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A REMARK ON THE BERNSTEIN-NIKOLSKII INEQUALITY*

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Abstract. Certain criteria for a function from an arbitrary Orlicz space having its spectrum contained in parallelepiped $\Delta_\sigma = \{\xi;\ |\xi_j| \leq \sigma_j,\ j = 1, \dots, n\}$ are established in this note.

1. Introduction

It is well-known that while trigonometric polynomials are good means of approximation for periodic functions, entire functions of exponential type may serve as a mean of approximation for nonperiodic functions, given on n-dimensional space. Let $1 \leq p \leq \infty$ and $\sigma = (\sigma_1, \dots, \sigma_n),\ \sigma_j > 0,\ j = 1, \dots, n$. Denote by $M_{\sigma,p}$ the space of all entire functions of exponential type σ which as functions of a real x belong to $L_p(\mathbb{R}^n)$. The Bernstein-Nikolskii inequality, which is very important in imbedding theory, approximation theory and applications, reads as follows ([3], p.114): Let $f(x) \in M_{\sigma,p}$. Then $D^\alpha f(x) \in L_p(\mathbb{R}^n)$ and

$$||D^\alpha f||_p \leq \sigma^\alpha ||f||_p, \tag{1}$$

for any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, where $D = (D_1, \dots, D_n),\ D_j = \frac{\partial}{\partial x_j},\ j = 1, \dots, n, D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n},\ \sigma^\alpha = \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$. It it natural to ask whether there is a function $f(x) \notin M_{\sigma,p}$ for which these inequalities (1) hold? Using the bounded average functions of $f(x)$ we will show by a very simple proof in this note (for a more general case) that the answer is negative. In other words, the Bernstein-Nikolskii inequality wholly characterizes the space $M_{\sigma,p}$. We

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emphasize that this technique is often helpful for us to study certain connections between properties of a function and its spectrum.

2. Results

Let $\Phi(t) : [0,+\infty) \to [0,+\infty]$ be an arbitrary Young function [2,4], i.e. $\Phi(0) = 0$, $\Phi(t) \geq 0$, $\Phi(t) \not\equiv 0$ and $\Phi(t)$ is convex. Denote by $L_\Phi(\mathbf{R}^n)$ the space of all functions $f(x)$ measurable on $\mathbf{R}^n$ such that

$$||f||_\Phi = \inf\{\lambda > 0 : \int_{\mathbf{R}^n} \Phi(|f(x)|/\lambda)dx \leq 1\} < \infty.$$

Then $L_\Phi(\mathbf{R}^n)$ with respect to the Luxemburg norm $||\cdot||_\Phi$ is a Banach space. $L_\Phi(\mathbf{R}^n)$ is called Orlicz space.

Recall that $||\cdot||_\Phi = ||\cdot||_p$ when $1 \leq p < \infty$ and $\Phi(t) = t^p$; and $||\cdot||_\Phi = ||\cdot||_\infty$ when $\Phi(t) = 0$ for $0 \leq t \leq 1$ and $\Phi(t) = \infty$ for $t > 1$. Orlicz spaces often arise in the study of nonlinear problems.

Denote by $M_{\sigma,\Phi}$ the space of all entire functions of exponential type σ which as functions of a real $x \in \mathbf{R}^n$ belong to $L_\Phi(\mathbf{R}^n)$. It is easy to check that $M_{\sigma,\Phi} \subset S'$, where S' is the dual space of the Schwartz space S of rapidly decreasing infinitely differentiable functions. And, as we shall show later that if $f \in M_{\sigma,\Phi}$, then $f \in L_\infty(\mathbb{R}^n)$, therefore, it follows from the Paley-Wiener-Schwartz theorem that

$$M_{\sigma,\Phi} = \{f \in L_\Phi(\mathbf{R}^n) : \mathrm{supp} Ff \subset \Delta_\sigma\},$$

where $Ff = \hat{f}$ is the Fourier transform of the function f and $\Delta_\sigma = \{\xi \in \mathbf{R}^n : |\xi_j| \leq \sigma_j, j = 1,\ldots,n\}$. And $M_{\sigma,\Phi}$, as a subspace of $L_\Phi(\mathbb{R}^n)$, is a Banach space.

Now we study some properties of the space $M_{\sigma,\Phi}$.

