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ON A GENERALIZATION OF EXCELLENT EXTENSIONS

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Abstract. We introduce almost excellent extensions of rings. If $S \geq R$ is such an extension, we study (1) the relation between properties of an S-module M_S and the R-module M_R; and (2) the relation between properties of the ring S and the ring R.

Let $S \geq R$ be a unitary ring extension. Then $S \geq R$ is a *finite normalizing extension* [4] if there is a finite subset $\{s_1, s_2, ..., s_n\} \subseteq S$ such that $S = \sum_{i=1}^{n} s_i R$ and $s_i R = R s_i$ for all $i = 1, ..., n$. We consider the following conditions:

(1) S is *right R-projective* [9]: that is, if M_S is a module and N_S is a submodule, then $N_R | M_R$ implies $N_S | M_S$, where $N|M$ means N is a summand of M; and

(2) S is a *free normalizing extension* of R with a basis that includes 1: that is, $S = \sum_{i=1}^{n} s_i R \geq R$ (where $s_1 = 1$) is a finite normalizing extension and S is free with basis $\{s_1, s_2, ..., s_n\}$ as both a right and left R-module.

The ring S is called an *excellent extension* of R in case the conditions (1) and (2) are satisfied. Excellent extensions were introduced by Passman [9], named by Bonami [3], and recently studied in [8], [13] and [6]. Examples include finite matrix rings [9], and crossed product $R * G$ where G is a finite group with $|G|^{-1} \in R$ [10]. We can weaken Condition (2) as follows:

(3) $S = \sum_{i=1}^{n} s_i R \geq R$ is a finite normalizing extension such that S_R is a projective R-modules. In this case, S_R is a generator by [5, Proposition 1.3].

We call the ring extension $S \geq R$ an *almost excellent extension* in case the conditions (1) and (3) are satisfied. Now we give two examples to illustrate our non-trivial generalization.

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EXAMPLE 1. If $S = \mathbf{Z}_6 \times \mathbf{Z}_2$ and $R = \{n(1,1) \in S \mid n \in \mathbf{Z}\}$, then $S \geq R$ is an almost excellent extension that is not an excellent extension.

EXAMPLE 2. Let $S \geq R$ be an excellent extension. If S has two ideals I and K such that $R \cap I = 0$ and $S = I \oplus K$, then the canonical embedding $R \hookrightarrow S/I$ is an almost excellent extension. If K_R is not a free R-module, then this almost excellent extension is not an excellent extension.

In this paper we use different proofs to generalized results of [8], [13], and [6] from excellent extensions to almost excellent extensions. Let $S \geq R$ be an almost excellent extension. We show that if M_S is a right S-module, then M_S is an injective (projective, PS-) module if and only if M_R is an injective (projective, PS-) module; and the equalities $\mathrm{Soc}(M_S) = \mathrm{Soc}(M_R)$, $Z(M_S) = Z(M_R)$ and $\mathrm{Rad}(M_S) = \mathrm{Rad}(M_R)$ hold. We also prove that if either ring is (i) right nonsingular, or (ii) right PS, or (iii) right PF, or (vi) PF, or (v) QF, or (vi) semiprimitive, of (vii) semisimple, or (viii) semilocal, or (ix) right V-ring, or (x) von Nemann regular, then so is the other.

Throughout this paper (except in Lemma 6), $S = \sum_{i=1}^{n} s_i R \geq R$ is an almost excellent extension. Then there is a positive integer t such that $R_R | S_R^t$, and $S_R | R_R^t$. If M_S is an S-module and $N_R \leq M_R$, we let $N s_i^{-1} = \{m \in M \mid m s_i \in N\}$. Then $b(N) = \cap_{i=1}^{n} N s_i^{-1}$ is the largest S-submodule of M contained in N. Recall that a module is a *PS-module* [7] if it has a projective socle, and a ring is a *right PS-ring* if it is a PS-module when considered as a right module over itself. The singular submodule of a module M is denoted by $Z(M)$, and M is called *nonsingular* if $Z(M) = 0$. A ring is a *right nonsingular ring* if it is a nonsingular module when considered as a right module over itself. For other terminologies and notations not defined here, we refer to the textbook of Anderson and Fuller [1].

