ACTA MATHEMATICA VIETNAMICA
Volume 19, Number 2, 1994, pp. 121-135

EXISTENCE OF GLOBAL MINIMAX SOLUTIONS OF THE CAUCHY PROBLEM FOR SYSTEMS OF FIRST-ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS[1]

TRAN DUC VAN AND NGUYEN DAC LIEM

The purpose of this paper is to present the existence results of global minimax solutions for some systems of first-order nonlinear partial differential equations (PDEs).

Since a classical solution of the nonlinear problem can fail to exist even in the cases where the data are analytic functions, we need to introduce concepts of generalized solutions.

In recent years many different methods have been created by Benton S. H., Cole V. J. D., Conway E. D., Crandall M. G., Doubnov B., Evans L. C., Fleming W. H., Glimm J., Hopf E., Kružkov S. N., Lax P. D., Lions P. L., Maslov V. P., Oleinhik O., Rozdestvenskii B. L., Subbotin A. I., Tsuji M. ... in the study of global generalized solutions of nonlinear PDEs. Especially, nonclassical theory of nonlinear PDEs represents a large portion of research in which the concept of global viscosity solutions introduced by Crandall and Lion [4,5] is used.

Another direction in this theory is motivated by differential games which leads the notion of global minimax solutions for the first-order nonlinear PDEs. The case of the Cauchy problem for a scalar nonlinear PDE of first-order was studied in great detail by Subbotin A. I., Subbotina N. N., Adiatulina ... (see, for example, [1,7,8]). As the terminology "minimax solution" indicates, solutions of nonlinear PDEs of first-order are closely connected with the minimax operations.

Received June 15th, 1994
[1]Supported in part by the National Basic Research Program in Natural Science, Vietnam.

In this paper for the first time we shall study minimax solutions of the Cauchy problems for some systems of first-order nonlinear PDEs. Namely, we are concerned with the problem

$$\frac{\partial u_k}{\partial t}(t,x) + H_k(t,x,u(t,x),\nabla_x u_k(t,x)) = 0, \tag{1}$$

$$(t,x) \in G := (0,T)\times \mathbb{R}^n, \quad k=1,\dots,m,$$

$$u(T,x) = u^0(x), \quad x\in \mathbb{R}^n. \tag{2}$$

Here $u=(u_1,\dots,u_m):\bar{G}\to\mathbb{R}^m$ represents the unknown function,

$$H := (H_1,\dots,H_m) : G\times\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^m,$$

$$u^0=(u_1^0,\dots,u_m^0):\mathbb{R}^n\to\mathbb{R}^m$$

are given functions, $\nabla_x u_k = (\frac{\partial u_k}{\partial x_1},\dots,\frac{\partial u_k}{\partial x_n})$.

We use the following notations:

$$S := \{p\in\mathbb{R}^n : \|p\|_n = 1\},$$

$$B := \{p\in\mathbb{R}^n : \|p\|_n \le 1\},$$

$$B_i(x,\delta) := \{x'\in\mathbb{R}^i : \|x'-x\|_i < \delta\},$$

where $\|.\|_i$ is the Euclide norm in $\mathbb{R}^i, i\in I\!N$. For $r=(r_1,\dots,r_m), s=(s_1,\dots,s_m)\in\mathbb{R}^m$ we write $r\le s$, if $r_k\le s_k$ for all $k=1,\dots,m$. For functions $u,v:A\to\mathbb{R}^m$ we also write $u\le v$ (on A) if $u_k(z)\le v_k(z)$ for all $z\in A,\ k=1,\dots,m$. Beside, let

$$H_k^{(s)}(t,x,r,p) := H_k(t,x,s_1,\dots,s_{k-1},r_k,s_{k+1},\dots,s_m,p),$$

$$H^{(s)} := (H_1^{(s)},\dots,H_m^{(s)}),$$

where $(t,x)\in G,\ p\in\mathbb{R}^n,\ r\in\mathbb{R}^m,\ s\in\mathbb{R}^m$.

We first assume that the function u^0 is continuous and H satisfies the conditions:

a) The function $(t,x,r,p)\to H(t,x,r,p)$ is continuous and positive-homogeneous with respect to the variable p,

$$H(t,x,r,\alpha p) = \alpha H(t,x,r,p), \quad \alpha\ge 0. \tag{3}$$

b) $H^{(r)}(t,x,l,p) \leq H^{(s)}(t,x,l,p), \quad r \leq s,$ (4)
$\forall (t,x,l,r,s,p) \in G \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times S,$

(the quasi-monotonicity condition with respect to the variable r).

c) $H^{(l)}(t,x,r,p) \geq H^{(l)}(t,x,s,p), \quad r \leq s,$ (5)
$\forall (t,x,l,r,s,p) \in G \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times S.$

d) $H^{(l)}(t,x,r,p) - H^{(l)}(t,x,s,p) \leq \lambda(s-r), \quad r \leq s,$ (6)
$\forall (t,x,l,r,s,p,) \in (0,T) \times C \times R \times R \times R \times S,$

where $\lambda = \lambda(C,R) \geq 0$ and C, R are bounded sets of $\mathbb{R}^n$, $\mathbb{R}^m$, respectively.

e) $\|H(t,x,r,p) - H(t,x,r,q)\|_m \leq \kappa(1+\|x\|_n)\|p-q\|_n,$ (7)
$\forall (t,x,r,p,q) \in G \times \mathbb{R}^m \times B \times B,$

where κ is a positive constant (the Lipschitz condition with respect to the variable p).

