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ON THE EXPONENTIAL DICHOTOMY OF THE SOLUTIONS OF COUNTABLE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction

The problem on the exponential dichotomy of the solutions of systems of differential equations was studied by many mathematicians. Some results for equations in Banach spaces were obtained (see, for example, [2], [3], [4]). In this paper, we study the exponential dichotomy of the solutions of countable systems of differential equations.

In the space m, the set of bounded sequences of numbers with the norm $\| x \| = \sup |x_n|$, we consider a countable system of linear differential equations:

$$\frac{dx_s}{dt} = \sum_{j=1}^{\infty} a_{sj}(t).x_j, \quad s = 1, 2, \dots \tag{1}$$

where $x = (x_1, x_2, \dots) \in M$, $\sup |x_n| < \infty$, $a_{sj}(t)$, $s, j = 1, 2, \dots$, are bounded and continuous fuctions of variables $t \in (-\infty, +\infty)$ and

$$a_s(t) = \sum_{j=1}^{\infty} |a_{sj}(t)| \leq a(t), \quad s = 1, 2, \dots \tag{2}$$

for $t \in (-\infty, +\infty)$.

It was shown in [1] that if the functions $a_s(t)$ and $a(t)$ are bounded and continuous for $t \in (-\infty, +\infty)$, the system (1) has in m only one solution which is bounded and equicontinuous for $t \in (-\infty, +\infty)$. We always assume this hypothesis for the system (1) – (2).

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DEFINITION 1. The solution of the system (1) is said to have the property of exponential dichotomy in the interval $(-\infty, +\infty)$ if for some $t_0 \in (-\infty, +\infty)$ the space m can be represented by a direct sum of two closed subspaces

$$m = m^+ \oplus m^- \tag{3}$$

such that

1) For the solution $x^+(t)$ of the system (1) with $x^+(t_0) \in m^+$ we have

$$\| x^+(t) \| \leq N_1 e^{-\nu_1(t-t_0)} \| x^+(t_0) \|, \quad t \geq t_0, \tag{4_1}$$

2) For the solution $x^-(t)$ of the system (1) with $x^-(t_0) \in m^-$ we have

$$\| x^-(t) \| \leq N_2 e^{\nu_2(t-t_0)} \| x^-(t_0) \|, \quad t \leq t_0 \tag{4_2}$$

where N_1, N_2, ν_1, ν_2 are some positive constants.

3)

$$S_n(m^+, m^-) \geq \gamma, \ \gamma > 0, \tag{5}$$

where $S_n(m^+, m^-)$ is the angle of the two spaces (see [4]).

REMARK. Since the exponential dichotomy of the solution does not depend on the starting point t_0, from now on we use the notations $m^+ \oplus m^-$ instead of $m^+(t_0) \oplus m^-(t_0)$ in (3) and $S_n(m^+, m^-)$ instead of $S_n\big(m^+(t_0), m^-(t_0)\big)$ in (5),...

The exponential dichotomy of systems of differential equations was studied by many authors (see, for example, [2] for n-dimensional spaces, [3] for general Banach spaces).

In this paper, we will study the exponential dichotomy of the system (1) by considering the exponential dichotomy of its shortened system (that is the system of a finite number of differential equations).

Explicitly, for the given system (1), we consider its shortened system for some fixed n :

$$\frac{dx_s^{(n)}}{dt} = \sum_{j=1}^{n} a_{sj}(t) x_j^{(n)}, \quad s = \overline{1, n}. \tag{6}$$

Suppose that the system (6) has the exponential dichotomy for $t \in (-\infty, +\infty)$. The problem is that under which conditions the system (1) has the exponential dichotomy too.

2. The exponential dichotomy depends on the shortened systems

Suppose that for some n, the system (6) (the shoretened system of system (1)) has the property of exponential dichotomy in the interval $(-\infty, +\infty)$, that is n-dimension space m_n corresponding to (6) by a direct sum of two closed subspaces

$$m_n = m_n^+ \oplus m_n^- \tag{7}$$

and there exist two positive numbers $\nu_1^{(n)}, \nu_2^{(n)}$ such that

1)

$$\| x^{(n)+}(t) \| \geq N_1^{(n)} e^{-\nu_1^{(n)}(t-s)} \| x^{(n)}(s) \|, \quad t \geq s, \tag{8_1}$$

for some solution $x^{(n)+}(t)$, with $x^{(n)-}(t_0) \in m_n^+$.

