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ON CONVERGENCE FOR SOME DIFFERENTIAL OPERATORS OF DISTRIBUTIONS

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The convergence of classical differential operators was considered by several authors (see, for instance, [1-4]). This work deals with the convergence for some differential operators of distributions.

1. Convergence of distributional differential operators

Let Ω be a connected set in the real n-dimensional space $\mathbf{R}^n$. We consider the following differential equations of distributions

$$P_\varepsilon(x, D)v^\varepsilon = f^\varepsilon, \tag{1}$$

where

$$P_\varepsilon(x, D) = \sum_{|\alpha|\leq m} a^\varepsilon_\alpha(x) D^\alpha \tag{2}$$

is a family of differential operators of order m with coefficients $a^\varepsilon_\alpha \in C^\infty(\Omega^*)$. Here $\Omega^* = \Omega$ if Ω is an open set, $\Omega^* = \Omega \setminus \partial\Omega$ if Ω is a closed set, $\partial\Omega$ is the boundary of Ω; $f^\varepsilon \in \mathcal{D}'(\Omega)$ - the space of distribution in Ω, $\varepsilon > 0, \alpha$ are multi-indices; and

$$\mathcal{D}^\alpha = \frac{\partial^{|\alpha|}}{x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \dots + \alpha_n.$$

Like in the classic analysis theory, it is easy to prove the following:

LEMMA 1. *Suppose that when $\varepsilon \to 0$ the sequences $\{u^\varepsilon\} \to u, \{f^\varepsilon\} \to f$ in $\mathcal{D}'(\Omega), \{a^\varepsilon_\alpha(x)\} \to a_\alpha(x)$ in $C^\infty(\Omega^*)$. Then the distribution u satisfies the*

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following equation

$$P(x,D)u = f, \tag{3}$$

where $P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$.

We now consider the convergence for the solutions of generalized Cauchy Problems. Let V be a closed convex acute cone, $\Gamma = \mathrm{int}V^*$, V^* is the conjugate cone of V. Let M be a Γ-like surface and M^+ a region lying above M (see [5], Section 4.4). Similarly [5] we use the term Generalized Cauchy Problem for the operators $P_\varepsilon(x,D)$ with the sources F^ε to describe the problem of finding a generalized solution u^ε in $\mathcal{D}'(\overline{M^+})$ of the equations

$$P_\varepsilon(x,D)u^\varepsilon = F^\varepsilon, \tag{4}$$

where $P_\varepsilon(x,D)$ are the hyperbolic operators relative to the cone Γ of order m, $F^\varepsilon \in \mathcal{D}'(\overline{M^+})$.

The Cauchy problem for a differential operator with variable coefficients and the right hand side $f \in \overset{o}{H}_{(k,s)}(\overline{\mathbf{R}_n^+})$ is considered in [6]. It is clear that all solutions of the classical Cauchy problem are contained among the solutions of Generalized Cauchy Problem.

The next theorems are followed by combining Lemma 1, Schwartz Theorem on a weakly bounded set and the theorem on a convergent sequence of linear functionals:

THEOREM 1. *Let $\{F^\varepsilon\}$ be a sequence of functionals from a weakly bounded set $A \subset \mathcal{D}'(\overline{M^+})$, i.e. $(F^\varepsilon,\varphi) < C_\varphi$ for all $F^\varepsilon \in A$ and $\varphi \in \mathcal{D}(\mathbf{R}^n)$, here C_φ are finite constants depending only on φ. Suppose that $\{a_\alpha^\varepsilon(x)\} \to a_\alpha(x)$, $\varepsilon \to 0$ in $C^\infty(M^+)$. Then the sequence of solutions $\{u^\varepsilon\}$ of Generalized Cauchy Problems (4) converges to a solution u of following Generalized Cauchy Problem*

$$P(x,D)u = F, \tag{5}$$

where the distribution F is the limit of the sequence $\{F^\varepsilon\}$.

THEOREM 2. *Suppose that the sequence $\{u^\varepsilon\}$ of solutions of Cauchy Problem (4) converges to a solution u of Cauchy Problem (5) in $\mathcal{D}'(\overline{M^+})$, then the sequence of right-hand sides $\{F^\varepsilon\}$ in (4) is a weakly bounded set in $\mathcal{D}'(\overline{M^+})$.*

REMARK. The previous results can be extended to the convergence for differential operators in the space of temperate distributions.

2. Convergence of mixed distributions

DEFINITION. *A mixed distribution on an open set $\overset{\circ}{\Omega}$ is a continuous positive homogenous and convex or cocave functional on the space $\mathcal{D}(\overset{\circ}{\Omega})$.*

From this definition it follows that any convex distribution [8] is a mixed distribution.