Let us define the sets

$$
\begin{aligned}
F(x) &:= \{f \in \mathbb{R}^n : \|f\|_n \leq \sqrt{2}\,\kappa\,(1+\|x\|_n)\},\\
(F_U)_k(t,x,r,q) &:= \{f \in F(x) : < f,q > \geq H_k(t,x,r,q)\},\\
(F_L)_k(t,x,r,p) &:= \{f \in F(x) : < f,p > \leq H_k(t,x,r,p)\},
\end{aligned}
$$

where $(t,x) \in G,\ r \in \mathbb{R}^m,\ p \in S,\ q \in S,\ k = 1,\dots,m.$

From (3) and (7) it is easily seen that these sets are nonempty, convex and compact. From the condition a), Propositions 1.4.9, 1.5.1 and Corollary 1.4.10 in [2] we deduce that the muti-valued function $x \to F(x)$ is globally Lipschitz continuous in $\mathbb{R}^n$, and the muti-valued functions $(t,x,r) \to (F_U)_k(t,x,r,q)$ and $(t,x,r) \to (F_L)_k(t,x,r,p)$ are continuous in $G \times \mathbb{R}^n$, where $p \in S,\ q \in S,\ k = 1,\dots,m.$

From (4)-(6) and $r \leq s$ we have

$$(F_U)_k^{(r)}(t,x,l,q) \supset (F_U)_k^{(s)}(t,x,l,q), \tag{9a}$$

$$(F_L)_k^{(r)}(t,x,l,p) \subset (F_L)_k^{(s)}(t,x,l,p), \tag{9b}$$

$$(F_U)_k^{(l)}(t,x,r,q) \subset (F_U)_k^{(l)}(t,x,s,q), \tag{10a}$$

$$(F_L)_k^{(l)}(t,x,r,p) \supset (F_L)_k^{(l)}(t,x,s,p), \tag{10b}$$

where $(t,x) \in G,\ l,r,s \in \mathbb{R}^m,\ p,q \in S,\ k=1,\dots,m$; and

$$(F_U)_k^{(l)}(t,x,s,q) \subset (F_U)_k^{(l)}(t,x,r,q) + \Lambda(s_k - r_k)B, \tag{11a}$$

$$(F_L)_k^{(l)}(t,x,r,p) \subset (F_L)_k^{(l)}(t,x,s,p) + \Lambda(s_k - r_k)B, \tag{11b}$$

for all $(t,x,l,r,s,p,q) \in (0,T)\times C\times R\times R\times R\times S\times S$, where $\Lambda = \sqrt{2}\lambda(C,R) \geq 0$ and C,R are bounded sets of $\mathbb{R}^n, \mathbb{R}^m$, respectively.

We can now proceed analogously to the proof of Subbotin A.I. in [7] (Chapter 1, section 2) to show that

$$(F_U)_k(t,x,r,q) \cap (F_L)_k(t,x,r,p) \neq \emptyset,$$

for all $(t,x) \in G,\ r \in \mathbb{R}^m,\ p \in S,\ q \in S,\ k=1,\dots,m$. Hence we obtain the equalities

$$H_k(t,x,r,w) = \sup_{q\in S} \min_{f\in(F_U)_k(t,x,r,q)} < f,w > \tag{12a}$$

$$H_k(t,x,r,w) = \inf_{p\in S} \max_{f\in(F_L)_k(t,x,r,p)} < f,w > \tag{12b}$$

for all $(t,x) \in G,\ r \in \mathbb{R}^m,\ w \in \mathbb{R}^n,\ k=1,\dots,m$.

Briefly, let (F_U) and (F_L) stand for the tubles of multi-valued functions $((F_U)_1,\dots,(F_U)_m)$ and $((F_L)_1,\dots,(F_L)_m)$, respectively, in which

$$(F_U)_k, (F_L)_k : G\times\mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}^n,\ k=1,\dots,m,$$

take nonempty convex and compact values.

Denote by $\mathcal{U}(H)$ and $\mathcal{L}(H)$ the sets of the tubles (F_U) and (F_L) such that $(F_U)_k(.,.,.,q),\ (F_L)_k(.,.,.,p)$: are continuous in $G\times\mathbb{R}^m$, for all $p \in S,\ q\in S$, and satisfy (9)-(12), respectively. Moreover,

$$(F_U)_k(t,x,r,q) \cup (F_L)_k(t,x,r,p) \subset c(1+\|x\|_n)B,$$

$$\forall (t,x,r,p,q) \in G\times\mathbb{R}^m\times S\times S,\ k=1,\dots,m,$$

where c is a positive number.

