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MORE ON APPLICATIONS AND EXTENSIONS OF P-COERCIVE VARIATIONAL INEQUALITIES*

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Introduction

This paper is a sequel to our two earlier papers, written jointly with K. Schmitt [ASV1], [ASV3], where the concept of P-coerciveness was introduced as an extension of the concept of compact-coerciveness due to Baiocchi, Gastaldi and Tomarelli (cf. [BGT] and [GT]). In [ASV1] and [ASV3], the authors restricted themselves to the case of Hilbert spaces. Several applications were given to obstacle problems, unilateral problems for elliptic equations. The equations considered there were semilinear, whereas the theory developped in [ASV3] could be applied to quasilinear equations, also.

The purpose of the present paper is two-fold. First, we give applications of the results of [ASV3] to some quasilinear problems. Second, since certain problems can be appropriately formulated as variational inequalities only in Banach spaces, we extend the concept of P-coerciveness to Banach spaces. Like our earlier papers [ASV1], [ASV3], our present investigation was motivated by the works of Baiocchi, Gastaldi and Tomarelli (loc. cit.).

This paper consists of two parts. Part 1 is devoted to applications of P-coercive variational inequalities to quasilinear elliptic problems. In Part 2, the concept of P-coerciveness is extended to the case of Banach spaces, a sufficient condition for existence is given for variational inequalities involving P-coercive nonlinear operators in Banach spaces, and applications to quasilinear equations are presented in a Banach space setting.

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1. Applications to quasilinear problems in Hilbert spaces

We shall consider in this part some examples of P-coercive variational inequalities involving essentially nonlinear operators. These arise from certain quasilinear boundary value problems.

1.1. EXAMPLE 1.

1.1.1. Let $\Omega \subset \mathbf{R}^n$ $(n \geq 1)$ be a bounded domain. We consider the problem

$$-\operatorname{div}\left(\frac{|\nabla u|^p}{1+|\nabla u|^p}\nabla u\right) + G(u) = f \quad \text{in } \Omega, \tag{1}$$

$$\partial u/\partial n = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where f is given on $\Omega, p > 0$, and $G(u) = F_0(u) + F(|u|)u$ with

$F_0 : \mathbf{R} \to \mathbf{R}$, increasing, continuous, $F_0(0) = 0$, and

$F : [0, \infty) \to [0, \infty)$, continuous and bounded.

We formulate the problem as a variational inequality as follows. Put

$$\psi(t) = \int_0^t F_0(\tau)d\tau, \ t \in \mathbf{R},$$

and

$$j(v) = \int_\Omega \psi(v(x))dx, \tag{3a}$$

$$< Au, v >= \int_\Omega \left[\frac{|\nabla u|^p}{1+|\nabla u|^p}\nabla u \nabla v + F(|u|)uv\right]. \tag{3b}$$

Then it can be shown that Problem (1)-(2) is equivalent to the following variational inequality

$$\begin{cases} < Au, v-u > \ +j(v) - j(u) \geq \int_\Omega f(v-u), \ \forall v \in H^1(\Omega), \\ u \in H^1(\Omega), \end{cases} \tag{4}$$

with j, A given by (3a), (3b) respectively.

1.1.2. Since ψ is convex and continuous, we have that

$$j : H^1(\Omega) \to \mathbf{R} \cup \{\infty\}$$

is convex and lower semi-continuous. Moreover, $j \geq 0$ but j is not semi-additive in general.

It is clear from the definition of A that $A : H^1(\Omega) \to [H^1(\Omega)]'$ and that A is nonnegative. We verify that A is j-P-coercive om $H^1(\Omega)$ (cf. Section 2.1, [ASV3]). Let $\{x_m\}$ be a sequence in $H^1(\Omega)$ such that

$$\begin{aligned} &\|x_m\| \to \infty, x_m/\|x_m\| \rightharpoonup 0 \quad \text{in } H^1(\Omega)\text{-weak, and} \\ &< Ax_m, x_m > /\|x_m\|^2 \to 0 \quad (\|.\| = \|.\|_{H^1(\Omega)}). \end{aligned} \tag{5}$$

We prove that

$$\limsup_{m\to\infty} \|x_m\|^{-1}[< Ax_m, x_m > +j(x_m)] > 0. \tag{6}$$

Suppose this is not true. Since $j \geq 0$, we must have

$$\lim \|x_m\|^{-1} < Ax_m, x_m >= 0. \tag{7}$$

But

$$\|x_m\|^{-1} < Ax_m, x_m > = \|x_m\|^{-1} \int_\Omega [\frac{|\nabla x_m|^{p+2}}{1+|\nabla x_m|^p} + F(|x_m|)x_m^2]$$

$$\begin{aligned} &= \int_\Omega \frac{|\nabla(x_m/\|x_m\|)|^{p+2}\|x_m\|^{p+1}}{1+|\nabla x_m|^p} + \|x_m\|^{-1} \int_\Omega F(|x_m|)|x_m|^2 \\ &\geq \|x_m\| \int_\Omega \frac{|\nabla w_m|^{p+2}}{\|x_m\|^{-p} + |\nabla w_m|^p} \\ &\geq 0 \quad (\text{here} \quad w_m = x_m/\|x_m\|). \end{aligned}$$

By (5) we have $\|x_m\| \geq 1$ for all m sufficiently large. Thus (7) implies that

$$\lim_{m\to\infty} \int_\Omega \frac{|\nabla w_m|^{p+2}}{1+|\nabla x_m|^p} = 0. \tag{8}$$

