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SOME RESULTS ON QUASI-CONTINUOUS MODULES

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Abstract. In [5] Mohamed and Müller introduced and gave some characterizations of quasi-continuous modules. Here we characterize these modules by extending property of uniform submodules. The following theorem is proven: Let $M = \oplus_{i \in I} M_i$ such that: (i) all M_i are uniform; (ii) this decomposition of M complements uniform direct summands; (iii) for all $i, j \in I, i \neq j$, M_i can not be proper embedded in M_j; and (iv) M has $(1 - C_1)$. Then M is a quasi-continuous module. As an application we show that, a ring R is QF iff R is semiperfect right continuous and every projective right R-module has $(1 - C_1)$.

1. Definitions and notations

Throughout this note all rings R are associative rings with identity and all modules are unitary right R-modules.

We consider the following conditions on a module M.

(C_1) Every submodule of M is essential in a direct summand of M.

(C_2) Every submodule isomorphic to a direct summand of M is itself a direct summand of M.

(C_3) If M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = 0$ then $M_1 \oplus M_2$ is a direct summand of M.

$(1 - C_1)$ Every uniform submodule of M is essential in a direct summand of M.

A module M is called continuous if it has (C_1) and (C_2), M is called quasi-continuous if it has (C_1) and (C_3). A module M is said to be an extending module if it satisfies condition (C_1), and M is said to have extending property of uniform submodules if it satisfies condition $(1 - C_1)$.

A submodule N of a module M is closed in M if it has no proper essential extensions in M.

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A ring R is called quasi-Frobenius (briefly, QF), if R is right artinian and right self-injective. It is known that a ring R is QF iff every projective R-module is injective, iff every injective R-module is projective (see [3, Theorem 24.20]).

A decomposition $M = \oplus_{i\in I} M_i$ is said to *complement uniform direct summands* if for every uniform direct summand U of M, there exists a subset J of I such that $M = U \oplus (\oplus_{j\in J} M_j)$.

For a submodule X of M, $X \subseteq^e M$ means that X is an essential submodule of M.

If $M = \oplus_{i\in I} M_i$ is a direct sum of modules M_i and J is a subset of I, then for convenience, we put $M(J) = \oplus_{j\in J} M_j$.

2. The results

LEMMA 1. *Let $M = \oplus_{i\in I} M_i$ be a decomposition that complements uniform direct summands. Then decomposition $M(J) = \oplus_{j\in J} M_j$ complements uniform direct summands for every subset J of I.*

PROOF. Let U be a uniform direct summand of $M(J)$. Then U is a uniform direct summand of M. By hypothesis, there exists a subset K of I such that $M = U \oplus M(K)$. Since U is a submodule of $M(J)$ we have by modularity:

$$M(J) = U \oplus X$$

where $X = M(K) \cap M(J)$. It is easy to see that $X = M(T)$ where $T = K \cap J$. Thus $M(J) = \oplus_{j\in J} M_j$ complements uniform direct summands.

THEOREM 2. *Let $M = \oplus_{i\in I} M_i$ be a decomposition such that: (i) all M_i are uniform; (ii) this decomposition of M complements uniform direct summands; and (iii) all $i, j \in I$, $i \neq j$, M_i can not be proper embedded in M_j. Then the following statements are equivalent:*

(i) M is a quasi-continuous module;

(ii) M has $(1 - C_1)$;

(iii) $M(J)$ is $M(K)$-injective for any subsets J and K of I such that $J \cap K = \emptyset$.

PROOF. $(i) \Rightarrow (ii)$ is trivial.

$(iii) \Rightarrow (i)$ by [5, Theorem 2.13].

$(ii) \Rightarrow (iii)$. By [5, Proposition 1.5], it suffices to prove that for each $k \in K$, $M(J)$ is M_k-injective. For this purpose, let U be a submodule of M_k and α be a homomorphism from U to $M(J)$. We show that α is extended to a homomorphism in $\mathrm{Hom}_R(M_k, M(J))$. Since $M(J) \oplus M_k$ has $(1-C_1)$, there exists a direct summand X of $M(J) \oplus M_k$ such that

$$\{x - \alpha(x);\ x \in U\} \subseteq^e X.$$

Since X is uniform and by Lemma 1, $M(J) \oplus M_k$ is a decomposition that complements uniform direct summands, there are only two cases:

1) $M(J) \oplus M_k = X \oplus M(J')$ for some subset J' of J. Then we have

$$M(J) \oplus M_k = X \oplus M(J') \subseteq X \oplus M(J) \subsetneq M(J) \oplus M_k.$$

Hence $X \oplus M(J') = X \oplus M(J)$. It follows that $J' = J$. Therefore $\pi|_{M_k}$ extends α, where $\pi : X \oplus M(J) \to M(J)$ is the canonical projection.