We note that $\mathcal{U}(H) \neq \phi$ and $\mathcal{L}(H) \neq \phi$. From (12) and the Hahn-Banach Theorem, it follows that

$$(F_U)_k(t,x,r,q) \cap (F_L)_k(t,x,r,p) \neq \emptyset \tag{13}$$

$$\forall (t,x,r,p,q) \in G \times \mathbb{R}^m \times S \times S,\ k = 1,\ldots,m.$$

From now on, we work only with $(F_U) \in \mathcal{U}(H), (F_L) \in \mathcal{L}(H)$ and $F(x) = c(1 + \|x\|_n)B, x \in \mathbb{R}^n$. For a function $v : \bar{G} \to \mathbb{R}^1$ we define

$$v^*(z) := \limsup_{\epsilon \downarrow 0} \{v(y) : \|z - y\|_{n+1} < \epsilon,\ y \in \bar{G}\},$$

$$v_*(z) := \liminf_{\epsilon \downarrow 0} \{v(y) : \|z - y\|_{n+1} < \epsilon,\ y \in \bar{G}\},$$

for all $z \in \bar{G}$. We note that v^* and v_* are upper and lower semicontinuous functions respectively, with values in $\mathbb{R}^1 \cup \{\pm\infty\}$, and $v_* \le v \le v^*$ on $\bar{G}$. Moreover if v is a locally bounded function then v^* and v_* take values in $\mathbb{R}^1$.

Briefly, we put $u^* = (u_1^*, \ldots, u_m^*)$ and $u_* = (u_{1*}, \ldots, u_{m*})$ for a function $u = (u_1, \ldots, u_m) : \bar{G} \to \mathbb{R}^m$. Further, for a locally bounded function $u : \bar{G} \to \mathbb{R}^m$ and $r_k \in \mathbb{R}^1$, we write

$$(F_U)_k^u(t,x,r_k,q) = (F_U)_k(t,x,u_{1*}(t,x),\ldots,u_{(k-1)*}(t,x),r_k,u_{(k+1)*}(t,x),\ldots, u_{m*}(t,x),q),$$

$$(F_L)_k^u(t,x,r_k,p) = (F_L)_k(t,x,u_1^*(t,x),\ldots,u_{k-1}^*(t,x),r_k,u_{k+1}^*(t,x),\ldots, u_m^*(t,x),p),$$

where $(t,x) \in G,\ p \in S\ q \in S,\ k = 1,\ldots,m$. We have

LEMMA 1. *The muti-valued functions $(t,x) \to (F_U)_k^u(t,x,r_k,q)$ and $(t,x) \to (F_L)_k^u(t,x,r_k,p)$ are upper semicontinuous in G.*

PROOF. The lemma is deduced from (9)-(11) and Proposition 1.4.9 in [2].

Here we note that the conditions (9a) and (9b) deduced from the quasi-monotonicity condition of the function H (the condition b)) with respect to the variable r are essential.

From (9), Lemma 1 and the fact that $(F_U)_k$, $(F_L)_k$ take nonempty convex compact values in $\mathbb{R}^n$, we get that the muti-valued functions

$$(t,x) \to (F_U)_k^u(t,x,r_k,q) \text{ and } (t,x) \to (F_L)_k^u(t,x,r_k,p)$$

satisfy the conditions of Theorem II3 in [7]. Denote by $(X_U)_k^u(t_0, x_0, r_k, q)$ and $(X_L)_k^u(t_0, x_0, r_k, p)$ the sets of all absolutely continuous functions $[0, T] \ni t \to x(t) \in \mathbb{R}^n$, satifying for almost all $t \in [0, T]$ the differential inclusions

$$\dot{x}(t) \in (F_U)_k^u(t, x(t), r_k, q),$$

and

$$\dot{x}(t) \in (F_L)_k^u(t, x(t), r_k, p),$$

respectively, and also the condition $x(t_0) = x_0$, where $(t_0, x_0) \in \bar{G}$. Note that both $(X_U)_k^u(t_0, x_0, r_k, q)$ and $(X_L)_k^u(t_0, x_0, r_k, p)$ are nonempty compact sets in $C([0, T], \mathbb{R}^n)$.

DEFINITION 1. *Let $u : \bar{G} \to \mathbb{R}^m$ be a locally bounded function. We call u a minimax supersolution of Problem (1)-(2) if for all $t_0 \in [0, T)$, $\tau \in (t_0, T]$, $x_0 \in \mathbb{R}^n$ and $k = 1, \ldots, m$, then*

$$\sup_{q \in S} \inf_{x(.) \in (X_U)_k^u(t_0, x_0, u_k(t_0, x_0), q)} \{u_k(\tau, x(\tau)) - u_k(t_0, x_0)\} \leq 0 \tag{14}$$

and

$$u(T, x) \geq u^0(x), \quad x \in \mathbb{R}^n. \tag{15}$$

DEFINITION 2. *Let $u : \bar{G} \to \mathbb{R}^m$ be a locally bounded function. We call u a minimax subsolution of Problem (1)-(2) if for all $t_0 \in [0, T)$, $\tau \in (t_0, T]$, $x_0 \in \mathbb{R}^n$ and $k = 1, \ldots, m$, then*

$$\inf_{p \in S} \sup_{x(.) \in (X_L)_k^u(t_0, x_0, u_k(t_0, x_0), p)} \{u_k(\tau, x(\tau)) - u_k(t_0, x_0)\} \geq 0 \tag{16}$$

and

$$u(T, x) \leq u^0(x), \quad x \in \mathbb{R}^n. \tag{17}$$

The sets of supersolutions and subsolutions will be denoted by Sol_U and Sol_L, respectively.