THEOREM 1. *If M_S is a right S-module, then*

(1) M_S is injective if and only if M_R is injective.

(2) M_S is projective if and only if M_R is projective.

(3) If N_S is a submodule of M_S, then N_S is essential in M_S if and only if N_R is essential in M_R.

(4) $Soc(M_S) = Soc(M_R)$. In particular, M_S is semisimple if and only if M_R is semisimple.

(5) $Z(M_S) = Z(M_R)$. Consequently, S is right nonsingular if and only if R is right nonsingular.

(6) M_S is PS-module if and only if M_R is a PS-module. In particular, S is a right PS-ring if and only if R is a right PS-ring.

(7) $Rad(M_S) = Rad(M_R)$. Consequently, $J(S) = SJ(R)$.

PROOF. (1) ($\Rightarrow$) Let

$$0 \to X_R \to Y_R$$

be exact. Since ${}_RS$ is projective,

$$0 \to (X \otimes_R S)_S \to (Y \otimes_R S)_S$$

is exact. Hence

$$\mathrm{Hom}_S((Y \otimes_R S)_S, M_S) \to \mathrm{Hom}_S((X \otimes_R S)_S, M_S) \to 0$$

is exact. By Adjoint Isomorphism,

$$\mathrm{Hom}_R(Y, \mathrm{Hom}_S(S, M)) \to \mathrm{Hom}_R(X, \mathrm{Hom}_S(S, M)) \to 0$$

is exact. Hence

$$\mathrm{Hom}_R(Y, M) \to \mathrm{Hom}_R(X, M) \to 0$$

is exact and M_R is injective.

(1) ($\Leftarrow$) Suppose that $M_S \leq N_S$. We will show that $M_S | N_S$. Since $M_R \leq N_R$ and M_R is injective, $M_R | N_R$. Now S is right R-projective, so $M_S | N_S$ and then M_S is injective.

(2) ($\Rightarrow$) There is an S-module N_S such that $M_S \oplus N_S \cong S_S^{(A)}$, where A is a set and $S_S^{(A)}$ is a direct sum of $|A|$ copies of S_S. As a right R-module, $S^{(A)}$ is projective, hence M_R is projective.

(2) ($\Leftarrow$) Let F_S be a free S-module with an epimorphism $f : F_S \to M_S$. Let $K = \mathrm{Ker}(f)$. We have an exact sequence of right S-modules

$$0 \to K \to F \to M \to 0.$$

Since M_R is projective, we have $K_R|F_R$. Hence $K_S|F_S$. Therefore $M_S \cong F/K$ is projective.

(3) One direction is obvious. For the other, let N_S be an essential submodule of M_S and let $T_R \leq M_R$ be maximal with respect to having 0 intersection with N_R. Then $N + T = N \oplus T$ is essential in M_R. By [2, Lemma 1.2], the right S-module $b(N+T)$ is also essential in M_R. Now

$$N \subseteq b(N+T) \subseteq N+T = N \oplus T,$$

hence

$$b(N+T) = N \oplus (T \cap b(N+T)).$$

But S is right R-projective. We have $N_S|b(N+T)$. Since N_S is essential in M_S, we have $N = b(N+T)$, which is essential in M_R.

(4) By [4, Theorem 4], each simple right S-module is a semisimple R-module. Hence $\mathrm{Soc}(M_S) \subseteq \mathrm{Soc}(M_R)$. From (3) we have the converse inclusion because the socle is the intersection of all essential submodules.

(5) The equality $Z(M_S) = Z(M_R)$ follows from the proof of [8, Lemma 2.1] by using our Theorem 1(4) instead of [8, Proposition 1.1]. Hence S_S is nonsingular if and only if S_R is nonsingular. Since $R_R|S_R^t$ and $S_R|R_R^t$, we see that S_R is nonsingular if and only if R_R is nonsingular.

(6) It follows from (4) and (2) that M_S is a PS-module if and only if M_R is a PS-module. The second assertion follows since $R_R|S_R^t$ and $S_R|R_R^t$.

