ACTA MATHEMATICA VIETNAMICA
Volume 19, Number 2, 1994, pp. 111-120

ON THE EXPLICIT REPRESENTATION OF GLOBAL SOLUTIONS OF THE CAUCHY PROBLEM FOR HAMILTON-JACOBI EQUATIONS[1]

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1. Introduction

This note is devoted to the explicit representation of global Lipschitz solutions of the Cauchy problem for Hamilton-Jacobi equations of the form

$$u_t + H(t, \nabla_x u) = 0\,, \ (t,x) \in \Omega, \tag{1.1}$$

with initial conditions

$$u(0,x) = \sigma(x)\,, \ x \in \mathbb{R}^n. \tag{1.2}$$

As it is known, the explicit formulas of global solutions of Problem (1.1)-(1.2) are constructed in the case where either $H(t,.)$ or $\sigma(.)$ is convex, (see [4], [6]). Here we try to release the convexity condition related to H and σ in establishing these formulas. Next, we prove that if the initial datum $\sigma(.)$ is convex, then a global Lipschitz solution of (1.1)-(1.2) can be determined by characteristics. Then we check that this solution is also a viscosity solution in the case where $\sigma(.)$ needs not be globally Lipschitz continuous. The result improves Theorem 3.1 in Bardi and Evans [1].

We use the following notations. Let $\Omega = (0,T) \times \mathbb{R}^n$, $||.||$ and $\langle .,. \rangle$ be the Euclidean norm and the scalar product in $\mathbb{R}^n$, respectively. Denote by $\mathrm{Lip}\,(\Omega)$ the set of all locally Lipschitz continuous functions u defined on Ω and set $\mathrm{Lip}\,\big([0,T) \times \mathbb{R}^n\big) := \mathrm{Lip}\,(\Omega) \cap \mathrm{C}\big([0,T) \times \mathbb{R}^n\big)$.

Received October 31st, 1994

[1]Supported in part by NCSR Program on Applied Mathematics and the National Basic Research Program in Natural Science, Vietnam.

DEFINITION 1.1. *A function $u(t,x)$ in $Lip([0,T)\times \mathbb{R}^n)$ is called a global Lipschitz solution of Problem (1.1)-(1.2) if $u(t,x)$ satisfies (1.1) almost everywhere in Ω and $u(0,x)=\sigma(x)$ for all $x\in\mathbb{R}^n$.*

2. The minimum of a family of global Lipschitz solutions.

Firstly we prove the following theorem.

THEOREM 2.1. *Let $(\sigma_\alpha(x))_{\alpha\in I}$ be a family of functions indexed by an arbitrary set I such that the equation (1.1) with initial datum $\sigma_\alpha(x)$ has a global Lipschitz solution $u_\alpha(t,x)$. Assume that for every $(t_0,x_0)\in[0,T)\times\mathbb{R}^n$ there exist an open neighbourhood $V(t_0,x_0)$ and a finite set $J\subset I$ such that $\inf\limits_{\alpha\in I} u_\alpha(t,x) = \min\limits_{\alpha\in J} u_\alpha(t,x)$ for all $(t,x)\in V(t_0,x_0)$. Then the function $u(t,x)=\inf\limits_{\alpha\in I} u_\alpha(t,x)$ is a global Lipschitz solution of (1.1)-(1.2) with $\sigma(x)=\inf\limits_{\alpha\in I}\sigma_\alpha(x)$.*

PROOF. Given any $(t_0,x_0)\in\Omega$, by assumption, there exist a finite set $J\subset I$, a neighbourhood $V=V(t_0,x_0)$ and a positive number K such that for all $(t,x)\in V$, $(t',x')\in V$ we have

$$|u_i(t',x')-u_i(t,x)|\le K(|t-t'|+||x-x'||),\qquad i\in J,$$

and

$$u(t,x)=\min_{i\in J} u_i(t,x).$$

Assume $u(t,x)\le u(t',x')$. Take $i_0\in J$ such that $u(t,x)=u_{i_0}(t,x)$. Then

$$u(t',x')-u(t,x)\le u_{i_0}(t',x')-u_{i_0}(t,x)\le K(|t-t'|+||x-x'||).$$

This means that $u(t,x)\in \mathrm{Lip}((0,T)\times\mathbb{R}^n)$. By the hypotheses, we also see that $u(t,x)\in \mathrm{C}([0,T)\times\mathbb{R}^n)$.