Denote by $\tilde{\mathcal{D}}(\overset{\circ}{\Omega})$ the set of all mixed distributions defined on $\overset{\circ}{\Omega}$. Suppose $\tilde{f}$ and $\tilde{g}$ are in $\tilde{\mathcal{D}}(\overset{\circ}{\Omega})$, we define their sum and product with a scalar as follows:

$$(\tilde{f}+\tilde{g},\varphi) = (\tilde{f},\varphi) + (\tilde{g},\varphi),$$
$$(\lambda\tilde{f},\varphi) = \lambda(\tilde{f},\varphi),$$

where $\varphi \in \mathcal{D}(\overset{\circ}{\Omega})$ and λ is a real number.

The mixed distributions $\tilde{f}, \tilde{g}$ are said to be homothetic if their linear combination $(\lambda\tilde{f}+\eta\tilde{g})$ belongs to $\tilde{\mathcal{D}}(\overset{\circ}{\Omega})$ for any real numbers λ and η. This means that the set of all homothetic mixed distributions on Ω is a linear set which is denoted by $\tilde{\mathcal{D}}_H(\overset{\circ}{\Omega})$. It is easy to verify that $\tilde{f}$ and $D^\alpha\tilde{f}$ are homothetic.

Let us define convergence in $\tilde{\mathcal{D}}_H(\overset{\circ}{\Omega})$: a sequence $\{\tilde{f}_k\}$, $k = 1,2,...$ in $\tilde{\mathcal{D}}_H(\overset{\circ}{\Omega})$ converges to $\tilde{f} \in \tilde{\mathcal{D}}_H(\overset{\circ}{\Omega})$ if for any basic function $\varphi \in \mathcal{D}(\overset{\circ}{\Omega})$ $(\tilde{f}_k,\varphi) \to (\tilde{f},\varphi)$, $k \to \infty$. In this case we write $\tilde{f}_k \to \tilde{f}, k \to \infty$ in $\tilde{\mathcal{D}}_H(\overset{\circ}{\Omega})$. The linear set $\tilde{\mathcal{D}}_H(\overset{\circ}{\Omega})$ equipped with this convergence is called the space of homothetic mixed distributions. $\tilde{\mathcal{D}}_H(\overset{\circ}{\Omega})$ is obviously a vector space. Note that arguing as in proof of the completeness of the space $\tilde{\mathcal{D}}_H(\overset{\circ}{\Omega})$ [8] we can show that $\tilde{\mathcal{D}}_H(\overset{\circ}{\Omega})$ is a complete space.

The definitions of differentiation, integration, multiplication by functions, convolution, ... of homothetic mixed distributions are analogous to that of convex distributions [8].

We now consider differential equations of homothetic mixed distributions:

$$P_\varepsilon(D)\tilde{u}^\varepsilon = \tilde{f}^\varepsilon, \tag{6}$$

where $\tilde{f}^\varepsilon \in \tilde{\mathcal{D}}_H(\Omega)$, $\{P_\varepsilon(D)\}$ is a family of differential operators of order m with constant coefficients C_α^ε :

$$P_\varepsilon(D) = \sum_{|\alpha|\leq m} C_\alpha^\varepsilon D^\alpha, \qquad \sum_{|\alpha|\leq m} |C_\alpha| \neq 0.$$

LEMMA 2. *Supppose that the sequence of fundamental solutions $\{\tilde{E}^\varepsilon\}$ of operators $P_\varepsilon(D)$ converges to $\tilde{E}$, $\{\tilde{f}^\varepsilon\} \to \tilde{f}$ in $\tilde{\mathcal{D}}_H(\Omega)$ and $\{C_\alpha^\varepsilon\} \to C_\alpha$, $\varepsilon \to 0$. Then $\tilde{E}$ is the fundamental solution of operator $P(D)$ and the sequence of solutions of the equations (6) $\{\tilde{u}^\varepsilon\}$ converges to a solution $\tilde{u}$ of the following equation*

$$P(D)u = f, \tag{7}$$

where $P(D) = \sum\limits_{|\alpha|\leq m} C_\alpha D^\alpha$.

PROOF. The solution of Equation (6) can be written in the following form, which is analogous to that of Convex Distributions [8]:

$$\tilde{u}^\varepsilon = \tilde{f}^\varepsilon * \tilde{E}^\varepsilon, \tag{8}$$

where $\tilde{E}^\varepsilon$ is a fundamental solution in $\tilde{\mathcal{D}}_H(\Omega)$ of the operator $P_\varepsilon(D)$.

It is easy to verify that if $\{C_\alpha^\varepsilon\} \to C_\alpha$ and $\{\tilde{E}^\varepsilon\} \to \tilde{E}$, $\varepsilon \to 0$, the homothetic mixed distribution $\tilde{E}$ will be a fundamental solution of the operator $P(D)$. Hence, passing to limit in (8), by virtue of the continuity of a convolution for its components, we obtain

$$\tilde{u}^\varepsilon \to \tilde{u} = \tilde{f} * \tilde{E}.$$

The last equality shows that the limit $\tilde{u}$ is a solution of Equation (7).