DEFINITION 3. *Let $u : \bar{G} \to \mathbb{R}^m$ be a locally bounded function. We call u a minimax solution of Problem (1)-(2) if it is simultaneously a minimax supersolution and a minimax subsolution of the same problem.*

THEOREM 1. *Suppose that $u \in C^1(G, \mathbb{R}^m) \cap C(\bar{G}, \mathbb{R}^m)$ is a global classical solution of Problem (1)-(2). Then u is also a minimax solution of the same problem.*

PROOF. We first prove that u is a minimax supersolution of Problem (1)-(2). Since u is a global classical solution, we have

$$\frac{\partial u_k}{\partial t}(t,x) + H_k(t,x,u(t,x),\nabla_x u_k(t,x)) = 0, \tag{18}$$

$$\forall (t,x) \in G,\ k = 1,\dots,m.$$

Suppose that $t_0 \in [0,T), x_0 \in \mathbb{R}^n, q \in S, k = 1,\dots,m$. Let $\bar{u} = (\bar{u}_1,\dots,\bar{u}_m)$ in which $\bar{u}_i = u_i$ on $\bar{G}$, $i \neq k$ and

$$\bar{u}_k(t,x) = \max\{u_k(t,x),\ u_k(t_0,x_0)\}, \quad (t,x) \in \bar{G}.$$

Since both $\bar{u}$ and $(t,x,r) \to (F_U)_k(t,x,r,q)$ are continuous in G and $G \times \mathbb{R}^m$ respectively, it follows that the composition of functions

$$(t,x) \to (F_U)_k(t,x,\bar{u}(t,x),q)$$

is also continuous in G. Set

$$F_k^0(t,x,\bar{u}(t,x),q) = \Big\{ f_0 \in (F_U)_k(t,x,\bar{u}(t,x),q);\ < f_0, \nabla_x u_k(t,x) > = \\ = \min_{f \in (F_U)_k(t,x,\bar{u}(t,x),q)} < f, \nabla_x u_k(t,x) > \Big\}.$$

From (12a) and (5) we have

$$\begin{aligned} < f_0, \nabla_x u_k(t,x) > &\leq H_k(t,x,\bar{u}(t,x),\nabla_x u_k(t,x)) \\ &\leq H_k(t,x,u(t,x),\nabla_x u_k(t,x)), \end{aligned} \tag{19}$$

$\forall f_0 \in F_k^0(t,x,\bar{u}(t,x),q)$, $(t,x) \in G$. It is easily seen that the set $F_k^0(t,x,\bar{u}(t,x),q)$ is nonempty convex and compact. Moreover, since $(t,x) \to (F_U)_k(t,x,\bar{u}(t,x),q)$ and $(t,x) \to \nabla_x u_k(t,x)$ are continuous, it follows that $(t,x) \to F_k^0(t,x,\bar{u}(t,x),q)$ is upper semicontinuous. From this we deduce that the set of solutions $X_k^0(t_0,x_0,\bar{u},q)$ of the differential inclusion

$$\dot{x}(t) \in F_k^0(t,x(t),\bar{u}(t,x(t)),q), \quad a.e.\ t \in [0,T],$$

satisfying the condition $x(t_0) = x_0$, is nonempty and compact.

Fix $y(.) \in X_k^0(t_0, x_0, \bar{u}, q)$. Then the function $[0, T] \ni t \to u_k(t, y(t)) \in \mathbb{R}^1$ is absolutely continuous in $(0, T)$.

From (18) and (19) we obtain

$$\begin{aligned}\frac{du_k}{dt}(t, y(t)) &= \frac{\partial u_k}{\partial t}(t, y(t)) + < \dot{y}(t), \nabla_x u_k(t, y(t)) > \\ &\leq \frac{\partial u_k}{\partial t}(t, y(t)) + H_k(t, y(t), u(t, y(t)), \nabla_x u_k(t, y(t))) = 0\end{aligned}$$

for almost all $t \in [0, T]$. Hence

$$u_k(t, y(t)) \leq u_k(t_0, x_0), \quad t_0 < t \leq T. \tag{20}$$

Thus

$$\begin{aligned}F_k^0(t, y(t), \bar{u}(t, y(t)), q) &\subset (F_U)_k(t, y(t), \bar{u}(t, y(t)), q) \\ &\subset (F_U)_k^u(t, y(t), u_k(t_0, x_0), q), \quad t_0 < t \leq T.\end{aligned}$$

Choose $z(.) \in (X_U)_k^u(t_0, x_0, u_k(t_0, x_0), q)$. We set

$$x(t) = \begin{cases} z(t) & 0 \leq t \leq t_0, \\ y(t) & t_0 < t \leq T. \end{cases}$$

Therefore $x(.) \in (X_U)_k^u(t_0, x_0, u_k(t_0, x_0), q)$. From this and (20), we conclude that u_k satisfies the condition (14). Hence u is a minimax supersolution of Problem (1)-(2).