Putting $\Omega_m = \{x \in \Omega / \ |\nabla w_m(x)| \geq 1\}$, we have

$$\frac{|\nabla w_m(x)|^{p+2}}{1+|\nabla w_m(x)|^p} \geq \begin{cases} 1/2.|\nabla w_m(x)|^2 & \text{if} \quad x \in \Omega_m, \\ 1/2.|\nabla w_m(x)|^{p+2} & \text{if} \quad x \in \Omega \setminus \Omega_m. \end{cases}$$

This and (8) give

$$\lim_{m\to\infty} \int_\Omega |\nabla w_m|^2 = \lim_{m\to\infty} \int_{\Omega\setminus\Omega_m} |\nabla w_m|^{p+2} = 0.$$

But since $p > 0$, we have, by Hölder's inequality,

$$\int_{\Omega\setminus\Omega_m} |\nabla w_m|^2 \leq |\Omega|^{p/(p+2)}(\int_{\Omega\setminus\Omega_m} |\nabla w_m|^{p+2,2/(p+2)}.$$

Thus $\lim_{m\to\infty} \int_\Omega |\nabla w_m|^2 = 0$. On the other hand, from (5) and the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, one has $\lim_{m\to\infty} \int_\Omega w_m^2 = 0$. Hence $||w_m|| \to 0 \quad (m \to \infty)$ which contradicts $||w_m|| = 1, \forall m$. This contradiction proves (6) and the j-P-coerciveness of A.

We verify next that A is pseudo-monotone of $H^1(\Omega)$.

Indeed, one has $A = A_1 + A_2$ with

$$< A_1 u, v > = \int_\Omega \frac{|\nabla u|^p}{1+|\nabla u|^p} \nabla u \nabla v,$$
$$< A_2 u, v > = \int_\Omega F(|u|)uv, \quad u, v \in H^1(\Omega).$$

By direct computation, we have

$$(\frac{|\nabla u|^p \nabla u}{1+|\nabla u|^p} - \frac{|\nabla v|^p \nabla v}{1+|\nabla v|^p})(\nabla u - \nabla v) \geq$$
$$\geq \frac{|\nabla u|^{p+2}}{1+|\nabla u|^p} + \frac{|\nabla v|^{p+2}}{1+|\nabla v|^p} - (\frac{|\nabla u|^p}{1+|\nabla u|^p} + \frac{|\nabla v|^p}{1+|\nabla v|^p})|\nabla u||\nabla v|$$
$$= (1+|\nabla u|^p)^{-1}(1_|\nabla v|^p)^{-1}[(|\nabla u| - |\nabla v|)^2 |\nabla u|^p |\nabla v|^p +$$
$$+ (|\nabla u| - |\nabla v|)(|\nabla u|^{p+1} - |nv|^{p+1}) \geq 0.$$

Thus A_1 is monotone on $H^1(\Omega)$.

One can verify, in the usual way, that A_1, A_2 (and hence A) are continuous and bounded. Moreover, if $u_m \to u$ in $L^2(\Omega)$, then

$$A_2 u_m \to A_2 u \quad \text{in } [H^1(\Omega)]'. \tag{9}$$

Consider now a sequence $\{u_m\}$ in $H^1(\Omega)$ such that

$$u_m \rightharpoonup u \quad \text{weakly in } H^1(\Omega), \tag{10}$$

and

$$\limsup < Au_m - Au, u_m - u > \leq 0. \tag{11}$$

We show that

$$\liminf < Au_m, u_m - v > \geq < Au, u - v > \quad , \quad \forall v \in H^1(\Omega).$$

Indeed, by (10) we have $u_m \to u$ in $L^2(\Omega)$ and by (9)

$$A_2 u_m - A_2 u \to 0 \quad \text{in } [H^1(\Omega)]'.$$

Since $\{u_m\}$ is bounded, this gives

$$< A_2 u_m - A_2 u, u_m - u > \to 0 \quad (m \to \infty). \tag{13}$$

Then, by (11)

$$\limsup \; < A_1 u_m - A_1 u, u_m - u > \leq 0.$$

Since A_1 is pseudo-monotone (in fact, it is monotone), this implies that

$$\liminf \; < A_1 u_m, u_m - v > \geq < A_1 u, u - v > , \quad \forall v \in H^1(\Omega).$$

This and (13) give us (12). Thus A is pseudo-monotone on $H^1(\Omega)$.

We can check that A is in general not coercive or monotone on $H^1(\Omega)$.

1.1.3. Now, by the convexity of ψ we can show that the following limits exits:

$$\psi^{\pm} = \lim_{t \to +\infty} \psi(t)/t \in [-\infty, \infty]$$

Applying the abstract result in [ASV3] (Theorem 3, [ASV3]), we have the following

PROPOSITION. *If*

$$|\Omega|\psi^- < \int_\Omega f < |\Omega|\psi^+ \quad (|\Omega| \; : \textit{Lebesgue measure of } \Omega), \tag{14}$$

then (4) has a solution.