Consider now the open covering $(V(t,x))_{(t,x)\in\Omega}$ of Ω. By Lindelöf's property, there exists a countable subcovering $(V_n)_{n\in\mathbb{N}}$, $V_n=V(t_n,x_n)$, of Ω. For every $n\in\mathbb{N}$ we have $u(t,x)=\min\limits_{i\in J_n} u_i(t,x)$ with all $(t,x)\in V_n$, where J_n is a finite subset of I.

Let $N_n \subset V_n$ with $\operatorname{mes}(N_n) = 0$ such that all the functions u_i $(i \in J_n)$ are differentiable and satisfy (1.1) at any point of $V_n \setminus N_n$. Put $N = \bigcup_{n=1}^{\infty} N_n$, then $\operatorname{mes}(N) = 0$. By virtue of Rademacher's Theorem, we may assume that $u(t,x)$ is differentiable at every $(t,x) \in \Omega \setminus N$.

Given $(t,x) \in V_n \setminus N$, there exists $i_0 \in J_n$ such that $u(t,x) = u_{i_0}(t,x)$. For every (t',x') close enough to (t,x) we have

$$u(t',x') - u(t,x) \leq u_{i_0}(t',x') - u_{i_0}(t,x). \tag{2.1}$$

Both u and u_{i_0} are differentiable at (t,x). Therefore, (2.1) implies that

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial u_{i_0}}{\partial t}(t,x) \qquad \text{and} \qquad \nabla_x u(t,x) = \nabla_x u_{i_0}(t,x).$$

It follows that $u(t,x)$ satisfies (1.1) in $V_n \setminus N$. Because n is arbitrarily chosen and $\bigcup_{n=1}^{\infty} (V_n \setminus N) = \Omega \setminus N$, it follows that $u(t,x)$ satisfies the equation (1.1) a.e. in Ω.

On the other hand, $u(0,x) = \inf_{\alpha \in I} u_\alpha(0,x) = \min_{i \in J} \sigma_i(x)$. Thus $u(t,x)$ is a global Lipschitz solution of Problem (1.1)-(1.2).

In [6] we proved the following result.

Assume that the following conditions hold for $H(t,q)$ and the convex function $\sigma(x)$.

(A0) : *$H(t,q)$ satisfies the Caratheodory condition and for every compact set $C \subset \mathbb{R}^n$ there exists a function $g_C \in L_\infty(0,T)$ such that for almost all $t \in (0,T)$*

$$\sup_{q \in C} |H(t,q)| \leq g_C(t).$$

(A1) : *For every $(t_0,x_0) \in [0,T) \times \mathbb{R}^n$, there exist positive constants r and K such that*

$$\langle x,p \rangle - \sigma^*(p) - \int_0^t H(\tau,p)d\tau < \max_{\|q\| \leq K} \{\langle x,q \rangle - \sigma^*(q) - \int_0^t H(\tau,q)d\tau\}$$

whenever $(t,x) \in [0,T) \times \mathbb{R}^n$, $|t-t_0| + \|x-x_0\| < r$ and $\|p\| > K$; σ^* being the conjugate of σ.

Then

$$u(t,x) = \max_{q\in\mathbb{R}^n}\{\langle x,q\rangle - \sigma^*(q) - \int_0^t H(\tau,q)d\tau\} \tag{2.2}$$

is a global Lipschitz solution of Problem (1.1)-(1.2).