We now consider Generalized Cauchy Problems for operators $P_\varepsilon(D)$ with the source $\tilde{F}^\varepsilon \in \tilde{\mathcal{D}}_H(\overline{M^+})$:

$$P_\varepsilon(D)\tilde{u}^\varepsilon = \tilde{F}^\varepsilon. \tag{9}$$

THEOREM 3. *Suppose that* $\{C_\alpha^\varepsilon\} \to C_\alpha$, $\varepsilon \to 0$. *Each of the following two conditions is necessary and sufficient in order that the sequence* $\{\tilde{u}^\varepsilon\}$ *of solutions of Problem (9) converges to a solution* u *of the following generalized Cauchy problem*

$$P(D)\tilde{u} = \tilde{F} \tag{10}$$

a. $\{\tilde{F}^\varepsilon\}$ *is a sequence from a weakly bounded set in* $\tilde{\mathcal{D}}_H(\overline{M^+})$;

b. The sequence of fundamental solutions of operators $P_\varepsilon(D)\{\tilde{E}^\varepsilon\}$ *is a weakly bounded set in* $\mathcal{D}_H(\overline{M^+})$; *where* F *is the limit of* $\{\tilde{F}^\varepsilon$.

PROOF. The solution of problem (9) can be written in the following form

$$\tilde{u}^\varepsilon = \tilde{E}^\varepsilon * \tilde{F}^\varepsilon.$$

Hence we have

$$(\tilde{u}^\varepsilon, \varphi) = (\tilde{F}^\varepsilon(x), (\tilde{E}^\varepsilon(y), \gamma(x)\beta(y)\varphi(x+y))), \tag{11}$$

or

$$(\tilde{u}^\varepsilon, \varphi) = (\tilde{E}^\varepsilon(y), (\tilde{F}^\varepsilon(x), \gamma(x)\beta(y)\varphi(x+y))), \tag{12}$$

where $\varphi \in \mathcal{D}(\mathbf{R}^n)$, γ and β are any functions in $C^\infty(\mathbf{R}^n)$ that are equal to 1 in the neighbourhood of supp $\tilde{F}^\varepsilon$ and of supp $\tilde{E}^\varepsilon$ respectively.

By virtue of (11) and (12) the theorem now easily follows.

Finally, we give an example for the convergence of solutions that has been mentioned in Theorem 3. In [9] we have constructed the exact solution for Problem of air pollution described by the following equation (see formulae (1'), (24) in [9]):

$$\frac{\partial F}{\partial t} - \lambda\Delta F + \overrightarrow{V}.\nabla F + \sigma F = f, \tag{13}$$

where $F = F(x,t)$ - the concentration of pollutant, $\overrightarrow{V}$- the wind velocity, $\lambda = \lambda(x)$ - the diffusion coefficient , $\sigma = \sigma(x)$ - the rate of chemical decay transformation and $f = f(x,t)$ - the power of source. If the considered region is unbounded, the concentration of pollutant vanishes at the infinity and the density distribution of masses f is a mixed distribution, the solution of (13) has the following form (for $f = f^\varepsilon$, $E = E^\varepsilon$) :

$$F^\varepsilon = f^\varepsilon * E^\varepsilon + [I(x) \times \delta(t)] * E^\varepsilon, \tag{14}$$

where E^ε is the fundamental solution of the differential equation corresponding to (13) (see the formula (20) in [9]) of the considered problem (for $\overrightarrow{V} = \overrightarrow{V}_\varepsilon$; $\lambda = \lambda_\varepsilon$, $\sigma = \sigma_\varepsilon$), $I(x)$- the initial condition, $\delta(t)$- the Dirac distribution. When the distributions f, I are continuously differentiable functions this solution has the form (see (34) in [9]):

$$F^\varepsilon = \int_0^t \int_{\mathbf{R}^n} \frac{f^\varepsilon(\xi,\tau)}{[4\lambda_\varepsilon \pi(t-\tau)]^{\frac{n}{2}}} exp\{-[\sigma_\varepsilon(t-\tau) + \frac{|(x-\xi) - \overrightarrow{V}_\varepsilon(t-\tau)|^2}{4\lambda_\varepsilon(t-\tau)}\} d\xi d\tau + \frac{\theta(t)e^{-\sigma_\varepsilon t}}{(4\lambda_\varepsilon \pi t)^{n/2}} \int_{\mathbf{R}^n} I(y) exp\{-\frac{|x-y-\overrightarrow{V}_\varepsilon t|^2}{4\lambda_\varepsilon t}\} dy.$$

From the last equality we see that when $\lambda_\varepsilon \to 1$, $\overrightarrow{V}_\varepsilon$ and $\sigma_\varepsilon \to 0$, our solutions converges to a solution of the Cauchy problem $\frac{\partial F}{\partial t} - \Delta F = f$ (see the solution (7.1) Section 15.7, p.256 in [5])

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