PROOF. Indeed, let $\{u_n\} \subset H^1(\Omega)$ be such that $||u_n|| \to \infty, u_n/||u_n|| \rightharpoonup w$ weekly in $H^1(\Omega)$ and

$$\limsup ||u_n||^{-1} < Au_n, u_n > \; j_\infty(w) \leq \int_\Omega f. \tag{15}$$

(see [ASV1] for a definition of j_∞). Since $j_\infty \geq 0$, this implies that

$$\lim ||u_n||^{-2} < Au_n, u_n > = 0.$$

Using argument similar to those in 1.1.2, we can conclude from this equality that

$$\lim_{n\to\infty} \int_\Omega |\nabla w_n|^2 = 0 \quad (w_n = u_n/||u_n||).$$

Since $w_n \rightharpoonup w$ weakly in $H^1(\Omega)$, we have

$$\int_\Omega |\nabla w|^2 \leq \liminf \int_\Omega |\nabla w_n|^2 = 0.$$

Hence $\nabla w = 0$ in Ω, and $w \in \mathbf{R}$. One has

$$j_\infty(w) = \lim_{t\to\infty} \int_\Omega \frac{\psi(tw)}{t} = \lim_{t\to\infty} |\Omega| \frac{\psi(tw)}{tw} t =$$

$$= \begin{cases} |\Omega|\psi^+ w & \text{if} \quad w > 0, \\ 0 & \text{if} \quad w = 0, \\ |\Omega|\psi^- w & \text{if} \quad w < 0. \end{cases}$$

By (15) we also have that

$$j_\infty(w) = \int_\Omega fw.$$

In view of (14) and (16), this holds if and only if $w = 0$. By the corollary of Theorem 3 [ASV3], (4) has a solution.

1.2. EXAMPLE 2.

Let Ω be as in Example 1.

1.2.1. Consider the following boundary value problem :

Find $u \not\equiv 0$ defined on Ω such that

$$\Delta(\frac{|\Delta u|^p}{1+|\Delta u|^p}\Delta u) + F(u) = f \quad \text{in} \quad H^2(\Omega), \tag{17}$$

$$\Delta u = 0 \quad \text{on} \quad \partial\Omega, \tag{18}$$

$$\frac{\partial}{\partial n}(\frac{|\Delta u|^p}{1+|\Delta u|^p}\Delta u) = 0 \quad \text{on} \quad \partial\Omega, \tag{19}$$

where $p > 0$ and $F(u) = \frac{d}{du}\phi(u)$, $\phi(u) = (\sum_{0\leq|\alpha|\leq 2} \int_\Omega |D^\alpha u|^2)^{1/2} = ||u||_{H^2(\Omega)}$.

Remark that because $H^2(\Omega)$ is a Hilbert space, ϕ is differentiable on $H^2(\Omega)\setminus\{0\}$ and that $< F(u), v >= ||u||^{-1}_{H^2(\Omega)}(u,v)_{H^2(\Omega)}$, where $< .,. >$ denotes the dual pairing between $H^2(\Omega)$ and $[H^2(\Omega)]'$.

It can be shown that the system (17)-(19) is equivalent to the following variational inequality

$$\begin{cases} < Au, v-u > \ +\phi(v)-\phi(u) \geq \int_\Omega f(v-u), \ \forall v \in H^2(\Omega), \\ u \in H^2(\Omega), \end{cases} \tag{20}$$

where $< Au, v >= \int_\Omega \frac{|\Delta u|^p}{1+|\Delta u|^p}\Delta u \Delta v$.

1.2.2. Note that ϕ is convex and continuous on $H^2(\Omega)$, and that $A : H^2(\Omega) \to [H^2(\Omega)]'$.

We prove that A is ϕ-P-coercive on $H^2(\Omega)$.

Indeed, let $P_0, P_1 : H^2(\Omega) \to \mathbf{R}$, $P_0 = P_1 = 0$. If $v \in H^2(\Omega), ||v||_{H^2(\Omega)} = 1$ and $\lambda \geq 1$, then

$$\lambda^{-1} < A(\lambda v), v > +P_0(v) + P_1(v) + \phi(v) \geq \phi(v) = ||v||_{H^2(\Omega)} = 1$$

By the remark following the definition of P-coerciveness in [ASV3], this proves that A is ϕ-P-coercive.

As in Example 1, we can verify that A is continuous, bounded, nonnegative and monotone on $H^2(\Omega)$.

We remark that although ϕ is semi-additive on $H^2(\Omega)$, A is not compact-coercive in the sense of [GT]. Indeed, we have

$$\begin{aligned} \mathcal{H} = \ker\ A &\equiv \{v \in H^2(\Omega) \ / \ < Av, v > = 0\} \\ &= \{v \in H^2(\Omega) \ / \ \Delta v = 0 \quad \text{in } \Omega\} \end{aligned}$$

is a closed, infinite dimensional subspace of $H^2(\Omega)$. Then we can choose a sequence $\{w_m\}$ in $\mathcal{H}$ such that $||w_m||_{H^2(\Omega)} = 1$, $\forall m$, and that $\{w_m\}$ does not contain any convergent subsequence. Put

$$v_m = m w_m, \ m = 1, 2, 3, \ldots$$

We have $||v_m||_{H^2(\Omega)} = m \to \infty$ $(m \to \infty)$. Since $\Delta v_m = m\Delta w_m = 0$ in Ω, one has, for $v_0 \in H^2(\Omega), m \in \mathbf{N}$,

$$< Av_m, v_m - v_0 > = \int_\Omega \frac{|\Delta v_m|^p}{1+|\Delta v_m|^p} \Delta v_m \Delta(v_m - v_0) = 0.$$

Then

$$||v_m||^{-1}_{H^2(\Omega)}[< Av_m, v_m - v_0 > + \phi(v_m)] = ||v_m||^{-1}_{H^2(\Omega)}\phi(v_m) = 1, \ \forall m.$$

But by the choice of $\{v_m\}$ and $\{w_m\}$, we see that

$$\{v_m/||v_m||_{H^2(\Omega)}\} = \{w_m\}$$

has no convergent subsequent in $H^2(\Omega)$. Hence A is not compact-coercive on $H^2(\Omega)$.