2)

$$\| x^{(n)-}(t) \| \geq N_2^{(n)} e^{\nu_2^{(n)}(t-s)} \|, \quad t \geq s, \tag{8_2}$$

for some solution $x^{(n)-}(t)$, with $x^{(n)-}(t_0) \in m_n^-$.

3)

$$S_n(m_n^+, m_n^-) \geq \gamma^{(n)} \tag{9}$$

where $N_1^{(n)}, N_2^{(n)}, \gamma^{(n)}$ are positive constants.

Then, by [2], there exists a matrix $U^{(n)}(t)$ satisfying following conditions:

$$\| U^{(n)}(t) \| \leq K^{(n)}, \ \| (U^{(n)})^{-1}(t) \| \leq K^{(n)}, \tag{10}$$

$$\| \frac{dU^{(n)}}{dt} \| \leq K^{(n)},$$

($K^{(n)}$ is a positive constant), such that the transformation

$$x^{(n)} = U^{(n)}(t).\eta^{(n)}, \tag{11}$$

transforms the system (6) into the system

$$\frac{d\eta^{(n)}}{dt} = Q^{(n)}(t).\eta^{(n)}, \tag{12}$$

where $\eta^{(n)} = \text{colon}\left(\eta_1^{(n)}, \eta_2^{(n)}, \ldots, \eta_n^{(n)}\right)$, $Q^{(n)}(t) = \text{diag}\left(Q^{(n)+}(t), Q^{(n)-}(t)\right)$, $Q^{(n)+}(t)$ is a triangle matrix of order k, $Q^{(n)-}(t)$ is a matrix of order $n-k$. Hence, system (12) can be written in the following form which has two blocks

$$\begin{aligned} \frac{d\xi^{(n)}}{dt} &= Q^{(n)+}(t).\xi^{(n)}, \\ \frac{d\zeta^{(n)}}{dt} &= Q^{(n)-}(t).\zeta^{(n)} \end{aligned} \tag{13}$$

where $\xi^{(n)} = \text{colon}\left(\xi_1^{(n)}, \cdots, \xi_k^{(n)}\right)$, $\zeta^{(n)} = \text{colon}\left(\zeta_1^{(n)}, \cdots, \zeta_{n-k}^{(n)}\right)$.

THEOREM 1. *Suppose that the system (1) satisfies the following conditions:*
1) The functions $a_{sj}(t),\ s=1,2,\dots,\ j=1,2,\dots,$ *are bounded and continuous for* $t\in(-\infty,+\infty)$.
2) The series $\sum\limits_{j=1}^{\infty}|a_{sj}(t)|,\ s=1,2,\dots,$ *uniformly converge for* $t\in(-\infty,+\infty)$.
Assume that the system (6) has the following properties for every n :
3) Its solution has the exponential dichotomy on $(-\infty,+\infty)$ *(see Definition 1).*
3_1) *The sequences of numbers* $\{N_k^{(n)}\},\ k=1,2,$ *converge when* $n\to\infty$.
3_2) $\inf\{\nu_1^{(n)}\}=\nu_1>0,\ \ \sup\{\nu_2^{(n)}\}=\nu_2<+\infty$.
3_3) $m_n^+\subset m_{n+1}^+,\ \ m_n^-\subset m_{n+1}^-$ *for* $n=1,2,\dots$
Then the system (1) has the exponential dichotomy on $(-\infty,+\infty)$.

PROOF. First, it is not hard to see that with the assumptions 1) and 2), the system (1) satisfies condition (2). Since the series $\sum\limits_{j=1}^{\infty}|a_{sj}(t)|,\ s=1,2,\dots,$ uniformly converge for $t\in(-\infty,+\infty)$, there exists a sequence of positive numbers $\epsilon_s(n)\to 0,\ s=1,2,\dots,$ such that for large n and $t\in(-\infty,+\infty)$ we have

$$\sum_{j=n+1}^{\infty}\sup|a_{sj}(t)|\geq H\epsilon_s(n),\ \ s=1,2,\dots$$

where H is a positive contant.