LEMMA 1. *Let $f \in M_{\sigma,\Phi}$. Then $f(x)$ is bounded on $\mathbb{R}^n$.*

PROOF. Without loss of generality we may assume that

$$\int_{\mathbb{R}^n} \Phi(|f(x)|)dx < \infty. \tag{2}$$

Let $\hat{\psi} \in C_0^\infty(\mathbb{R}^n)$ and $\hat{\psi} = 1$ in some neighbourhood of supp$\hat{f}$. Then (see, for example, [1])

$$f(x) = \int f(y)\psi(x-y)dy.$$

Further, let M_1, M_2 be positive numbers such that $\overline{\Phi}(\|\psi\|_\infty/M_1)$ and $\|\psi\|_\infty \leq M_2$. Then the Young inequality and $\overline{\Phi}(\lambda t) \leq \lambda\overline{\Phi}(t)$ for all $0 \leq \lambda \leq 1,\ t \geq 0$ yield

$$\begin{aligned} |f(x)|/M_1M_2 &\leq \int \Phi(|f(y)|)dy + \int \overline{\Phi}(|\psi(x-y)|/M_1M_2)dy \\ &\leq \int \Phi(|f(y)|)dy + \overline{\Phi}(\|\psi\|_\infty/M_1)\int |\psi(y)|/M_2 dy \ < \infty, \end{aligned}$$

where

$$\overline{\Phi}(t) = \sup_{s\geq 0}\{ts - \Phi(s)\}$$

is the complementary to $\Phi(t)$ function.

The proof is complete.

REMARK 1. Let $1 \leq p < \infty$. It was proved in [3] that

$$\lim_{|x|\to\infty} f(x) = 0$$

for all $f \in M_{\sigma,p}$. Now we consider this property for $M_{\sigma,\Phi}$. Clearly, this conclusion is false if $\Phi(\lambda) = 0$ for some $\lambda > 0$ because, in this case, $M_{\sigma,\Phi}$ contains all constant functions.

LEMMA 2. *Let $\Phi(t) > 0$ for $t > 0$. Then*

$$\lim_{|x|\to\infty} f(x) = 0$$

for all $f \in M_{\sigma,\Phi}$.

PROOF. Assume the contrary that there are a function $f \in M_{\sigma,\Phi}$, a constant $c > 0$ and a sequence $|x^m| \to \infty$ such that

$$|f(x^m)| \geq 2c, \quad m = 1, 2, \ldots \tag{3}$$

Without loss of generality we may assume that (2) holds and $|x_1^m| \to \infty, m \to \infty$. Since

$$f(x) - f(x^m) = \int_{x_1^m}^{x_1} \frac{\partial}{\partial t_1} f(t)dt,$$

and by Lemma 1 and the Bernstein-Nikolskii inequality, we get

$$f(x) - f(x^m) \leq \sigma_1 \|f\|_\infty |x_1 - x_1^m| \tag{4}$$

for all $x \in \mathbf{R}^n$ and $m \geq 1$.

Putting $r = c/\sigma_1 \|f\|_\infty$, we get from (3)-(4)

$$|f(x)| \geq c \quad \text{for} \quad |x_1 - x_1^m| \leq r \quad \text{and } m \geq 1. \tag{5}$$

On the other hand, without loss of generality we may assume that

$$x_1^{m+1} - x_1^m \geq r, \ m \geq 1.$$

Then, from (2) and (5) we obtain

$$\infty > \int_{\mathbf{R}^n} \Phi(|f(x)|)dx \geq \sum_{m=1}^{\infty} \int_{|x-x^m| \leq r} \Phi(|f(x)|)dx$$
$$\geq \sum_{m=1}^{\infty} \Phi(r) \text{mes}\, B(x^m, r) = \sum_{m=1}^{\infty} \pi r^n \Phi(r) = \infty,$$

where $B(x^m, r)$ is the ball of radius r centered at x^m, which is impossible.

We obtain the following result:

THEOREM 1. *Let $f(x) \in S'$. In order that $f(x) \in M_{\sigma,\Phi}$, it is necessary and sufficient that there exists a constant $C = C(f)$ such that*

$$||D^\alpha f||_\Phi \leq C\sigma^\alpha, \quad \alpha \geq 0. \tag{6}$$

PROOF. **Necessity**. Let $f(x) \in M_{\sigma,\Phi}$. It follows from Lemma 1 that $f(x)$ is bounded on $\mathbf{R}^n$. Then, in the same way as in [3] we easily get the Bernstein-Nikolskii inequality for the Luxemburg norm:

$$||D^\alpha f||_\Phi \leq \sigma^\alpha ||f||_\Phi, \quad \alpha \geq 0.$$

Therefore, we have (6).