Suppose now that the function $\sigma(x)$ can be represented in the form

$$\sigma(x) = \inf_{\alpha\in I}\sigma_\alpha(x),$$

where $(\sigma_\alpha)_{\alpha\in I}$ is a family of convex functions. Applying Theorem 2.1 we obtain the following results for representation of global Lipschitz solutions with nonconvex data.

COROLLARY 2.2. *Assume (A0)-(A1) for $H(t,q)$ and $\sigma_\alpha(x)$, for any $\alpha\in I$. Further, suppose that the hypotheses of Theorem 2.1 hold for all $u_\alpha(t,x)$ defined by*

$$u_\alpha(t,x) = \max_{q\in\mathbb{R}^n}\{\langle x,q\rangle - \sigma^*_\alpha(q) - \int_0^t H(\tau,q)d\tau\}.$$

Then the function $u(t,x) = \min_{\alpha\in I} u_\alpha(t,x)$ is a global Lipschitz solution of Problem (1.1)-(1.2) with $\sigma(x) = \inf_{\alpha\in I}\sigma_\alpha(x)$.

COROLLARY 2.3. *Assume (A0). Suppose that $\sigma_1(x),\ldots,\sigma_k(x)$ are convex and globally Lipschitz continuous. Then*

$$u(t,x) = \min_{i\in\{1,\ldots,k\}}\max_{q\in\mathbb{R}^n}\{\langle x,q\rangle - \sigma^*_i(q) - \int_0^t H(\tau,q)d\tau\}$$

is a global Lipschitz solution of Problem (1.1)-(1.2) where $\sigma(x) = \min_{i\in\{1,\ldots,k\}}\sigma_i(x)$.

EXAMPLE 2.1. We consider the Cauchy problem

$$u_t + |u_x^2 - 1| = 0, \qquad (t,x)\in(0,T)\times\mathbb{R}^1,$$

$$u(0,x) = e^{-|x|} = \min\{e^x, e^{-x}\}, \quad x\in\mathbb{R}^1.$$

Then $u(t,x) = \min\limits_{i=1,2} \max\limits_{q\in\mathbb{R}^1} \{xq - h_i(q) - t|q^2-1|\}$ with

$$h_1(q) = \begin{cases} q\ln q - q & q>0, \\ 0 & q=0, \\ +\infty & q<0, \end{cases}$$

and

$$h_2(q) = \begin{cases} -q\ln(-q) + q & q<0, \\ 0 & q=0, \\ +\infty & q>0, \end{cases}$$

is a global Lipschitz solution.

EXAMPLE 2.2. Let $H(t,q)$ be a continuous function on $[0,T)\times\mathbb{R}^n$ and $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}^1$, $i=1,\ldots,k$. Put $\sigma(x) = \min\{\langle a_i,x\rangle + b_i,\ i=1,\ldots,k\}$. Then the problem (1.1)-(1.2) has a global Lipschitz solution of the form

$$u(t,x) = \min_{i\in\{1,\ldots,k\}} \{\langle a_i,x\rangle + b_i - \int_0^t H(\tau,a_i)d\tau\}.$$

3. Construction of global Lipschitz solutions via characteristics.

In this section we will show that the formula (2.2) can be obtained by means of characteristics if the given data σ and H belong to class C^2. For sake of simplicity, we use the notations $H_p(t,p) = \nabla_p H(t,p)$, $\nabla_x\sigma = \sigma'$.

Consider the characteristic differential equations of Problem (1.1)-(1.2) (see [2]),

$$\dot{x} = H_p\ ; \qquad \dot{v} = \langle H_p, p\rangle - H\ ; \qquad \dot{p} = 0\ ,$$

with initial conditions

$$x(0) = y\ ; \qquad v(0) = \sigma(y)\ ; \qquad p(0) = \sigma'(y)\ , \quad y\in\mathbb{R}^n.$$