Now we show that the strong Cauchy condition for x is satisfied. Indeed, suppose that $x=(x_1,x_2,\dots,x_{n+1},\dots)$ and $x=(x_1,x_2,\dots,\bar{x}_{n+1},\dots)$ are two arbitrary points of the space m which have the same first n coordinates. We denote

$$\Delta_n x=\sup\{|x_{n+1}-\bar{x}_{n+1}|,|x_{n+2}-\bar{x}_{n+2}|,\dots\}\tag{14}$$

Put $x_j=\bar{x}_j,\ j=1,2,\dots,n,$ for $s=1,2,\dots$. We have

$$\begin{aligned}\Big|\sum_{j=1}^{\infty}a_{sj}(t)x_j-\sum_{j=1}^{n}a_{sj}(t)x_j-\sum_{j=n+1}^{\infty}a_{sj}(t)x_j\Big|&\leq\sum_{j=n+1}^{\infty}|a_{sj}(t)||x_j-\bar{x}_j|\\&\leq\Delta_n x\sum_{j=n+1}^{\infty}\sup|a_{sj}(t)|\\&\leq H\epsilon_s(n)\Delta_n x,\end{aligned}$$

for $t\in(-\infty,+\infty)$ (where $\epsilon_s(n)\to 0$, when $n\to\infty$).

Let $x(t) = x(t, t_0, x_0)$ and $x^{(n)}(t) = x^{(n)}(t, t_0, x^{(n)})$ be the solutions of the system (1) and (6), respectively, where $x(t_0, t_0, x_0) = x_0$ and $x^{(n)}(t_0, t_0, x_0^{(n)}) = x_0^{(n)}$ $\left(x_0 = (x_1^0, x_2^0, \dots) \in m,\ x_0^{(n)} = (x_1^0, x_2^0, \dots, x_n^0).\right)$

By [1, pp. 13–19] we can conclude that

$$\lim x_s^{(n)}(t, t_0, x_0^{(n)}) = x_s(t, t_0, x_0) \tag{15}$$

uniformly converge for $t \in (-\infty, +\infty)$, $s = 1, 2, \cdots$; that is, $x^{(n)}(t, t_0, x_0^{(n)})$ converge to $x(t, t_0, x_0)$ uniformly.

We now consider the solution of the system (6). Suppose that there is given $x^{(n)+}(t)$ with $x^{(n)+}(t_0) \in m_n^+$. We have

$$\| x^{(n)+}(t) \| \leq N_1^{(n)} \exp \nu_1^{(n)}(t - s) \| x^{(n)+}(s) \|, \quad t \geq s \tag{16}$$

By letting $n \to \infty$, from the above proof and the hypotheses 3_1) and 3_2) we get

$$\| x^+(t) \| \leq N_1 c^{-\nu_1(t-s)} \| x^+(s) \|, \quad t \geq s \tag{17}$$

if $N_1 = \lim N_1^{(n)} > 0$. If $N_1 = 0$, we replace N_1 by a positive number.

Moreover, since $x^{(n)+}(t_0) \in m_n^+, x^+(t_0) \in m^+$, where

$$m^+ = \bigcup_{n+1}^{\infty} m_n^+, \ m_n^+ \subset m_{n+1}^+, \quad n = 1, 2, \dots$$

Indeed, if we set $x^+(t_0) = (x_1^{0+}, x_2^{0+}, \dots, x_n^{0+}, \dots)$ then $x^+(t_0) \in m^+$. Moreover, the limit in (17) is uniform for t, so $x^+(t_0) \in m^+$ where $x^+(t, t_0, x_0^+) = \lim x^{(n)+}(t, t_0, x_0^{(n)+})$.

Analogously, we have:

$$\| x^-(t) \| \leq N_2 c^{\nu_2(t-s)} \| x^-(s) \|, \quad t \leq s, \tag{18}$$

$(0 < N_2 \geq \lim N_2^{(n)},\ x^-(t_0) \in m^-)$. It is clear that $m = m^+ \oplus m^-$. Finally, we consider $S_n(m^+, m^-)$ (see [4]):

$$\begin{aligned} S_n(m^+, m^-) &= \inf \| x^{(n)+} + x^{(n)-} \|, \\ &\| x^{(n)+} \| = 1 \quad (x^{(n)\pm} \in m^+), \\ &\| x^{(n)-} \| = 1. \end{aligned} \tag{19}$$