By an argument analogous to the above proof, we show that u is also a minimax subsolution of the same problem. Theorem 1 is completely proved.

THEOREM 2. *The global minimax solution u of Problem (1)-(2) satisfies the equation (1) at each point (t, x) where u is differentiable.*

PROOF. By definition, u belongs to Sol_U. Let $q \in S$, $k = 1, \ldots, m$. We assume that u is differentiable at $(t_0, x_0) \in G$. From (14), for $\epsilon > 0$ and $\delta \in (0, T - t_0)$, then there exists $x_\delta(.) \in (X_U)_k^u(t_0, x_0, u_k(t_0, x_0), q)$ such that

$$u_k(t_0 + \delta, x_\delta(t_0 + \delta)) - u_k(t_0, x_0) < \epsilon\delta \tag{21}$$

Since $(t,x) \to (F_U)^u_k(t,x,u_k(t_0,x_0),q)$ is upper semicontinuous at (t_0,x_0), then there exists $\gamma_j > 0$ such that for every $t \in B_1(t_0,\gamma_j)$, $x \in B_n(x_0,\gamma_j)$ we have

$$(F_U)^u_k(t,x,u_k(t_0,x_0),q) \subset (F_U)_k(t_0,x_0,u(t_0,x_0),q) + \frac{1}{j}B,$$

for any $j \in \mathbb{N}$.

From the fact that $\gamma_j > 0$ and $x_\delta(.)$ belongs to compact set

$$(X_U)^u_k(t_0,x_0,u_k(t_0,x_0),q) \subset C([0,T],\mathbb{R}^n)$$

for every $\delta \in (0,T-t_0)$, it follows that there exists $\delta_j \in (0,T-t_0)$ such that

$$x_{\delta_j}(t) \in B_n(x_0,\gamma_j),\ \forall t \in B_1(t_0,\delta_j),$$

and $\delta_j \to 0$ as $j \to \infty$. Therefore

$$\dot{x}_{\delta_j}(t) \in (F_U)^u_k(t,x_{\delta_j}(t),u_k(t_0,x_0),q) \subset (F_U)_k(t_0,x_0,u(t_0,x_0),q) + \frac{1}{j}B,$$

for almost all $t \in B_1(t_0,\delta_j)$. Applying Lemma 12 in [6] we get

$$\frac{1}{\delta_j}\big(x_{\delta_j}(t_0+\delta_j) - x_0\big) = \int_{t_0}^{t_0+\delta_j} \dot{x}_{\delta_j}(t)dt \in (F_U)_k(t_0,x_0,u(t_0,x_0),q) + \frac{1}{j}B$$

Since $\left\{(F_U)_k(t_0,x_0,u(t_0,x_0),q)+\frac{1}{j}B\right\}_{j\geq 1}$ is a decreasing sequence of sets, tends to the compact set $(F_U)_k(t_0,x_0,u(t_0,x_0),q)$, we find that

$$\frac{1}{\delta_j}\big(x_{\delta_j}(t_0+\delta_j) - x_0\big) = f_0 + g_j, \tag{22}$$

where $f_0 \in (F_U)_k(t_0,x_0,u(t_0,x_0),q)$ and $\|g_j\|_n \to 0$ as $j \to \infty$. From (21)-(22) and the fact that u_k is differentiable at (t_0,x_0), we see that

$$\frac{1}{\delta_j}\Big(u_k(t_0+\delta j,x_{\delta_j}(t_0+\delta_j)) - u_k(t_0,x_0)\Big) = \\ \frac{\partial u_k}{\partial t}(t_0,x_0) + < \nabla_x u_k(t_0,x_0), f_0 + g_j > + \alpha_j < \epsilon, \tag{23}$$

where $\alpha_j \to 0$ as $j \to \infty$. Letting $j \to \infty$ in (23) we get

$$\frac{\partial u_k}{\partial t}(t_0,x_0) + \min_{f \in (F_U)_k(t_0,x_0,u(t_0,x_0),q)} < \nabla_x u_k(t_0,x_0), f > \\ \leq \frac{\partial u_k}{\partial t}(t_0,x_0) + < \nabla_x u_k(t_0,x_0), f_0 > < \epsilon,$$

for any $q \in S$. Consequently,

$$\frac{\partial u_k}{\partial t}(t_0,x_0) + H_k(t_0,x_0,u(t_0,x_0),\nabla_x u_k(t_0,x_0)) = \\ \frac{\partial u_k}{\partial t}(t_0,x_0) + \sup_{q \in S} \min_{f \in (F_U)_k(t_0,x_0,u(t_0,x_0),q)} < \nabla_x u_k(t_0,x_0), f > \leq \epsilon.$$

Since both $\epsilon > 0$ and k can be arbitrarily chosen, we have

$$\frac{\partial u_k}{\partial t}(t_0,x_0) + H_k(t_0,x_0,u(t_0,x_0),\nabla_x u_k(t_0,x_0)) \leq 0, \quad k = 1,\dots,m.$$

Analogously, from $u \in Sol_L$ we get

$$\frac{\partial u_k}{\partial t}(t_0,x_0) + H_k(t_0,x_0,u(t_0,x_0),\nabla_x u_k(t_0,x_0)) \geq 0, \quad k = 1,\dots,m.$$

Therefore u satisfies equation (1) at (t_0,x_0).