(7) If N_S is a maximal submodule of M_S then M/N is a semisimple R-module by [4, Theorem 4]. Hence $\mathrm{Rad}(M_R) \subseteq N$ and so $\mathrm{Rad}(M_R) \subseteq \mathrm{Rad}(M_S)$. To show the converse inclusion, we let N_R be a maximal submodule of M_R. By [2, Lemma 1.1], the right R-module M/Ns_i^{-1} is either simple or zero. Now $b(N) = \cap_{i=1}^n Ns_i^{-1}$ is the largest S-submodule of M_S contained in N and there is a monomorphism of R-modules $M/b(N) \to \oplus_{i=1}^n M/Ns_i^{-1}$. Hence $M/b(N)$ is a semisimple R-module, and $M/b(N)$ is a semisimple S-module by (4). Then $\mathrm{Rad}(M_S) \subseteq b(N) \subseteq N$ and so $\mathrm{Rad}(M_S) \subseteq \mathrm{Rad}(M_R)$. We have proved the equality $\mathrm{Rad}(M_S) = \mathrm{Rad}(M_R)$. Finally, we have $J(S) = \mathrm{Rad}(S_S) = \mathrm{Rad}(S_R) = SJ(R)$, where the last equality holds since S_R is projective.

Our Theorem 1 generalizes [6, Lemma 1.4], [8, Proposition 1.1 and Corollary 2.3], and [13, Theorem].

THEOREM 2. *(1) S_S is injective (finitely cogenerated) if and only if R_R is injective (finitely cogenerated). Consequently. S is right PF if and only if R is right PF.*

(2) S is PF if and only if R is PF.

(3) S if QF if and only if R is QF.

PROOF. (1) ($\Rightarrow$) Since S_S is injective (finitely cogenerated), S_R is injective (finitely cogenerated) by Theorem 1 (1) [5, Proposition 2.5 (2)]. Now $R_R|S_R^t$, hence R_R is injective (finitely cogenerated).

(1) ($\Leftarrow$) Since R_R is injective (finitely cogenerated) and $S_R|R_R^t$, S_R is injective (finitely cogenerated) by Theorem 1 (1).

(2) ($\Rightarrow$) Since S is PF-ring, i.e., ${}_SS_S$ defines a Morita duality (see [14] for an introduction of Morita duality), R has a right Morita duality by [5, Theorem 2.6 (2)], and so R_R is linearly compact. By (1) R_R is finitely cogenerated injective cogenerator. Therefore ${}_RR_R$ defines a Morita duality.

(2) ($\Leftarrow$) Since ${}_RR_R$ induces a Morita duality, R_R is linearly compact. Hence S_R is linearly compact because $S_R|R_R^t$. Therefore S_S is linearly compact. By (1) S_S is a finitely cogenerated injective cogenerator, and so ${}_SS_S$ defines a Morita duality.

(3) Since $S_R|R_R^t$, R_R is artinian if and only if S_R is artinian. By [4, Theorem 4] or [11, Proposition 5], S_R is artinian if and only if S_S is artinian. Now (3) follows from (2).

Recall that a ring is *semiprimitive* if its (Jacobson) radical is zero, and it is *semilocal* if it is semisimple modulo its radical. Our Theorem 2 (3) and Theorem 3 (2) are generalizations of [3, Theorem 2 (1)(2)].

THEOREM 3. *(1) S is semiprimitive if and only if R is semiprimitive.*

(2) S is semisimple if and only if R is semisimple.

(3) S is semilocal if and only if R is semilocal.

PROOF. (1) This follows from Theorem 1 (7).

(2) ($\Rightarrow$) If S is a semisimple ring, then S_R is a semisimple R-module by Theorem 1 (4). Hence R_R is semisimple.

(2) ($\Leftarrow$) Let M_S be an S-module. Since R is semisimple, M_R is injective. By Theorem 1 (1), M_S is injective, and so S is semisimple.