From the hypothesis of the theorem it follows that

$$\int_t^{t+1} \| A(s) \| ds \leq M, \quad (M > 0). \tag{20}$$

Indeed, we have

$$\int_t^{t+1} \| A(s) \| ds = \int_t^{t+1} \sup \sum_{j=1}^{\infty} \| a_{sj}(s) \| ds \leq \int_t^{t+1} a(s) ds \leq M.$$

($M > 0$, exists by the assumptions 1) and 2)). Then by [4, p. 237], inequality (5) is a consequence of inequalities (17) and (18). It implies that

$$S_n(m^+, m^-) \geq \gamma > 0.$$

And it is the limit of (19) when $n \to \infty$. This completes the proof of the theorem.

THEOREM 2. *Let the system (1) satisfy the following conditions:*

1) The functions $a_{sj}(t)$, $a_{sj} = 1, 2, \ldots$, are bounded and continuous for $t \in (-\infty, +\infty)$,

2) The series $\sum_{j=1}^{\infty} |a_{sj}(t)|$, $s = 1, 2, \ldots$, uniformly converge for $t \in (-\infty, +\infty)$.

Assume moreover that for every n the shortened system (6) (of the system (1)) satisfies the follwing conditions:

3) Its solutions have the property of exponential dichotomy for $t \in (-\infty, +\infty)$.

4) The sequence of matrix $U^{(n)}(t)$ in the transformation (11) and the sequence of their inverse matrix $(U^{(n)})^{-1}$ uniformly regularly converge (in the meaning of [5]).

Then the system (1) can be represented in the form

$$\frac{d\eta}{dt} = Q(t)\eta, \tag{21}$$

where $Q(t) = diag(Q^+(t), Q^-(t))$, or

$$\frac{d\xi}{dt} = Q^+(t).\xi,$$
$$\frac{d\zeta}{dt} = Q^-(t).\zeta,$$

where

$$\xi = colon(\xi_1, \xi_2, \ldots),$$
$$\zeta = colon(\zeta_1, \zeta_2, \ldots),$$

$Q^+(t)$ *are triangle matrices.*

PROOF. 1) Let $x = x(t,t_0,x_0)$ and $x^{(n)} = x^{(n)}(t,t_0,x_0^{(n)})$ be the solutions of the system (1) and the system (6), respectively, and

$$x(t,t_0,x_0) = x_0,\ x^{(n)}(t,t_0,x_0^{(n)}) = x_0^{(n)},$$

where $x_0 = (x_1^0, x_2^0, \dots) \in m$ and $x_0^{(n)} = (x_1^0, x_2^0, \dots, x_n^0)$. From the proof of Theorem 1 we get that

$$\lim x_s^{(n)}(t,t_0,x_0^{(n)}) = x_s(t,t_0,x_0), \quad s = 1,2,\dots,$$

uniformly converge for $t \in (-\infty,+\infty)$. This means that $x^{(n)}(t,t_0,x_0^{(n)})$ uniformly converge by coordinate to $x(t,t_0,x_0)$.

2) From hypothesis 4) it is easy to see that the solution $\eta^{(n)}(t)$ of the system (12) uniformly converge to $\eta(t)$ by coordinate (see [5]), that is

$$\eta(t) = \lim \eta^{(n)}(t), \quad s = 1,2,\dots, t \in (-\infty,+\infty);$$

where $\eta(t) = \operatorname{colon}\big(\eta_1(t), \eta_2(t), \dots\big)$, $U^{(n)}(t)$ uniformly converge for t to $U(t)$ by coordinate, so $(U^{(n)})^{-1}(t)$ to $U^{-1}(t)$ and

$$\| U(t) \| \le K, \quad \| U^{-1}(t) \| \le K,$$

for $t \in (-\infty,+\infty)$ such that $x(t) = U(t).\eta(t)$, or

$$\eta(t) = U^{-1}(t).x(t). \tag{22}$$

3) On the other hand, the matrix $Q^{(n)}(t)$ in the system (12) has a form of triangle blocks

$$Q^{(n)}(t) = \operatorname{diag}(Q^{(n)+}, Q^{(n)-})$$

where

$$Q^{(n)}(t) = U^{(n)-1}(t)\left(A^{(n)}(t)U^{(n)}(t) - \frac{dU^{(n)}}{dt}\right).$$