Further, to prove the existence of the minimax solutions we need the following auxiliary results.

LEMMA 2. *$Sol_U \neq \emptyset$ and $Sol_L \neq \emptyset$.*

PROOF. Assume $(t,x) \in \bar{G}$. We denote by $X(t,x)$ the set of solutions of the differential inclusion

$$\dot{x}(t) \in F(x(t)), \quad a.e.\ t \in [0,T],$$

which satisfies the condition $x(t) = x$. From the Theorem 18 in [3] and F is Lipschitz continuous, we deduce that $X(t,x)$ is a nonempty compact set in $C([0,T],\mathbb{R}^n)$ and $(t,x) \to X(t,x)$ is continuous on $\bar{G}$. Moreover set

$$D(t,x,\tau) = \{x(\tau) : x(.) \in X(t,x)\}, \quad \tau \in [0,T],$$

we deduce that $(t,x) \to D(t,x,\tau)$ is continuous and $D(t,x,\tau)$ is a nonempty compact set. Let $\psi = (\psi_1,\ldots,\psi_m) : \bar{G} \to \mathbb{R}^m$ with

$$\psi_k(t,x) = \max\{u_k^0(y),\ y \in D(t,x,T)\}, \tag{24}$$

$$\forall (t,x) \in \bar{G},\ k = 1,\ldots,m.$$

We will now show that $\psi \in Sol_U$. Suppose $t_0 \in [0,T)$, $\tau \in (t_0,T]$, $x_0 \in \mathbb{R}^n$, $q \in S$ and $k = 1,\ldots,m$. Fix $x(.) \in (X_U)_k^{\psi}(t_0,x_0,\psi_k(t_0,x_0),q)$. Then $x(.) \in X(t_0,x_0)$ and $x(\tau) \in D(t_0,x_0,\tau)$. Therefore

$$D(\tau,x(\tau),T) \subset D(t_0,x_0,T.)$$

Thus we get

$$\begin{aligned}\psi_k(\tau,x(\tau)) &= \max\{u_k^0(y) :\ y \in D(\tau,x(\tau),T)\}\\ &\le \max\{u_k^0(y) :\ y \in D(t_0,x_0,T)\} \ = \ \psi_k(t_0,x_0).\end{aligned}$$

This means ψ satisfies (14). The condition (15) is deduced from the fact that $D(T,x,T) = \{x\}$, $x \in \mathbb{R}^n$. Thus, $\psi \in Sol_U$.

By an argument analogous to the above proof, we can show that the continuous function $\varphi = (\varphi_1,\ldots,\varphi_m)$ in which

$$\varphi_k(t,x) = \min\{u_k^0(y),\ y \in D(t,x,T)\}, \tag{25}$$

$$\forall (t,x) \in \bar{G},\ k = 1,\ldots,m,$$

belongs to Sol_L.

We note that $\psi(T,x) = \varphi(T,x) = u^0(x)$, $x \in \mathbb{R}^n$. Since u_k^0 and $(t,x) \to D(t,x,T)$ are continuous on $\mathbb{R}^n$ and $\bar{G}$, respectively, we conclude that the marginal function ψ_k is also continuous on $\bar{G}$, $k = 1,\ldots,m$. This means that the function ψ is continuous . Analogously, φ is a continuous function.

By the definition of ψ and φ in (24) and (25), respectively, we get

LEMMA 3. *If $u \in Sol_U$ and $v \in Sol_L$ then $u \ge \varphi$ and $v \le \psi$ on $\bar{G}$*

Let us now define $\omega = (\omega_1, \ldots, \omega_m) : \bar{G} \to \mathbb{R}^m$, in which

$$\omega_k(t,x) = \inf\{v_k(t,x) : v = (v_1, \ldots, v_m) \in Sol_U\}, \quad k = 1, \ldots, m.$$

(This definition is correct by Lemma 3.)

LEMMA 4. $\omega \in Sol_U$.

PROOF. Let $u = \omega_*$. It is sufficient to show that $u \in Sol_U$.