1.2.3. Let $\mathcal{H} = \{v \in H^2(\Omega) \ / \ \Delta v = 0 \ \text{ in } \ \Omega\}$ be the set of all harmonic function on Ω. Then a sufficient condition for the existence of a solution of (20) is that

$$\int_\Omega f.w < ||w||_{H^2(\Omega)}, \ \forall w \in \mathcal{H} \setminus \{0\}. \tag{21}$$

Indeed, suppose $\{u_n\}$ and w satisfy condition (18) in Theorem 3 [ASV3]. We have

$$0 \le \limsup ||u_n||^{-2}_{H^2(\Omega)} < Au_n, u_n > \le \lim ||u_n||^{-1}_{H^2(\Omega)} \int_\Omega fw = 0.$$

Then

$$\lim_{n\to\infty} \int_\Omega \frac{|\Delta w_n|^{p+2}}{||u_n||^{-p}_{H^2(\Omega)} + |\Delta w_n|^p} = 0.$$

By arguments similar to those following (8), we obtain, from this equality, that

$$\lim \int_\Omega |\Delta w_n|^2 = 0, \ (w_n = u_n/||u_n||_{H^2(\Omega)}).$$

Since $(u, v) \mapsto \int_\Omega \Delta u \Delta v$ is a nonnegative, continuous bilinear form on $H^2(\Omega)$, we have, from "$w_m \rightharpoonup w$ in $H^2(\Omega)$", that

$$\int_\Omega |\Delta w|^2 \le \liminf \int_\Omega |\Delta w_m|^2 = 0.$$

Hence $\Delta w = 0$ in Ω, i.e.,

$$w \in \mathcal{H}. \tag{22}$$

Since $\phi_\infty = \phi$, from (18) of [ASV3] (with $j = \phi$), one has

$$\phi(w) \leq \int_\Omega fw.$$

By (21), (22) this can happen only if $w = 0$. Hence our result follows from the corollary of Theorem 3 [ASV3].

REMARK 1. The boundary conditions in Examples 1 and 2 can be replaced by inhomogeneous or unilateral ones. These lead to variational inequalities similar to (4) and (20).

2. P-coercive variational inequalities in Banach spaces

There are many problems that can be appropriately formulated as variational inequalities only in Banach spaces. Some of these involve noncoercive nonlinear operators.

2.1. AN ABSTRACT RESULT.

2.1.1. Let $(V, ||.||)$ be a reflexive Banach space, V' its dual and let $< .,. >$ be the pairing between V and V'. We consider the following variational inequality

$$\begin{cases} < Au, v-u > \ +j(v) - j(u) \geq < f, v-u >, \ \forall v \in K, \\ u \in K. \end{cases} \tag{23}$$

Here, $f \in V'$, $A: V \to V'$ is bounded, hemi-continuous and pseudo-monotone, $j: V \to \mathbf{R} \cup \{\infty\}$ is a convex, lower semi continuous functional, $j(0) = 0, K = D(j) = \{v \in V / \ j(v) < \infty\}$ is closed and convex in V.

Let $||.||_0$ be a norm V, equivalent to $||.||$, such that V' with the dual norm $||.||_0^*$ corresponding to $||.||_0$ is strictly convex (these norms $||.||_0$ always exist by Theorem 2.6, [L]). Let $\Phi : [0, \infty] \to [0, \infty)$ be a continuous, strictly increasing function such that

$$\Phi(0) = 0 \quad \text{and} \quad \Phi(r) \to \infty \quad \text{as} \quad r \to \infty.$$

We denote by $J = J(\Phi, ||.||_0): V \to V'$ the duality mapping corresponding to Φ and $||.||_0$, i.e.,

$$< J(u), u > = ||J(u)||_0^* ||u||_0 \quad \text{and} \quad ||J(u)||_0^* = \Phi(||u||_0).$$

We know (Propositions 2.1 - 2.4, [L]) that J exists uniquely, is bounded, hemi-continuous, monotone and coercive on V. Moreover (cf. the proof of Proposition 2.4, [L]), J is continuous from the strong topology of V to the weak-star topology of V'.

The operator A is said to be j-P-coercive with respect to J if the following condition is fulfilled:

There exists $v_0 \in K$ such that for all sequences $\{v_n\} \subset K$ satisfying :

$$\begin{cases} (i) & ||v_n|| \to \infty, \\ (ii) & ||v_n||^{-2} < Av_n, v_n > \to 0, \\ (iii) & w_n = v_n/||v_n|| \rightharpoonup w \quad \text{weakly in } V \quad \text{and}, \\ & \qquad < J(w_n), w > = 0, \quad \forall n. \end{cases} \tag{24}$$

We always have

$$\limsup_{n\to\infty} ||v_n||^{-1}[< Av_n, v_n - v_0 > \; + j(v_n)] > 0.$$

REMARK 2.

(a) If V is a Hilbert space and $J = I$ is the identity mapping of V (I is the duality mapping corresponding to $||.||_0 = ||.||$ and $\Phi(r) = r$, $r \in [0, \infty)$), then the above definition reduces to that presented in [ASV3].

Indeed, if J is weakly continuous (i.e. J is continuous from the weak topology of V to the weak-star topology of V', this holds in the particular case $J = I$), then the condition (24 iii) is equivalent to:

$$w_n = v_n/||v_n|| \rightharpoonup 0 \quad \text{weakly in } V. \tag{24 iii'}$$

It is clear that (24 iii') $\Rightarrow$ (24 iii). Conversely, if (24 iii) holds, then by the weak continuity of J we have

$$J(w_n) \rightharpoonup J(w) \quad \text{weak}^* \text{ in } V'.$$

Thus $0 = < J(w), w > = \Phi||w||_0)||w||_0$, i.e., $w = 0$ and we obtain (24 iii').