It is clear that the sequence $Q^{(n)}(t)$ uniformly regularly converge for t to the matrix $Q(t)$, that is

$$\lim Q^{(n)}(t) = Q(t), \text{ uniformly for } t \in (-\infty,+\infty).$$

By the assumption 3) we may assume that the system (6) has exponential dichotomy corresponding to m^+ of k-dimensions and to m^- of $(n-k)$-dimensions. Then the system (12) has the following form

$$\frac{d\xi^{(n)}}{dt} = Q^{n+}(t).\xi^{(n)},$$

$$\frac{d\zeta^{(n)}}{dt} = Q^{(n)-}(t).\zeta^{(n)}.$$

We consider the system

$$\frac{d\xi_s^{(n)}(t)}{dt} = \sum_{j\geq s}^{\infty} q_{sj}^{(n)+}(t)\xi_j^{(n)}(t), \quad s = 1, 2, \ldots,$$

where $q_{sj}^{(n)+}(t).\xi_j^{(n)} = 0$ if $j > k$. Since

$$\| \xi^{(n)}(t) \| \leq \| U^{(n)-1}(t) \| \| x^{(n)}(t) \| \leq D < \infty, \quad t \in (-\infty, +\infty),$$

$\sum_{j\geq s}^{\infty} q_{sj}^{(n)+}(t).\xi_j^{(n)}(t)$ uniformly converge for $t \in (-\infty, +\infty)$. Moreover, $\{q_{sj}^{(n)+}(t).\xi_j^{(n)}(t)\}$ unifomly converge for j when $n \to \infty$ for each $t \in (-\infty, +\infty)$. We have

$$\lim \frac{d\xi_s^{(n)}}{dt} = \lim \sum_{j\geq s}^{\infty} q_{sj}^{(n)+}(t).\xi_j^{(n)}(t)$$

$$= \sum_{j\geq s}^{\infty} q_{sj}^{+}(t).\xi_j(t)$$

uniformly for $t \in (-\infty, +\infty)$, $s = 1, 2, \ldots$, that is

$$\frac{d\xi_s(t)}{dt} = \sum_{j\geq s}^{\infty} q_{sj}^{+}(t).\xi_j(t), \quad s = 1, 2, \ldots \tag{23}$$

Analogously,

$$\frac{d\zeta_s(t)}{dt} = \sum_{j\geq s}^{\infty} q_{sj}^{-}(t).\zeta_j(t), \quad s = 1, 2, \ldots \tag{24}$$

We denote

$$Q^+(t) = [q_{sj}^+(t)], \quad j \geq s, \ s = 1, 2, \ldots,$$

$$Q^-(t) = [q_{sj}^-(t)], \quad j \geq s, \ s = 1, 2, \ldots$$

Then from (21) and (22) we obtain the desired result. The theorem is proved.

3. The stability of the exponential dichotomy

We consider the system

$$\frac{dy_s}{dt} = \sum_{j=1}^{\infty} a_{sj}(t).y_j + F_s(t, y_1, y_2, \dots) \quad (s = 1, 2, \dots) \tag{25}$$

or

$$\frac{dy}{dt} = A(t)y + F(t, y). \tag{26}$$

Besides the assumptions on the system (1), we assume that in the domain

$$Z = \{(t, y) : t \in (-\infty, +\infty),\ y \in m,\ \| y \| < \infty\}$$

the function F_s satisfy the following conditions

$$\| F_s(t, y_1, y_2, \dots) - F_s(t, \bar{y}_1, \bar{y}_2, \dots) \| \le \alpha \sup\{|y_1 - \bar{y}_1|, |y_2 - \bar{y}_2|, \dots\}, \quad s = 1, 2, \dots, \tag{27}$$

where α is a positive constant, and

$$\| F_s(t, y_1, y_2, \dots) \| \le D \| y \|^q, \quad s = 1, 2, \dots, \tag{28}$$

where D is a positive constant, $q > 1$.

LEMMA. *If the system (1) has exponential dichotomy on $(-\infty, +\infty)$, then there exists a transformation*

$$x = U(t).y \tag{29}$$

which transforms (1) into a diagonal block system with

$$\| U(t) \| \le K, \ \| U^{-1}(t) \| \le K, \quad \| \frac{dU}{dt} \| \le K, \tag{30}$$

where K is a positive constant.

