text{ for } j-1 \leq i < n-2\\
\gamma_{n-2}(U_P) &= \gamma_1(U_0)\\
\gamma_{n-1}(U_P) &= \gamma_j(U_0)\\
\gamma_n(U_P) &= \gamma_n(U_0).
\end{aligned}$$

Suppose that after x_P has surrounded counterclockwise the multiple point $\bar{P}$, the paths $\gamma_i(U_P)$ are deformed into $\alpha_i(U_P)$. We have

$$\begin{aligned}
\alpha_i(U_P) &= \gamma_i(U_P) \text{ for } i \leq n-3\\
\alpha_{n-2}(U_P) &= w_1\\
\alpha_{n-1}(U_P) &= w_2\\
\alpha_n(U_P) &= w_3,
\end{aligned}$$

for certain words w_1, w_2, w_3 of $\gamma_1(U_0), \gamma_j(U_0), \gamma_n(U_0)$.

Now moving x_P by the same path γ back to x_0, the paths $\alpha_i(U_P)$, $1 \leq i \leq n$, are deformed into $\alpha_i(U_0)$, $1 \leq i \leq n$, respectively. Remind that this time we have to use (B) to relate $\alpha_i(U_0)$ and $\alpha_i(U_P)$. We have

$$\alpha_i(U_0) = w_1\gamma_1(U_0)^{-1}\gamma_i(U_0)\gamma_1(U_0)w_1^{-1} \text{ for } 1 < i < j$$
$$\alpha_i(U_0) = w_2w_1\gamma_1(U_0)^{-1}\gamma_j(U_0)^{-1}\gamma_i(U_0)\gamma_j(U_0)\gamma_1(U_0)w_1^{-1}w_2^{-1} \text{ for } j < i < n$$
$$\alpha_1(U_0) = w_1$$
$$\alpha_j(U_0) = w_2$$
$$\alpha_n(U_0) = w_3.$$

Therefore, among generators $\gamma_1(U_0), \ldots, \gamma_n(U_0)$ we have following relations

$$\gamma_i(U_0) = w_1\gamma_1(U_0)^{-1}\gamma_i(U_0)\gamma_1(U_0)w_1^{-1} \text{ for } 1 < i < j$$
$$\gamma_i(U_0) = w_2w_1\gamma_1(U_0)^{-1}\gamma_j(U_0)^{-1}\gamma_i(U_0)\gamma_j(U_0)\gamma_1(U_0)w_1^{-1}w_2^{-1} \text{ for } j < i < n$$
$$\gamma_1(U_0) = w_1$$
$$\gamma_j(U_0) = w_2$$
$$\gamma_n(U_0) = w_3.$$

Using the last three rows we substitute $\omega_1, \omega_2, \omega_3$ in the first two rows and get trivial relations. As a consequence of Lemma 3.1, from last the three rows we obtain the required relations $R(P)$. Now an application of Van Kampen's Theorem (see [4]) completes the proof of the theorem.

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