2) $M(J) \oplus M_k = X \oplus M(J_1) \oplus M_k$ for some subset J_1 of J. Let

$$\pi_k : X \oplus M(J_1) \oplus M_k \longrightarrow M_k$$

be the canonical projection and let $A = (X \oplus M(J_1)) \cap M(J)$.

If $A \neq 0$ and suppose that $A \cap M_j \neq 0$ for all $j \in J$, then by [2, Proposition 3.6], A is essential in $M(J)$. It is easy to check that $X \oplus A$ is essential in $M_k \oplus M(J)$. Hence $M_k \cap (X \oplus M(J_1)) \neq 0$, a contradiction. Consequently there exists $j \in J$ such that $M_j \cap A = 0$. Hence $M_j \cap \ker\pi_k = 0$ and thus $M_j \simeq \pi_k(M_j)$. By hypothesis we have $\pi_k(M_j) = M_k$. Therefore

$$X \oplus M(J_1) \oplus M_k = X \oplus M(J_1) \oplus M_j = X \oplus M(J_2),$$

where $J_2 = J_1 \cup \{j\} \subseteq J$. Hence we may use Case 1 to show that α is extended to a homomorphism in $\mathrm{Hom}_R(M_k, M(J))$.

If $A = 0$, then $M(J_1) = 0$. From this we see that $M(J) \oplus M_k = X \oplus M_k$. It is easy to see that $M(J)$ is uniform and

$$M(J) \oplus M_k = M_j \oplus M_k = X \oplus M_k,$$

where $J = \{j\}$. Hence we have $\pi_k(M_j) = M_k$ and $M_j \oplus M_k = X \oplus M_j$. Therefore $\pi|_{M_k}$ extends α, where $\pi : X \oplus M_j \longrightarrow M_j$ is the canonical projection. This proves (iii) and the proof of Theorem 2 is complete.

COROLLARY 3. *Let $M = U_1 \oplus \cdots \oplus U_n$ be a finite direct sum of uniform modules U_i with $End(U_i)$ local $(1 \leq i \leq n)$ such that $U_i \oplus U_j$ is extending for all $1 \leq i \neq j \leq n$. If U_i cannot be proepr embedded in U_j for all $i \neq j$, then M is a quasi-continuous module.*

PROOF. Since End(U_i) is local for all $(1 \leq i \leq n)$, $U_i \oplus U_j$ is a decomposition that complements uniform direct summands for all $i \neq j$. Then by Theorem 2, $U_i \oplus U_j$ is quasi-continuous and hence M is quasi-continuous module by [15, Corollary 2.14].

COROLLARY 4. *Let P be any projective module over a right continuous semiperfect ring. Then P is quasi-continuous if and only if P has $(1 - C_1)$.*

PROOF. Let R be a right continuous semiperfect ring and P a projective right R-module. Since R is semiperfect, by [1, Chapter 27] or [3, Theorem 22.23], R contains a complete set of primitive orthogonal idempotents $\{e_1, e_2, \ldots, e_n\}$ and

$$R = e_1R \oplus \cdots \oplus e_nR,$$

where each End(e_iR) is a local ring. Since R is right continuous, we see that e_iR is uniform and cannot be proper embedded in e_jR for all $1 \leq i, j \leq n$. By [1, Theorem 27.11] we have

$$P = \oplus_{i \in I} P_i, \tag{1}$$

for some set I, where each P_i is isomorphic to some e_iR in $\{e_1R, \ldots, e_nR\}$. Consequently P_i is uniform and cannot be proper embedded in P_j for all $i, j \in I$.

Now P obviously has $(1 - C_1)$, if P is quasi-continuous.

Conversely, assume that P have $(1 - C_1)$. We will show that the decomposition (1) of P complements uniform direct summands. Let U be a unform direct summand of P. Let π_i be the projection of P onto P_i. Since U is uniform, we have $U \cap \ker\pi_k = 0$ for some $k \in I$. On the other hand, since U is a direct summand of P, U is a uniform projective right R-module. Hence U is isomorphic to some $e_j R$ in $\{e_1 R, \ldots, e_n R\}$. Then we have $\pi_k(U) = P_k$ and therefore

$$P = U \oplus P(I - \{k\}).$$

Now by Theorem 2, P is quasi-continuous.

As an application we have the following result:

THEOREM 5. *A semiperfect ring R is QF if and if R is right continuous and every projective right R-module has $(1 - C_1)$.*

PROOF. It follows by [6, Theorem 4.2] and Corollary 4.

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