It should be mentioned that the Bernstein-Nikolskii inequality was proved in [3] for general norms $\| \cdot \|_p$ but only, loosely speaking, for bounded on $\mathbf{R}^n$ functions of exponential type. So, our contribution here is only Lemma 1.

Sufficiency. Without loss of generality we may assume that $\Phi(t)$ is left continuous. Actually, in the contrary case, there exists a point $t_0 > 0$ such that

$$\lim_{t \to t_0 -} \Phi(t) < \Phi(t_0) \leq \infty, \text{ and } \Phi(t) = \infty \text{ for } t > t_0.$$

We put

$$\psi(t) = \begin{cases} \Phi(t), & t \neq t_0, \\ \lim_{t \to t_0 -} \Phi(t), & t = t_0. \end{cases}$$

Then $\psi(t)$ is a left continuous Young function and $\| \cdot \|_\psi = \| \cdot \|_\Phi$. Therefore, we can replace $\Phi(t)$ by $\psi(t)$.

Assume that (6) holds. It is easily seen that $f(x) \in C^\infty(\mathbf{R}^n)$. Actually, let $g(x) \in L_\Phi(\mathbf{R}^n)$. Since $\Phi(t) \not\equiv 0$, we get

$$\Phi(\gamma/(\|g\|_\Phi + \epsilon)) > 0$$

for some numbers $\gamma, \epsilon > 0$. Further, it follows from the definition of $\Phi(t)$ that

$$a\Phi(t) \leq \Phi(at)$$

for all $a \geq 1$ and $t \in [0, \infty)$. Therefore,

$$\begin{aligned} &\Phi(\gamma/(\|g\|_\Phi + \epsilon)) \int_{|g(x)| \geq \gamma} (|g(x)|/\gamma) dx \\ &\leq \int_{|g(x)| \geq \gamma} \Phi(|g(x)|/(\|g\|_\Phi + \epsilon)) dx \\ &\leq \int_{\mathbf{R}^n} \Phi(|g(x)|/(\|g\|_\Phi + \epsilon)) dx \leq 1. \end{aligned}$$

Hence, $g(x) \in L_{1,\text{loc}}(\mathbf{R}^n)$. Therefore, by (6) we get $D^\alpha f(x) \in L_{1,\text{loc}}(\mathbf{R}^n)$ for all $\alpha \geq 0$. Thus, $f(x) \in C^\infty(\mathbf{R}^n)$ by virtue of the Sobolev imbedding theorem.

Further, we remark that it is difficult to imply directly from (6) that $f(x)$ (and all $D^\alpha f(x)$) is bounded. Next we construct the approximative bounded functions of the function $f(x)$: Let $r > 0$, we put

$$f_r(x) = \frac{1}{\text{mes } B(0,r)} \int_{B(0,r)} f(x+t)dt. \tag{7}$$

Then by Jensen's inequality we get

$$\Phi(\frac{|D^\alpha f_r(x)|}{||D^\alpha f||_\Phi + \epsilon}) \le \frac{1}{\text{mes } B(0,r)} \int_{B(0,r)} \Phi(\frac{|D^\alpha f_r(x)|}{||D^\alpha f||_\Phi + \epsilon})dt \le \frac{1}{\text{mes } B(0,r)}$$

for $\epsilon > 0$ and $\alpha \ge 0$. Therefore, taking account of the left continuity of $\Phi(t)$ and (6), we have

$$\sup_{x\in\mathbf{R}^n} |D^\alpha f_r(x)| \le \lambda_r ||D^\alpha f||_\Phi \le C\lambda_r \sigma^\alpha, \alpha \ge 0, \tag{8}$$

where $\lambda_r = \sup\{t : \Phi(t) \le 1/\text{mes } B(0,r)\}$. Therefore, the Taylor series

$$\sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} D^\alpha f_r(0).z^\alpha$$

converges for any point $z \in \mathbf{C}^n$ and represents $f_r(x)$ in $\mathbf{R}^n$. Hence, taking account of (8), we have

$$|f_r(z)| \le C\lambda_r \exp(\sum_{j=1}^n \sigma_j |z_j|), \ z \in \mathbf{C}^n.$$

Therefore, $f_r(z)$ is an entire function of exponential type σ. Hence, it follows from the Paley-Wiener-Schwartz theorem that

$$\text{supp} F f_r \subset \Delta_\sigma, \qquad r > 0. \tag{9}$$

On the other hand, it obviously follows from (7) that f_r converges weakly to f in S' and therefore, Ff_r also converges weakly to Ff in S'. Consequently, it follows readily from (9) that $\text{supp}Ff \subset \Delta_\sigma$. The proof is complete.