(3) By Theorem 1 (7), we have $J(S) = SJ(R)$. Hence we have a ring monomorphism $f : R/J(R) \to S/J(S)$, via $r + J(R) \mapsto r + J(S)$. The ring extension $S/J(S) \geq \mathrm{Im}(f)$ ($\cong R/J(R)$) is also an almost excellent extension. Now the result follows from (2).

A ring is a *right V-ring* if each simple right module is injective. By a different approach, we are able to generalize [8, Proposition 3.4] or [6, Theorem 1.3] as follows.

THEOREM 4. *S is a right V-ring if and only if R is a right V-ring.*

PROOF. ($\Leftarrow$). Let T_S be simple. Then T_R is semisimple of finite length [4, Theorem 4]. Hence T_R is injective and then T_S is injective by Theorem 1 (1).

($\Rightarrow$). Let I be a maximal right ideal of R. We will show that R/I is an injective R-module. Since $Rs_i = s_iR$ and I is maximal right ideal of R, we have that the right R-module $(Rs_i/Is_i)_R$ is either simple or zero. It follows that the right R-module $T_i = (\sum_{j\neq i} Is_j + Rs_i)/IS$ is either simple or zero, hence $S/IS = \sum_{i=1}^{n} T_i$ is a semisimple R-module of finite length. By Theorem 1 (4), S/SI is a semisimple S-module of finite length. Hence S/IS is an injective S-module. By Theorem 1 (1), S/IS is a semisimple injective R-module. Since I is a maximal right ideal of R, we know that IS is a proper right ideal of S by [11, Corollary 3 (i)]. Hence $R \cap IS = I$, and we have a monomorphism of R-modules, $R/I \to S/IS$, via $r + I \mapsto r + IS$. So R/I is an injective R-module.

A ring R is called a (von Neumann) *regular* ring if $r \in rRr$ for each $r \in R$. It is well-known that a ring is regular if and only if all left (or right) modules are flat. The next result is a generalization of [8, Corollary 3.3].

THEOREM 5. *S is a regular ring if and only if R is a regular ring.*

PROOF. ($\Leftarrow$) Since R_R^t is a regular module (see [17]) and $S_R | R_R^t$, S_R is a regular module. If $s \in S$, then $sS_R = \sum_{i=1}^n ss_i R$ is a finitely generated submodule of S_R. Hence $sS_R | S_R$ and then $sS_S | S_S$, i.e., S is a regular ring.

($\Rightarrow$) Let M_R be an R-module. Since S is regular, $M \otimes_R S$ is a flat right S-module. Hence M_R is flat by [12, Proposition 2.1]. Thus R is regular.

Recall that a ring R is a *right SF-ring* [6], [15], [16] if each simple right R-module is flat. A regular ring is a right (and left) SF-ring, but it is an open question that whether or not a right (or two-sided) SF-ring is a regular ring. Partial answers have been obtained in [15], [16].

LEMMA 6. *Let $S \geq$ be a unitary ring extension such that S_R is a flat module. If M_S is flat, then M_R is flat.*

PROOF. Let F_S be a free S-module with an epimorphism $f : F_S \to M_S$. Let $K = \mathrm{Ker}(f)$. We have an exact sequence of S-modules

$$0 \to K \to F \to M \to 0.$$

This is also an exact sequence of R-modules and F_R is a flat module. If I is a left ideal of R then SI is a left ideal of S. Since M_S is flat, by [1, Lemma 19.18] we have $K \cap F(SI) = K(SI)$, i.e., $K \cap FI = KI$. Hence M_R is flat by [1, Lemma 19.18] again.

The following result generalizes [6, Theorem 1.2], and our proof given here is also relatively simple.

PROPOSITION 7. *If S is a right SF-ring, then so is R.*

PROOF. If I is a maximal right ideal of R, then S/SI is a semisimple S-module of finite length by the proof of Theorem 4 ($\Rightarrow$). Hence S/IS is a semisimple flat S-module which is a semisimple flat R-module by Theorem 1 (4) and the above lemma. Now R/I is embedded in S/IS, so R/I is a flat R-module.

We are unable to settle the converse of Proposition 7. A counterexample would produce a counterexample to the open question we mentioned preceding Lemma 6.

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