Then

$$x = x(t,y) = y + \int_0^t H_p(\tau,\sigma'(y))d\tau, \tag{3.1}$$

$$v = v(t,y) = \sigma(y) + \int_0^t \langle H_p(\tau,\sigma'(y)), \sigma'(y)\rangle d\tau - \int_0^t H(\tau,\sigma'(y))d\tau. \tag{3.2}$$

Combining (3.1) with (3.2), we can rewrite $v(t,y) = \varphi(t,x(t,y),y)$, where

$$\varphi(t,x,y) = \sigma(y) + \langle x-y, \sigma'(y)\rangle - \int_0^t H(\tau,\sigma'(y))d\tau. \tag{3.3}$$

We put now

$$\tilde{u}(t,x) = \sup_{y\in\mathbb{R}^n} \varphi(t,x,y).$$

In the case where the function σ is convex we can use the Legendre transformation on σ to rewrite

$$\tilde{u}(t,x) = \sup_{y\in\mathbb{R}^n} \{\langle x, \sigma'(y)\rangle - \sigma^*(\sigma'(y)) - \int_0^t H(\tau,\sigma'(y))d\tau\}. \tag{3.4}$$

We have a comparing result as follows:

PROPOSITION 3.1. *Suppose that H is continuous in $[0,T)\times\mathbb{R}^n$, σ is convex and belongs to the class $C^1(\mathbb{R}^n)$. Moreover if $u(t,x)$ given by (2.2) is a global Lipschitz solution of Problem (1.1)-(1.2), then $\tilde{u}(t,x) = u(t,x)$, $\forall (t,x) \in$ $[0,T)\times\mathbb{R}^n$.*

PROOF. It is obvious that $\tilde{u}(t,x) \le u(t,x)$ for all $(t,x) \in [0,T)\times\mathbb{R}^n$. We have to show that $u(t,x) \le \tilde{u}(t,x)$ for any $(t,x)\in[0,T)\times\mathbb{R}^n$. To this end, let $E = \{\sigma'(y) \mid y \in \mathbb{R}^n\}$. Then by [5, Corollary 26.4.1] we have

$$\mathrm{ri}\,(\mathrm{Dom}\,\sigma^*) \subset E \subset \mathrm{Dom}\,\sigma^*, \tag{3.5}$$

where $\mathrm{ri}\,(\mathrm{Dom}\,\sigma^*)$ is the relative interior of $\mathrm{Dom}\,\sigma^*$.

Suppose that $q_0 \in \mathrm{Dom}\,\sigma^*$ with

$$u(t,x) = \langle x, q_0\rangle - \sigma^*(q_0) - \int_0^t H(\tau,q_0)d\tau.$$

By (3.5) and the density in Dom σ^* of its relative interior, we can take a sequence $z_n = \sigma'(y_n) \in E$ such that $z_n \to q_0$. Then

$$\begin{aligned} u(t,x) &\leq \lim_{n\to\infty} \{\langle x, z_n\rangle - \sigma^*(z_n) - \int_0^t H(\tau, z_n)d\tau\} \\ &\leq \lim_{n\to\infty} \{\langle x, \sigma'(y_n)\rangle - \sigma^*(\sigma'(y_n)) - \int_0^t H(\tau, \sigma'(y_n))d\tau\} \\ &\leq \sup_{y\in\mathbb{R}^n} \{\langle x, \sigma'(y)\rangle - \sigma^*(\sigma'(y)) - \int_0^t H(\tau, \sigma'(y))d\tau\} \\ &= \tilde{u}(t,x). \end{aligned}$$

The proof is then complete.

REMARK 3.1. Suppose that the conditions (A0)-(A1) hold. Then the formula (2.2) always attains maximum at some point $q \in \mathbb{R}^n$. But the same is not true for the formula (3.4). We show this by the following example.