The conditions (24 i,ii,iii') are exactly those in the definition in [ASV3] the P-coerciveness in Hilber spaces.

(b) Our definition of P-coerciveness strictly contains that of compact-coerciveness of Gastaldi and Tomarelli ([GT]).

In fact, suppose A is compact-coercive in the sense of [GT] and that $\{v_n\}$ satisfies (24). If (25) does not hold, i.e.,

$$\limsup ||v_n||^{-1}[< Av_n, v_n - v_0 > \ +j(v_n)] \leq 0.$$

Then there exists $C > 0$ such that

$$||v_n||^{-1}[< Av_n, v_n - v_0 > \ +j(v_n)] \leq C, \quad \forall n.$$

By the compact-coerciveness condition, there is a subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that the sequence $\{w_{n_k}\}$ strongly converges in V. Together with (24 iii), this implies that

$$w_{n_k} \to w \quad \text{(strongly) in } V.$$

Then $J(w_{n_k}) \rightharpoonup J(w)$ weak* in V', and $0 = < J(w_{n_k}), w > \to < J(w), w > = 0$. Hence $w = 0$, i.e., $w_{n_k} \to 0$ in V (strong). This contradicts the fact that $||w_{n_k}|| = 1, \forall k$ and proves (25).

We have shown that if A is compact-coercive then it is j-P-coercive with respect to any duality mapping J. Example 3 to be given latter provides an example of a P-coercive operator that is not compact-coercive in the sense of [GT].

2.1.2. As for the case of P-coercive variational inequalities in Hilbert spaces, we have the following sufficient condition for the existence of a solution of (23).

THEOREM 2.1. *Let A be nonnegative and j-P-coercive with respect to a duality mapping J. Suppose that the following condition holds:*

If $w \in rcK$ is such that there exists a sequence $\{u_n\} \subset K$ with $||u_n|| \to \infty$ and

$$u_n/||u_n|| \rightharpoonup w \quad \textit{weakly in } V, \textit{ and}$$

$$\limsup ||u_n||^{-1} < Au_n, u_n > +j_\infty(w) \leq < f, w >, \tag{26}$$

then

$$-w \in rcK, \ \max\{0, j_\infty(w)\} \leq - < f, v > \quad and \tag{27}$$

$$< Au_{n_k}, w > = 0 \quad for \ a \ subsequence \ \{u_{n_k}\} \subset \{u_n\}$$

Then (23) has a solution.

PROOF. The proof of this theorem is similar to that of Theorem 3 [ASV3]. Hence we only sketch a brief outline of it.

For $\varepsilon > 0$, consider the regularized variational inequality :

$$\begin{cases} < Au_\varepsilon, v - u_\varepsilon > +\varepsilon < J(u_\varepsilon), v - u_\varepsilon > +j(v) - j(u_\varepsilon) \geq \\ \geq \ f, v - u_\varepsilon > \quad , \quad \forall v \in K, \\ u_\varepsilon \in K. \end{cases} \tag{28}$$

Since A is pseudo-monotone and since J is monotone and hemi-continuous, $A + \varepsilon J$, by Remark 2.12, [L], is also pseudo-monotone. We have moreover

$$||u||_0^{-1}[< Au, u > +\varepsilon < J(u), u) > +j(u)] \geq$$
$$\geq \varepsilon\Phi(||u||_0) + ||u||_0^{-1}(< L, u > +\alpha), \forall u \in V$$

for some $L \in V', \alpha \in \mathbf{R}$. Since $\Phi(||u||_0) \to \infty$ as $||u||_0 \to \infty, A + \varepsilon J$ is coercive on V. Hence (28) always has a solution $u_\varepsilon \in K$.

We prove that $\{u_\varepsilon\}$ ($\varepsilon > 0$, small) is bounded in V. Suppose by contradiction that there is a sequence $\{\varepsilon_n\} \searrow 0$ such that $||u_n|| = ||u_{\varepsilon_n}|| \to \infty, \ n \to \infty$. By choosing a subsequence, we can assume furthermore that

$$w_n = u_n/||u_n|| \rightharpoonup w \quad \text{weakly in } V.$$

Substituting $v = 0$ into (28), remarking that $< J(u_\varepsilon), u_\varepsilon > \geq 0$ and dividing both sides of (28) (with $\varepsilon = \varepsilon_n$) by $||u_n||$, and then letting $n \to \infty$ in the inequality thus obtained, we see that w satisfies (26). By hypothesis, (27) holds for w. Now, substituting $v = u_{n_k} \pm \lambda w \in K \quad (\lambda > 0)$ into (28), we have

$$\varepsilon_{n_k} < J(u_{n_k}), w > +\lambda^{-1}j(u_{n_k} + \lambda w) - \lambda^{-1}j(u_{n_k}) \geq < f, w >,$$

and

$$-\varepsilon_{n_k} < J(u_{n_k}), w > \; + \lambda^{-1} j(u_{n_k} - \lambda w) - \lambda^{-1} j(u_{n_k}) \geq \; - < f, w > .$$

Letting $\lambda \to \infty$, we obtain

$$\varepsilon_{n_k} < J(u_{n_k}), w > \geq < f, z > \; - j_\infty(w) \geq \limsup \frac{< Au_{n_k, u_{n_k}} >}{||u_{n_k}||} \geq 0,$$

and

$$-\varepsilon_{n_k} < J(u_{n_k}), w > \geq - < f, w > \; - j_\infty(-w) \geq 0.$$

Then $< J(u_{n_k}), w) > = 0, \; \forall k$.