By definition, we have $\varphi \le u \le \psi$ on $\bar{G}$. Thus $u(T,x) = u^0(x), x \in \mathbb{R}^n$. This means u satisfies (15). We note that u_k is lower semicontinuous on $\bar{G}$, $k = 1, \ldots, m$. Suppose $\epsilon > 0$, $t_0 \in [0,T)$, $\tau \in (t_0, T]$, $x_0 \in \mathbb{R}^n$, $q \in S$ and $k = 1, \ldots, m$. By definition, there exist sequences $\{(t_i, x_i)\}_{i \ge 1}$ and $\{v_i\}_{i \ge 1}$, where $(t_i, x_i) \in [0, \tau] \times \mathbb{R}^n$ and $v_i \in Sol_U$, $i = 1, 2, \ldots$ such that $(t_i, x_i) \to (t_0, x_0)$ and $v_k^i(t_i, x_i) \to u_k(t_0, x_0)$ as $i \to \infty$. Since $v_i \in Sol_U$, we find $x_i(.)$ contained in $(X_U)_k^v(t_i, x_i, v_k^i(t_i, x_i), q)$ and

$$v_k^i(t_i, x_i) + \epsilon > v_k^i(\tau, x_i(\tau)), \quad i = 1, 2, \ldots. \tag{26}$$

From (9)-(12), both ψ and φ are continuous and the fact that $\{v_k^i(t_i, x_i)\}_{i \ge 1}$ is a convergent sequence, we deduce that there exists a positive constant Λ such that

$$\dot{x}_i(t) \in (F_U)_k^u(t, x_i(t), u_k(t_0, x_0), q) + \Lambda\, |v_k^i(t_i, x_i) - u_k(t_0, x_0)|\, B.$$

for a.e. $t \in [0,T]$, $i = 1, 2, \ldots$, where $\Lambda = \Lambda(C, R) \ge 0$ and C, R are bounded sets of $\mathbb{R}^n, \mathbb{R}^m$, respectively.

Using Theorem II5 in [7] we claim that the set of limit points of the sequence $\{x_i(.)\}_{i \ge 1}$ is not empty, and it is contained in the set $(X_U)_k^u(t_0, x_0, u_k(t_0, x_0), q)$. Without loss of generality we can assume

$$x_i(.) \to x(.) \in (X_U)_k^u(t_0, x_0, u_k(t_0, x_0), q), \quad i \to \infty.$$

Thus

$$\lim_{i \to \infty} x_i(\tau) = x(\tau).$$

From (26) we have

$$u_k(t_0, x_0) + \epsilon > \varliminf_{i\to\infty} v_k^i(\tau, x_i(\tau))$$
$$\geq \varliminf_{i\to\infty} u_k(\tau, x_i(\tau)) \geq u_k(\tau, x(\tau)).$$

This means that u satisfies (14). Thus $\omega \in Sol_U$.

LEMMA 5. $\omega \in Sol_L$.

PROOF. It is sufficient to show that ω satisfies (16).

Fix $\theta \in (0, T]$, $k = 1, \ldots, m$. Suppose $t \in [0, \theta]$, $x \in \mathbb{R}^n, r_k \in \mathbb{R}^1$, $p \in S$. Let

$$(D_L)_k^\omega(t, x, r_k, \theta, p) = \{x(\theta) : x(.) \in (X_L)_k^\omega(t, x, r_k, p)\},$$

and

$$\bar{\rho}(t, x, r_k) = \sup\{\omega_k(\theta, y) : y \in (D_L)_k^\omega(t, x, r_k, \theta, p)\} - r_k.$$

Since $(D_L)_k^\omega(t, x, r_k, \theta, p)$ is a compact set, $\omega_k(\theta, .) \leq \psi_k(\theta, .)$ in $\mathbb{R}^n$ and ψ_k is continuous, we deduce that

$$\bar{\rho}(t, x, r_k) < +\infty, \quad \forall (t, x, r_k) \in [0, \theta] \times \mathbb{R}^n \times \mathbb{R}^1.$$

Set $\rho(t, x, r_k) = \lim_{\epsilon\downarrow 0} \inf\{\bar{\rho}(t, x, r'_k) : |r_k - r'_k| < \epsilon,\ r'_k \in \mathbb{R}\}$. We see that ρ is strictly decreasing in r_k. From this we are able to define the function $[0, \theta] \times \mathbb{R}^n \ni (t, x) \to r_k(t, x) \in \mathbb{R}^1$ as follows

$$r_k(t, x) = \min\{r_k \in \mathbb{R}^1 : \rho(t, x, r_k) \leq 0\}.$$

Here we note that $r_k(\theta, x) = \omega_k(\theta, x), \quad x \in \mathbb{R}^n$. Suppose that $t_0 \in [0, \theta)$, $\tau \in (t_0, \theta]$, $x_0 \in \mathbb{R}^n$, $q \in S$. Then the multi-valued function

$$(t, x) \to (F_U)_k^\omega(t, x, r_k(t_0, x_0), q) \cap (F_L)_k^\omega(t, x, r_k(t_0, x_0), p)$$

satisfies the conditions of Theorem Π3 in [7]. Hence

$$(X_U)_k^\omega(t_0, x_0, r_k(t_0, x_0), q) \cap (X_L)_k^\omega(t_0, x_0, r_k(t_0, x_0), p) \neq \emptyset.$$

Assume $x(.) \in (X_U)_k^\omega(t_0, x_0, r_k(t_0, x_0), q) \cap (X_L)_k^\omega(t_0, x_0, r_k(t_0, x_0), p)$, we get

$$(D_L)_k^\omega(t_0, x_0, r_k(t_0, x_0), \theta, p) \supset (D_L)_k^\omega(\tau, x(\tau), r_k(t_0, x_0), \theta, p)$$