To check $f(x) \in M_{\sigma,\Phi}$, the following result is more convenient:

THEOREM 2. *A function $f(x)$ belongs to $M_{\sigma,\Phi}$ if and only if*

$$\limsup_{|\alpha|\to\infty} (\sigma^{-\alpha} ||D^\alpha f||_\Phi)^{1/|\alpha|} \le 1. \tag{10}$$

PROOF. The "if" part follows readily from Theorem 1. Further, we suppose that inequality (10) holds. Given $\epsilon > 0$. There exists a constant C_ϵ such that

$$||D^\alpha f||_\Phi \leq C_\epsilon (1+\epsilon)^{|\alpha|}\sigma^\alpha, \ \alpha \geq 0.$$

Therefore, taking account of Theorem 1, we get

$$\text{supp}Ff \subset \Delta_{(1+\epsilon)\sigma}.$$

Thus,

$$\text{supp}Ff \subset \bigcap_{\epsilon>0} \Delta_{(1+\epsilon)\sigma} = \Delta_\sigma.$$

REMARK 3. Theorem 2 gives us ability to estimate more roughly than Theorem 1. For example, if we have

$$||D^\alpha f||_\Phi \leq C|\alpha|^4\sigma^\alpha, \ \alpha \geq 0.$$

Then (10) is valid although (6) does not hold. Further, we notice that the root $1/|\alpha|$ in (10) cannot be replaced by any $1/|\alpha|t(\alpha)$, where $0 < t(\alpha)$, $\lim_{|\alpha|\to\infty} t(\alpha)$

$= \infty$. Actually, let $f(x) = e^{i2\sigma x}$. Then $f(x) \in M_{2\sigma,\infty}$. At the same time,

$$\lim_{|\alpha|\to\infty} (\sigma^{-\alpha}||D^\alpha f||_\infty)^{1/|\alpha|t(\alpha)} = \lim_{|\alpha|\to\infty} 2^{1/t(\alpha)} = 1.$$

REMARK 4. Let $s \in \mathbf{R}^n$. Denote by $\Delta(s) = (s_1, s_1 + 2\pi) \times \cdots \times (s_n, s_n + 2\pi)$ and

$$||f||_{\Phi,\Delta(s)} = \inf\{\lambda > 0 : \int_{\Delta(s)} \Phi(|f(x)|/\lambda)dx \leq 1\}.$$

Further, denote by $M^*_{\sigma,\Phi}$ the space of all entire functions of exponential type σ such that

$$|||f|||_\Phi = \sup_{s\in\mathbf{R}^n} ||f||_{\Phi,\Delta(s)} < \infty.$$

Then Theorems 1-2 still hold when we replace $M_{\sigma,\Phi}$ by $M^*_{\sigma,\Phi}$ and $||\cdot||_\Phi$ by $|||\cdot|||_\Phi$.

REMARK 5. Lemma 1 holds, but Lemma 2 does not hold for $M^*_{\sigma,\Phi}$.

REMARK 6. R. O'Neil and W. Luxemburg require the left continuity in the definition of a Young function [2,4]. It has already been shown that doing with $||\cdot||_\Phi$, we may always assume that $\Phi(t)$ is left continuous. Therefore, results obtained in [2,4] still hold when we drop this restriction.

REFERENCES

[1] L. Hormander, *The analysis of linear partial differential operators I*, Grundlehren 256, Springer, Berlin, Heidelberg, New York, Tokyo, 1983.

[2] W. Luxemburg, *Banach function spaces*, (Thesis), Technische Hogeschool te Delft., The Netherlands, 1955.

[3] S.M. Nikolskii, *Approximation of functions of several variables and imbedding theorems*, "Nauka", Moscow, 1977.

[4] R. O'Neil, *Fractional integration in Orlicz space I.*, Trans. Amer. Math. Soc. **115**(1965), 300-328.

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