By direct computation, we have, for all k,

$$< J(w_{n_k}), w > = \Phi(||u_{n_k}||_0^{-1}[\Phi(1)]^{-1} < J(u_{n_k}), w > = 0.$$

On the other hand, as in the proof of Theorem 3 [ASV3], one has

$$\lim_{k \to \infty} ||u_{n_k}||^{-2} < Au_{n_k}, u_{n_k} > = 0.$$

Thus the sequence $\{u_{n_k}\}$ satisfies the conditions (24 i,ii,iii). By the j-P-coerciveness of A with respect to J, we can conclude that

$$\limsup_{k \to \infty} [||u_{n_k}||^{-1}[< Au_{n_k}, u_{n_k} - v_0 > \; + j(u_{n_k})] > 0. \tag{29}$$

Next, substituting $v = v_0 \in K$ into (28) (with $\varepsilon = \varepsilon_{n_k}$), dividing both sides of the inequality by $||u_{n_k}||$ and letting $k \to \infty$, we have

$$\limsup_{k \to \infty} ||u_{n_k}||^{-1}[< Au_{n_k}, u_{n_k} - v_0 > \; + j(u_{n_k})] \leq$$

$$\leq \lim_{k \to \infty} < f, -||u_{n_k}||^{-1} v_0 + w_{n_k} > = < f, w > \leq 0.$$

This contradicts (29) and proves the boundedness of $\{u_\varepsilon\}$ $(\varepsilon > 0$, small).

The remainder of the proof proceeds along similar lines as in the proof of Theorem 3 [ASV3].

From Theorem 2.1, we deduce the following

COROLLARY 2.2. *Let A be nonnegative and j-P-coercive with respect to J. Then one of the two following conditions is sufficient for the existence of a solution of (23) :*

(i) (26) does not hold for any $w \in rcK$.

(ii) $w = 0$ is the only element of rcK that satisfies (26).

We consider now some examples of P-coercive variational inequalities in Banach spaces.

2.2. EXAMPLE 3.

2.2.1. Let Ω be a bounded domain in $\mathbf{R}^n$, let $1 < \alpha < \infty$ and let f be defined on Ω. We consider the Dirichlet problem for the following quasilinear equation

$$-div(a(u)\nabla u) = f \quad \text{in } \Omega \tag{30}$$

with

$$u = 0 \quad \text{on } \partial\Omega, \tag{31}$$

where

$$a(u) = [1 + (\int_\Omega |\nabla u|^\alpha)^{1-1/\alpha}]^{-1} |\nabla u|^{\alpha-2}.$$

Using Green's formular, we can formulate (30)-(31), in the usual way, as the following variational inequality :

$$\begin{cases} < Au, v > = \int_\Omega fv, \ \forall v \in W_0^{1,\alpha}(\Omega), \\ u \in W_0^{1,\alpha}(\Omega). \end{cases} \tag{32}$$

Here $f \in L^{\alpha'}(\Omega)$ (α' id the conjugate exponent of α) and A is defined by

$$< Au, v > = \int_\Omega a(u)\nabla u \nabla v, \quad u, v \in W_0^{1,\alpha}(\Omega).$$

We consider on $W_0^{1,\alpha}(\Omega)$ the usual norm :

$$||u|| = ||u||_{W_0^{1,\alpha}(\Omega)} = (\int_\Omega |\nabla u|^\alpha)^{1/\alpha}.$$

By direct verification, we can see that

$$A : W_0^{1,\alpha}(\Omega) \to W^{-1,\alpha'}(\Omega) \quad (\equiv [W_0^{1,\alpha}(\Omega)]')$$

is continuous and bounded.

A is moreover monotone on $W_0^{1,\alpha}(\Omega)$. Indeed, we have, for $u, v \in W_0^{1,\alpha}(\Omega)$,

$$< Au - Av, u - v > = \int_\Omega \left(\frac{|\nabla u|^{\alpha-2}\nabla u}{1+||u||^{\alpha-1}} - \frac{|\nabla v|^{\alpha-2}\nabla v}{1+||v||^{\alpha-1}}\right)(\nabla u - \nabla v)$$

$$\geq \frac{||u||^\alpha}{1+||u||^{\alpha-1}} + \frac{||v||^\alpha}{1+||v||^{\alpha-1}} - \frac{||u||^{\alpha-1}||v||}{1+||u||^{\alpha-1}} - \frac{||u||\ ||v||^{\alpha-1}}{1+||v||^{\alpha-1}}$$
$$= \frac{(||u||^{\alpha-1} - ||v||^{\alpha-1})(||u|| - ||v||)}{(1+||u||^{\alpha-1})(1+||v||^{\alpha-1})} \geq 0.$$

2.2.2. We prove that A is P-coercive with respect to any duality mapping J, but is not compact-coercive in the sense of [GT].

For $v \in W_0^{1,\alpha}(\Omega)$, we first have that

$$< Av, v > = (1 + ||v||^{\alpha-1})^{-1} \int_\Omega |\nabla v|^\alpha \qquad (33)$$
$$= (1 + ||v||^{\alpha-1})^{-1} ||v||^\alpha.$$

Suppose $\{v_n\} \subset W_0^{1,\alpha}(\Omega)$ satisfies (24). From (33) and the fact that $||v_n|| \to \infty$ as $n \to \infty$, we have, for $v_0 = 0$,

$$\limsup_{n\to\infty} ||v_n||^{-1} < Av_n, v_n > = \lim_{n\to\infty} (1 + ||v_n||^{\alpha-1})||v_n||^{\alpha-1} = 1 > 0.$$

Hence, we have (25) and thus the P-coerciveness of A on $W_0^{1,\alpha}(\Omega)$.