PROOF. Since the system (1) has exponential dichotomy, we may assume that $m^+ = \{x_s^+(t) : x_s^+(t) = x_s^+(t, t_0, x_0^+)$ is the solution of the system (1) satisfying (4_1), $s = 1, 2, \dots\}$,
$m^- = \{x_r^-(t) : x_r^-(t) = x_r^+(t, t_0, x_0^-)$ is the solution of the system (1) satisfying (4_2), $r = 1, 2, \dots\}$.

Denoting

$$u_1^+(t) = \frac{x_1^+(t)}{\| x_1^+(t) \|}, \quad u_2^+(t) = \frac{x_2^+(t) - \left(x_2^+(t), u_1^+(t)\right)u_1^+(t)}{\| x_2^+(t) - \left(x_2^+(t), u_1^+(t)\right)u_1^+(t) \|}$$

and

$$v_n^+ = x_n^+(t) - \sum_{j=1}^{n-1}\Big(x_n^+(t), u_j^+(t)\Big)u_j^+(t), \quad n \geq 2. \tag{31}$$

we have

$$u_n^+(t) = \frac{v_n^+}{\| v_n^+ \|}, \quad n = 2, 3, \dots \tag{32}$$

It is clear that $\| u_n^+(t) \| = 1$, $n = 1, 2, \dots$, and $(u_j^+, u_k^+) = \delta_{jk}$, $j, k = 1, 2, \dots$, where δ_{jk} is the Kronecker symbol.

By the above mentioned analogous method, for $u_m^-(t)$, $m = 1, 2, \dots$, where

$$u_1^-(t) = \frac{v^-}{\| v_1^- \|}, \dots, \quad u_m^-(t) = \frac{v^-}{\| v_m^- \|}, \quad m = 1, 2, \dots$$

we also get

$$\| u_m^-(t) \| = 1, \ (u_j^-, u_k^-) = \delta_{jk}, \quad j, k = 1, 2, \dots$$

We denote

$$U(t) = \text{colon}\left(u_1^+, u_2^+, \dots, u_1^-, u_2^-, \dots\right). \tag{33}$$

Then by the hypothesis of the lemma, there exists a positive constant K such that

$$\| U(t) \| \leq K, \ \| U^{-1}(t) \| \leq K.$$

On the other hand,

$$\dot{u}_n^\pm(t) = \frac{\dot{v}_n^\pm}{\| v_n^\pm \|} - \frac{v_n^\pm(\dot{v}_n^\pm, v_n^\pm)}{\| v_n^\pm \|^3}, \quad n = 1, 2, \dots \tag{34}$$

So we have

$$\| \frac{dU}{dt} \| \leq K.$$

Put

$$\Theta(t) = \left(x_1^+, x_2^+, \dots, x_1^-, x_2^-, \dots\right). \tag{35}$$

It is the fundamental matrix of solutions of the system (1). We denote

$$U(t) = \Theta(t).S(t). \tag{36}$$

It is clear that $S(t)$ is a diagonal block matrix of the form

$$S(t) = \begin{pmatrix} S^+(t) & 0 \\ 0 & S^-(t) \end{pmatrix}$$

Where $S^{\pm}(t)$ are the infinite matrix. Indeed, since u_s^+, $s = 1, 2, \dots, u_r^-$, $r = 1, 2, \dots$ only depend on x_s^+ and x_r^-, $s, r = 1, 2, \dots$, respectively. By differentiating (36) we get

$$\frac{dU}{dt} = \frac{d\Theta}{dt}S + \Theta\frac{dS}{dt} = AU + US^{-1}\frac{dS}{dt}.$$

Hence

$$Q(t) = U^{-1}\left(AU - \frac{dU}{dt}\right) = -S^{-1}\frac{dS}{dt}$$

is a diagonal block form. If we put

$$x = U(t).\eta, \tag{37}$$

then the system (1) will be transformed into the diagonal block form

$$\frac{d\eta}{dt} = Q(t).\eta, \tag{38}$$

or

$$\begin{aligned} \tfrac{d\xi}{dt} &= Q^+(t).\xi, \\ \tfrac{d\zeta}{dt} &= Q^-(t).\zeta \end{aligned} \tag{39}$$

where

$$\xi = \text{colon}\,(\xi_1, \xi_2, \dots), \quad \zeta = \text{colon}\,(\zeta_1, \zeta_2, \dots).$$

So the lemma is proved.