Let

$$H(q) = \begin{cases} 1 & q > 1, \\ q - q\ln q & q \in [0,1], \\ 0 & q < 0, \end{cases}$$

and $\sigma \in C^1$ be the function given by

$$\sigma(x) = \begin{cases} e^x & x \leq 0, \\ x + 1 & x > 0. \end{cases}$$

Then

$$\sigma^*(q) = \begin{cases} q\ln q - q & q \in [0,1], \\ +\infty & q \notin [0,1], \end{cases}$$

and

$$\tilde{u}(t,x) = \sup_{y\leq 0}\{x.e^y - \sigma^*(e^y) - tH(e^y)\}.$$

It is easily seen that the value $\tilde{u}(1,-2) = \sup_{y\leq 0}\{-2e^y\} = 0$ can not be attained in (3.4) at any point y.

From now on we make the following assumption:

(A2) : *Assume that the formula (3.4) always attains maximum at some point* $y \in \mathbb{R}^n$.

This assumption is fulfilled, for example, if $\operatorname{Dom}\sigma^*$ is an open set (cf. (3.5)) or σ is a co-finite strictly convex function (i.e., $\frac{\sigma(\lambda x)}{\lambda} \to \infty$ as $\lambda \to \infty$, for all $x \in \mathbb{R}^n$), (see [5, Ths. 26.3, 26.5, 26.6]).

THEOREM 3.2. *Assume (A0)-(A1)-(A2). Let H and σ be of class C^2 in $\mathbb{R}^n$ and $\sigma'' = (\sigma_{x_i x_j})_{i,j}$ is a positive definite matrix. Then the global Lipschitz solution $u(t,x)$ in (2.2) can be given as the largest value of $\varphi(t,x,y)$, the maximum being taken over all y such that characteristic curves $x(.,y)$ starting from y meet each other at x at the time-point t.*

PROOF. As it is shown in Proposition 3.1, $u(t,x) = \tilde{u}(t,x)$ for all $(t,x) \in \Omega$. By (3.3) we have

$$\nabla_y \varphi(t,x,y) = \sigma''(y).(x - y - \int_0^t H_p(\tau, \sigma'(y))d\tau).$$

From (A2), the maximum in (3.4) attains at some $y \in \mathbb{R}^n$, which must be a stationary point of $\varphi(t,x,.)$, i.e., $\nabla_y \varphi(t,x,y) = 0$. Because $\sigma''(y)$ is positive definite, then

$$x - y - \int_0^t H_p(\tau, \sigma'(y))d\tau = 0.$$

Therefore y is a root of the equation (3.1). Consequently, the maximum in the formula (3.4) is not attained if it is taken not over all y but only over those stationary y. This completes the proof.

4. Connection with the viscosity solutions

In [1] Bardi and Evans proved that the global Lipschitz solution

$$u(t,x) = \max_{q \in \mathbb{R}^n} \{\langle x, q\rangle - \sigma^*(q) - tH(q)\} \tag{4.1}$$

is also a unique viscosity solution of the problem

$$u_t + H(\nabla_x u) = 0, \quad \text{in } \Omega, \tag{4.2}$$

$$u(0,x) = \sigma(x) \quad x \in \mathbb{R}^n, \tag{4.3}$$

where H is continuous and $\sigma(x)$ is convex and globally Lipschitz continuous. Here we extend slightly this result to the case where $\sigma(x)$ is convex (but needs not to be globally Lipschitz continuous). For the definition of viscosity solution we refer the readers to [3].

We need a condition similar to Section 2 (A1).

(A'1) : *For every* $(t_0, x_0) \in [0,T) \times \mathbb{R}^n$, *there exist positive numbers* r *and* K *such that*

$$\langle x, p \rangle - \sigma^*(p) - tH(p) < \max_{||q|| \le K} \{\langle x, q \rangle - \sigma^*(q) - tH(q)\},$$

whenever $(t,x) \in [0,T) \times \mathbb{R}^n$, $|t - t_0| + ||x - x_0|| < r$ *and* $||p|| > K$.