Remark now that since $W_0^{1,\alpha}(\Omega)$ is infinite dimensional, there exists a sequence $\{w_n\}$ in $W_0^{1,\alpha}(\Omega)$ with $||w_n|| = 1, \forall n$, such that $\{w_n\}$ does not contain any convergent subsequence. Putting $v_n = nw_n, n = 1, 2, 3, \ldots$, one has $||w_n|| \to \infty$ and for all $v_0 \in W_0^{1,\alpha}(\Omega)$, by Hölder inequality,

$$\lim |v_n||^{-1} < Av_n, v_n - v_0 > \leq \lim(1 + ||v_n||^{\alpha-1})||v_n||^{\alpha-1} +$$
$$+ \lim(1 + ||v_n||^{\alpha-1})||v_n||^{\alpha-2}||v_0|| = 1 + 0 = 1.$$

But $\{||v_n||^{-1}v_n = v_n\}$ has no convergent subsequence in $W_0^{1,\alpha}(\Omega)$. Hence A is not compact-coercive.

2.2.3. Applying Theorem 2.1, we see that a sufficient condition for the existence of a solution of (32) is that

$$||f||_* < 1. \tag{34}$$

In fact, if w satisfies (26), then

$$||w|| \leq \liminf || \, ||u_n||^{-1} u_n|| = 1,$$

and by (33) we have

$$1 = \lim ||u_n||^{-1} < Au_n, u_n > \leq < f, w > \leq ||f||_* ||w|| \leq ||f||_*$$

This contradicts (34) and proves that there is no $w \in W_0^{1,\alpha}(\Omega)$ that satisfies (26). Our existence result follows from Corollary 2.2.

EXAMPLE 4.

2.3.1. Let $1 < \alpha < \infty$ and let F be increasing on $\mathbf{R}$, $F(0) = 0$.

We consider the following Neumann-type quasilinear problem

$$\begin{cases} -div[(1 + |\nabla u|^2)^{\alpha/2-1} \nabla u] + F(u) = f & \text{in } \Omega, \\ (1 + |\nabla u|^2)^{\alpha/2-1} \partial u / \partial n = g & \text{on } \partial\Omega, \end{cases} \tag{35}$$

with f and g given respectively on Ω and $\partial\Omega$.

We formulate (35) as a variational inequality.

As in Example 1, we put

$$\psi(t) = \int_0^t F(tau), \; t \in \mathbf{R}, \quad \text{and}$$

$$j(v) = \int_\Omega \psi(v(x)) \quad (v \text{ defined on } \Omega).$$

Then ψ is convex, nonnegative and continuous on $\mathbf{R}$. Moreover $\psi' = F$ if F is continuous. We then have, for u, v defined on Ω,

$$\int_\Omega F(u)(v - u) \leq \int_\Omega [\psi(v) - \psi(u)] = j(v) - j(u).$$

Now, let v be defined on Ω by (35),

$$-\int_\Omega div \; [(1 + |\nabla u|^2)^{\alpha/2-1} \nabla u](v - u) + \int_\Omega F(u)(v - u) = \int_\Omega f(v - u).$$

But

$$\int_{\Omega} div\ [(1+|\nabla u|^2)^{\alpha/2-1}\nabla u](v-u)$$

$$= \int_{\partial\Omega}(1+|\nabla u|^2)^{\alpha/2-1}(v-u)\partial u/\partial n - \int_{\Omega}(1+|\nabla u|^2)^{\alpha/2-1}\nabla u\nabla(v-u)$$

$$= \int_{\partial\Omega} g(v-u) - \int_{\Omega}(1+|\nabla u|^2)^{\alpha/2-1}\nabla u\nabla(v-u).$$

These imply that

$$< Au, v-u > \ +j(v)-j(u) \geq < h, v-u >, \tag{36}$$

with $< Au, v > = \int_{\Omega}(1+|\nabla u|^2)^{\alpha/2-1}\nabla u\nabla v$, and $< h, v > = \int_{\Omega} fv + \int_{\partial\Omega} gv$.

Suppose conversely that (36) holds for all v defined on Ω. Let w be defined on Ω. Substituting $v = u+\theta w, 0<\theta<1$, into (36), we have

$$< Au, w > \ + \int_{\Omega}\frac{1}{\theta}[\psi(u+\theta w)-\psi(u)] \geq < h, w > .$$

Letting $\theta \to 0$ and considering $-w$ instead of w, we arrive at

$$< Au, v > \ + \int_{\Omega} F(u)w = < h, w > . \tag{37}$$

With $w \in C_0^{\infty}(\Omega)$, applying Green's formula, we obtain from (37) $-div\ [(1+|\nabla u|^2)^{\alpha/2-1}\nabla u] + F(u) = f$ in Ω (in the distributional sense), i.e., the first equation in (35) holds.

Now, applying Green's formula again, we have, by this equation,

$$< Au, w > \ + \int_{\partial\Omega}(1+|\nabla u|^2)^{\alpha/2-1}w\partial u/\partial n + \int_{\Omega} F(u)w = \int_{\Omega} fw.$$

Comparing with (37), this gives

$$\int_{\partial\Omega}(1+|\nabla u|^2)^{\alpha/2-1}(\partial u/\partial n)w = \int_{\partial\Omega} gw,\ \forall w.$$

Hence, we have the boundary condition in (35).

These arguments show that (35) is (formally) equivalent to

$$\begin{cases} < Au, v-v > \ +j(v)-j(v) \geq < h, v-u > ,\ \forall v \in W^{1,\alpha}(\Omega), \\ u \in W^{1,\alpha}(\Omega). \end{cases} \tag{38}$$

2.3.2. Note that $A: W^{1,\alpha}(\Omega) \to [W^{1,\alpha}(\Omega)]'$ is continuous, bounded, and nonnegative and, moreover, $h \in [W^{1,\alpha}(\Omega)]'$ if $f \in L^{\alpha'}(\Omega)$ and $g \in L^{\alpha'}(\partial\Omega)$.