THEOREM 3. *Suppose that*
1. The system (1) has the exponential dichotomy on $(-\infty, +\infty)$.
2. The functions F_s in the region Z satisfy the conditions (27) and (28).
Then the system (25) also has the exponential dichotomy for $t \in (-\infty, +\infty)$.

PROOF. Without loss of generality we can write the system (1) in the form (39) corresponding to two closed subspaces m^+ and m^- :

$$\begin{aligned} \frac{du}{dt} &= Q^+(t).u \\ \frac{dv}{dt} &= Q^-(t).v. \end{aligned} \tag{40}$$

That is, the matrix $A(t)$ can be presented in the form

$$A(t) = \begin{pmatrix} Q^+(t) & 0 \\ 0 & Q^-(t) \end{pmatrix}.$$

If we put

$$F(t,y) = \text{colon}\,(F^+(t,y), F^-(t,y)),$$

then the system (25) can be written in the form

$$\begin{aligned} \tfrac{du}{dt} &= Q^+(t)u + F^+(t,u_1,u_2,\dots,v_1,v_2,\dots), \\ \tfrac{dv}{dt} &= Q^-(t)v + F^-(t,u_1,u_2,\dots,v_1,v_2,\dots). \end{aligned} \tag{41}$$

Let $x_s^+(t) = x_s^+(t,t_0,x_0^+)$ be the solution of the system (25) corresponding to $F(t,y) = 0$ and satisfies $x_s^+(t_0,t_0,x_0^+) = x_s^{+0} \in m^+,\ s = 1,2,\dots$. By [1, pp. 52–55] the solution $x_s^+(t)$ of the system (41) is of the form

$$x_s^+(t) = x_s^+(t,t_0,x_0^+) + \int_{t_0}^{t} x_s^+(t,r,F_1^+(r,x_1^+(r),\dots),F_2^+(r,\dots))dr, \tag{42}$$

where $F^+ = \text{colon}\,(F_1^+, F_2^+, \dots)$.

We will estimate the solution (42). By the exponential dichotomy of the system (1) and the assumption on function F, we have

$$\begin{aligned} \| x_s^+(t) \| &\le \| x_s^+(t,t_0,x_0^+) \| + \int_{t_0}^{t} \| x_s^+(t,r,F_1^+(r),x_1^+(r)\dots),\dots)dr \| \\ &\le \| x_s^+(t_0) \| N_1 e^{-\nu_1(t-t_0)} + \int_{t_0}^{t} N_1 D e^{-\nu_1(t-t_1)} \| x^+(t_1) \|^q \, dt_1 \\ &\le N_1 \| x_s^+(t_0) \| e^{-\nu_1(t-t_0)} + \\ &\quad + DN_1^{q+1} \| x_s^+(t_0) \|^q e^{-\nu_1(t-t_0)} \int_{t_0}^{t} e^{\nu_1(1-q)r} dr \\ &\le N_1 \| x_s^+(t_0) \| e^{-\nu_1(t-t_0)} + DN_1^{q+1} \| x_s^+(t_0) \|^q \frac{e^{-\nu_1(t-t_0)}}{\nu_1(q-1)}, \end{aligned}$$

or

$$\| x_s^+(t) \| \le C_s \| x_s^+(t_0) \|^{-\nu_1(t-t_0)}, \quad \text{for } t \ge t_0,\ (s = 1,2,\dots), \tag{43}$$

where

$$C_s = N_1 \left(\frac{DN_1^q \| x_s^+(t_0) \|^{q+1}}{\nu_1(q-1)} + 1 \right).$$

Hence we have

$$\| x^+(t) \| \leq C_1 \| x^+(t_0) \| e^{-\nu_1(t-t_0)}, \quad \text{for } t \geq t_0, \tag{44}$$

where $C_1 = \sup\{C_s\} > 0$. Analogously, we also have

$$\| x^-(t) \| \leq C_2 \| x^-(t_0) \| e^{\nu_2(t-t_0)}, \quad \text{for } t \leq t_0, \tag{45}$$

where C_2 is a positive constant. Now from (44) and (45) and the properties of function $F(t, y)$ we deduce the inequality $S_n(m^+, m^-) \geq \gamma > 0$.

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