Thus

$$\rho(\tau, x(\tau), r_k(t_0, x_0)) \leq \rho(t_0, x_0, r_k(t_0, x_0)) \leq 0.$$

In accordance with the definition of ρ we find that

$$r_k(t_0, x_0) \geq r_k(\tau, x(\tau)). \tag{27}$$

Here we note that $x(.) \in (X_U)_k^\omega(t_0, x_0, r_k(t_0, x_0), q)$. Set $u = (u_1, \dots, u_m)$ with

$$u_i(t,x) = \omega_i(t,x), \quad (t,x) \in \bar{G},\ i \neq k, i = 1, \dots, m;$$

$$u_k(t,x) = \begin{cases} \min\{r_k(t,x), \omega_k(t,x)\} & (t,x) \in [0,\theta) \times \mathbb{R}^n, \\ \omega_k(t,x) & (t,x) \in [\theta, T] \times \mathbb{R}^n. \end{cases}$$

From (27) and $\omega \in Sol_U$, we can check that $u \in Sol_U$. By the definition of ω, we have $u_k \geq \omega_k$ on $\bar{G}$. Thus $r_k \geq \omega_k$ on $[0, \theta) \times \mathbb{R}^n$.

Assume $t_0 \in [0,T)$, $\theta \in (t_0, T]$, $x_0 \in \mathbb{R}^n$, $p \in S$ and $k = 1, \dots, m$. By the above argument, we have $r_k(t_0, x_0) \geq \omega_k(t_0, x_0)$. Thus

$$\bar{\rho}(t_0, x_0, \omega_k(t_0, x_0)) \geq \rho(t_0, x_0, \omega_k(t_0, x_0)) \geq 0.$$

This means that

$$\sup\{\omega_k(\theta, y) : y \in (D_L)_k^\omega(t_0, x_0, \omega_k(t_0, x_0)), \theta, p)\} - \omega_k(t_0, x_0) \geq 0,$$

$$\sup_{x(.) \in (X_L)_k^\omega(t_0, x_0, \omega_k(t_0, x_0), p)} \{\omega_k(\theta, x(\theta)) - \omega_k(t_0, x_0)\} \geq 0.$$

Therefore ω satisfies (16).

From the Lemma 4 and 5, we have

THEOREM 3. *Suppose that the conditions a)-e) are satisfied and that $u_0(x)$ is continuous. Then there exists a minimax solution of Problem (1)-(2).*

EXAMPLE. Let

$$\mathcal{H} = \{h : \mathbb{R}^1 \to \mathbb{R}^1 \,|\, h \text{ is bounded, increasingand globally Lipschitz continuous}\}$$

We note that $\mathcal{H} \neq \emptyset$, (for example, the function arctan $\in \mathcal{H}$).

$H = (H_1, \ldots, H_m) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$H_k(r,p) \;=\; \sum_{i=1}^{n} f_{ki}(r)\,|p_i|, \quad k = 1, \ldots, m,$$

where

$$f_{ki}(r) = h_{ki1}(r_1) + \cdots + h_{ki(k-1)}(r_{k-1}) + h_{kik}(-r_k) + h_{ki(k+1)}(r_{k+1}) + \cdots + h_{kim}(r_m),$$

$k = 1, \ldots, m,\ i = 1, \ldots, n$ and $h_{kij} \in \mathcal{H},\ k, j = 1, \ldots, m\ i = 1, \ldots, n$.

We can check that the function H satisfies the conditions a)-e) of Problem (1)-(2).

REFERENCES

[1] Adiatullina R. E. and Subbotin A. I., *Generalized solutions of equations of the type* $\frac{\partial\varphi}{\partial t} + H(t, x, \varphi, \nabla_x\varphi) = 0$, Differential Equations **28**(1992), 643-649.

[2] Aubin J. -P. and Frankowska H., *Set-Valued Analysis*, Birkhäuser, 1990.

[3] Blagodatskykh V. I., *Some results on the theory of differential inclusions*, Lecture, Summer School on ordinary differential equations, Part 11, Stara Lesnă -High Tatras, September, 1974.

[4] Crandall M. G., Ishii H. and Lions P. L., *User's guide to viscosity solutions of second order partial differential equations.* Bull. Amer. Math. Soc. **27**(1992), 1-67.

[5] Crandall M. G. and Lions P. L., *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277**(1983), 1-42.

[6] Philippov A. F., *Differential Equations with discontinuous right part*, (Russian), Nauka, Moscow, 1985.

[7] Subbotin A. I., *Minimax inequalities and Hamilton-Jacobi equations* (Russian), Nauka, Moscow, 1991.

[8] Subbotina N. N., *The method of Cauchy characteristics and generalized solutions of Hamilton-Jacobi-Bellman Equations*, Soviet Math. Dokl. **44**(1992), 501-506.

INSTITUTE OF MATHEMATICS

P.O. BOX 631 BO HO, HANOI VIETNAM