On the other hand, since the mapping

$$\xi \mapsto (1+\xi^2)^{\alpha/2-1}\xi, \ \xi \in \mathbf{R}^n,$$

is the gradient of the convex functional

$$\xi \mapsto \alpha^{-1}(1+\xi^2)^{\alpha/2}\xi, \ \xi \in \mathbf{R}^n,$$

we see directly that A is monotone on $W^{1,\alpha}(\Omega)$.

We check the j-P-coerciveness of A.

Let $\{v_n\} \subset W^{1,\alpha}(\Omega)$ satisfy (24). Suppose by contradiction that (25) is false. Then since $A, j \geq 0$, we have

$$\lim_{n\to\infty} ||v_n||^{-1} < Av_n, v_n > = 0.$$

(In what follows, $||.||$ denotes the usual norm of $W^{1,\alpha}(\Omega)$). By (24 i), this implies

$$\lim ||v_n||^{-\alpha} < Av_n, v_n > = 0. \tag{39}$$

Consider the case $1 < \alpha \leq 2$. We have

$$\begin{aligned}
||v_n||^{-\alpha} < Av_n, v_n > &= \int_\Omega \Big(\frac{|\nabla v_n|^2}{1+|\nabla v_n|^2}\Big)^{1-\alpha/2}|\nabla w_n|^\alpha \\
&= \int_\Omega \Big(\frac{|\nabla w_n|^2}{||v_n||^{-2}+|\nabla w_n|^2}\Big)|\nabla w_n|^\alpha \\
&\geq \int_\Omega \Big(\frac{|\nabla w_n|^2}{1+|\nabla w_n|^2}\Big)^{1-\alpha/2}|\nabla w_n|^\alpha \\
&\geq \int_{\Omega_n} ((1/2)|\nabla w_n|^2)^{1-\alpha/2}|\nabla w_n|^\alpha + \int_{\Omega\backslash\Omega_n} (1/2)^{1-\alpha/2}|\nabla w_n|^\alpha \\
&= (1/2)^{1-\alpha/2}\Big(\int_{\Omega_n} |\nabla w_n|^2 + \int_{\Omega\backslash\Omega_n} |\nabla w_n|^\alpha\Big),
\end{aligned}$$

(here $\Omega_n = \{x \in \Omega / |\nabla w_n(x)| \leq 1\}$).

From (39) and Hölder's inequality (remarking that $\alpha \leq 2$) we obtain

$$\lim_{n\to\infty} \int_{\Omega_n} |\nabla w_n|^\alpha = \lim_{n\to\infty} \int_{\Omega\backslash\Omega_n} |\nabla w_n|^\alpha = 0.$$

Thus

$$\lim_{n\to\infty} \int_\Omega |\nabla w_n|^\alpha = 0. \tag{40}$$

Since $w_n \rightharpoonup w$ in $W^{1,\alpha}(\Omega)$-weak, we have

$$w_n \to w \quad \text{(strongly) in} \quad L^\alpha(\Omega). \tag{41}$$

On the other hand, since the functional

$$v \mapsto (\int_\Omega |\nabla v|^\alpha)^{1/\alpha}$$

is convex and continuous on $W^{1,\alpha}(\Omega)$, we have, by a well-known result,

$$(\int_\Omega |\nabla w_n|^\alpha)^{1/\alpha} \leq \liminf (\int_\Omega |\nabla w_n|^\alpha)^{1/\alpha} = 0.$$

Hence $\nabla w = 0$ a.e. in Ω, i.e., $w \in \mathbb{R}$, and

$$\lim_{n\to\infty} \int_\Omega |\nabla (w_n - w)|^\alpha = 0.$$

This and (41) imply that $w_n \to w$ strongly in $W^{1,\alpha}(\Omega)$ and $J(w_n) \rightharpoonup J(w)$ in $[W^{1,\alpha}(\Omega)]'$ weak*. Then $< J(w), w > = \lim_{n\to\infty} < J(w_n), w > = 0$, i.e., $w = 0$. Hence $w_n \to 0$ in $W^{1,\alpha}(\Omega)$, which contradicts the fact that $||w_n|| = 1, \ \forall n$. This contradiction proves (25) in the case $1 < \alpha \leq 2$.

Now, if $\alpha > 2$ then

$$||v_n||^{-\alpha} < Av_n, v_n > \ ||v_n||^{-\alpha} \int_\Omega |\nabla v_n|^{-\alpha-2} |\nabla v_n|^2 = ||v_n||^{-\alpha} \int_\Omega |\nabla v_n|^\alpha$$
$$= \int_\Omega |\nabla w_n|^\alpha.$$

Hence (14) also follows from (13) in this case. The remainder of the proof proceeds along similar lines as for the case $1 < \alpha \leq 2$.

We have proved that A is $j - P$-coercive with respect to any duality mapping J.

Finally, $j : W^{1,\alpha}(\Omega) \to \mathbb{R} \cup \{\infty\}$ is convex (by the convexity of ψ) and lower semi continuous (by the continuity of ψ). But j is not semi-additive in general.

2.3.3. In view of Theorem 2.1, we see that if

$$|\Omega|\psi^- < \int_\Omega f + \int_{\partial\Omega} g < |\Omega|\psi^+,$$

then (38) has a solution in $W^{1,\alpha}(\Omega)$.

The proof of this proposition, which is similar to those of the previous examples, is omitted.

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