PROPOSITION 4.1. *Let* $\sigma(x)$ *be a convex function on* $\mathbb{R}^n$. *Assume (A'1). Then the global Lipschitz solution* $u(t,x) = \max_{q \in \mathbb{R}^n} \{\langle x, q \rangle - \sigma^*(q) - tH(q)\}$ *is also a viscosity solution of Problem (4.2)-(4.3).*

PROOF. Let $\sigma^*(q)$ be the conjugate function of $\sigma(x)$. We put

$$\sigma_n(x) = \sup_{||q|| \le n} \{\langle x, q \rangle - \sigma^*(q)\}.$$

Then $\sigma_n(x)$ is a globally Lipschitz continuous function with Lipschitz constan n. It is easily seen that

$$\sigma_n^*(q) = \begin{cases} \sigma^*(q) & ||q|| \le n, \\ +\infty & ||q|| > n. \end{cases}$$

From Theorem 3.1 in [1], $u_n(t,x) = \max_{||q|| \le n} \{\langle x, q \rangle - \sigma^*(q) - tH(q)\}$ is the unique viscosity solution of (4.2) with the initial condition $u(0,x) = \sigma_n(x)$.

We see that $\{\sigma_n(x)\}_n$ (resp. $\{u_n(t,x)\}_n$) converges uniformly on any compact of $\mathbb{R}^n$ (resp. Ω) to $\sigma(x)$ (resp. $u(t,x)$) because of Dini's Theorem. Applying Theorem 1.4 in [3], we deduce that $u(t,x)$ is a viscosity solution of Problem (4.2)-(4.3).

REMARK 4.1. Suppose that H is a bounded continuous function on $\mathbb{R}^1$ and $\sigma(x) = x^2/2$. Since $x^2/2$ is not globally Lipschitz continuous, we can not conclude, from Theorem 3.1 [1], that (4.1) is a viscosity solution in this case. Nevertheless, applying Proposition 4.1, we see that the function

$$u(t,x) = \max_{q \in \mathbb{R}^1} \{\langle x, q\rangle - q^2/2 - tH(q)\}$$

is a viscosity solution of the problem (4.2)-(4.3).

REMARK 4.2. Global quasi-classical solutions of the Cauchy problem for first-order nonlinear partial differential equations were studied in [7,8]. Some results on their explicit representation will be published elsewhere.

The authors would like to thank Prof. Ha Tien Ngoan for his assistance.

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ACTA MATHEMATICA VIETNAMICA
Volume 19, Number 2, 1994, pp. 121-135

EXISTENCE OF GLOBAL MINIMAX SOLUTIONS OF THE CAUCHY PROBLEM FOR SYSTEMS OF FIRST-ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS[1]

TRAN DUC VAN AND NGUYEN DAC LIEM

The purpose of this paper is to present the existence results of global minimax solutions for some systems of first-order nonlinear partial differential equations (PDEs).

Since a classical solution of the nonlinear problem can fail to exist even in the cases where the data are analytic functions, we need to introduce concepts of generalized solutions.

In recent years many different methods have been created by Benton S. H., Cole V. J. D., Conway E. D., Crandall M. G., Doubnov B., Evans L. C., Fleming W. H., Glimm J., Hopf E., Kružkov S. N., Lax P. D., Lions P. L., Maslov V. P., Oleinhik O., Rozdestvenskii B. L., Subbotin A. I., Tsuji M. ... in the study of global generalized solutions of nonlinear PDEs. Especially, nonclassical theory of nonlinear PDEs represents a large portion of research in which the concept of global viscosity solutions introduced by Crandall and Lion [4,5] is used.

Another direction in this theory is motivated by differential games which leads the notion of global minimax solutions for the first-order nonlinear PDEs. The case of the Cauchy problem for a scalar nonlinear PDE of first-order was studied in great detail by Subbotin A. I., Subbotina N. N., Adiatulina ... (see, for example, [1,7,8]). As the terminology "minimax solution" indicates, solutions of nonlinear PDEs of first-order are closely connected with the minimax operations.

Received June 15th, 1994
[1]Supported in part by the National Basic Research Program in Natural Science, Vietnam.