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A VERSION AT INFINITY OF THE KUIPER-KUO THEOREM

HA HUY VUI

Abstract. We prove that if $P(x)$ is a tame polynomial of several variables and if $Q(x)$ is a polynomial of degree lower than one plus the Lojasiewicz number at infinity of $P(x)$, then the Milnor fibrations at infinity of $P(x)$ and $P(x)+Q(x)$ are equivalent. This fact can be considered as a version at infinity of the Kuiper-Kuo theorem. We also relate the Lojasiewicz number at infinity to the phenomenon of singularities at infinity.

1.

Let us consider the topology at infinity of polynomials of several complex variables. By analogy with the finite determinacy properties of the local case, one can expect the existence of polynomials, whose topology at infinity does not depend on their monomials of low degrees.

In this note we show that the tame polynomials ([1]) have a such determinacy property. For them we can define a natural equivalent relation at infinity and prove that their equivalent classes do not depend on their monomials of degrees lower than one plus their Lojasiewicz's numbers at infinity. This fact can be considered as a version at infinity of the Kuiper-Kuo theorem ([5,6]). In addition to this result, we also give a relation connecting the Lojasiewicz number at infinity to the phenomenon of singularities at infinity. Namely, we prove that if a polynomial has critical values, corresponding to the singularities at infinity, then its Lojasiewicz's number at infinity must be lower than -1.

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2. Lojasiewicz's number at infinity

Let $P(x) \in C[x]$ be a polynomial of n complex variables. For $r > 0$ we define

$$\varphi(r) = \inf_{||x||=r} ||gradP(x)||.$$

Let

$$L_\infty(P) = \lim_{r\to\infty} \frac{\ln\varphi(r)}{\ln r}.$$

We call this number $L_\infty(P)$ the Lojasiewicz's number at infinity of P.

PROPOSITION 2.1. *The Lojasiewicz number at infinity is rational.*

PROOF. Let

$$A = \{x \in C^n;\ ||gradP(x)|| = \inf_{||x||=||y||} ||gradP(y)||\}.$$

Then A is an unbounded semi-algebraic set. By a version at infinity of Curve Selection Lemma, there exists a real meromorphic curve $x(\tau), \tau \in (o, \epsilon]$ such that $x(\tau) \in A, \lim_{\tau\to 0} ||x(\tau)|| = \infty$. Let $||gradP(x(\tau))|| = \tau^k\theta(\tau)$ and $||x(\tau)|| = \tau^q\mathcal{X}(\tau)$ with $\theta(0) \neq 0$ and $\mathcal{X}(0) \neq 0$. Then

$$L_\infty(P) = \lim_{\tau\to 0} \frac{\ln(\tau^k\theta(\tau))}{\ln(\tau^q\mathcal{X}(\tau))}$$

or $L_\infty(P) = k/q$.

Let us consider $P(x)$ as a map from C^n to C. Then there exists a finite set $A(P) \subset C$ such that the map

$$P : C^n - P^{-1}(A(P)) \to C - A(P).$$

is a locally trivial fibration ([10,11]). The smallest such set $A(P)$ consists of: (i) critical values of P; (ii) critical values corresponding to the singularities at infinity of P.

DEFINITION 2.2. *The value t_0 is regular at infinity, if there are $\delta > 0, R > 0$ such that*

$$P : P^{-1}(D_\delta) - B_R \to D_\delta$$

is a trivial fibration. If the value t_0 is not regular at infinity, then it is called a critical value corresponding to the singularities of P.

THEOREM 2.3. *If P(x) has critical values corresponding to the singularities at infinity, then its Lojasiewicz's number at infinity must be lower than -1.*

REMARK 2.4. In [3,4] one showed that if $n = 2$, then the converse of Theorem 2.3 is also true.

LEMMA 2.5. *Let $t_0 \in C$. Suppose that $L_\infty(P) \geq 1$. Then there exist $\delta_0 > 0$ and $R_0 > 0$ such that for any $t \in D_{\delta_0}$, the hypersurface $P^{-1}(t)$ will transversally intersect any sphere S_R with $R > R_0$.*

PROOF. By the contrary assume that there exist $\lambda_k \in C, x^k \in C^n$ such that $||x^k|| \to \infty, P(x^k) \to t_0$ as $k \to \infty$, and

$$gradP(x^k) = \lambda_k \bar{x}^k.$$

For $\delta > 0$, let

$$V = \{x \in C^n | P(x) \leq \delta, \exists \lambda \in C \text{ such that } gradP(x) = \lambda \bar{x}\}.$$

Then V is a semi-algebraic set, which is obvious unbounded. Let $x(\tau)$ be a meromorphic curve, $x(\tau) \in V, \tau \in (0, \epsilon]$ and $||x(\tau)|| \to \infty$ as $\tau \to 0$. We have

$$\frac{dP(x(\tau))}{d\tau} = \lambda(\tau) < \bar{x}(\tau), \frac{dx(\tau}{d\tau} > .$$

It follows that

$$\frac{1}{\lambda(\tau)} \frac{d\overline{P}(x(\tau)}{d\tau} + \frac{1}{\overline{\lambda}(\tau)} \frac{d\overline{P}(x(\tau))}{d\tau} = \frac{d||x(\tau)||^2}{d\tau}.$$

We get

$$\frac{2}{|\lambda(\tau)|} |\frac{dP(x(\tau))}{d\tau}| \geq \frac{d||x(\tau)||^2}{d\tau},$$

or

$$|\frac{dP(x(\tau))}{d\tau}| \geq \frac{||gradP(x(\tau))||}{2||x(\tau)||} \frac{d||x(\tau)||^2}{d\tau}. \qquad (1)$$

Let $P(x(\tau)) = t_0 + a_1\tau^s + terms\ of\ higher\ degrees$, and let $||x(\tau)|| = \tau^q + terms\ of\ higher\ degrees$. Since

$$||gradP(x(\tau))|| \geq c_1||x(\tau)||^{L_\infty(P)}$$

for some $c_1 > 0$, we get from (1)

$$s \leq q(L_\infty(P) + 1).$$

Since $q < 0$ and $L_\infty(P) \geq -1$, this implies $s \leq 0$, which is impossible if δ is small enough. Thus Lemma 2.5 is proved.

PROOF OF THEOREM 2.3. We shall show that if $L_\infty(P) \geq -1$, then every value t_0 is regular at infinity. Let δ_0 and R_0 be as in Lemma 2.5. Then we can construct in $P^{-1}(D_{\delta_0}) - B_{R_0}$ two smooth vector fields $U(x)$ and $V(x)$ such that

$$< gradP(x), U(x) > = 1,$$

$$< \bar{x}, U(x) > = 0,$$

and

$$< gradP(x), V(x) > = \sqrt{-1},$$

$$< \bar{x}, V(x) > = 0.$$

Let $W(\tau)$ be a solution of $\frac{dW(\tau)}{d\tau} = U(W(\tau))$. Then $\frac{d||W(\tau)||^2}{d\tau} = 0$. Thus, $||W(\tau)|| = const$ and in particular, $W(\tau)$ can be continued for any τ. Moreover, $\frac{dP(W(\tau))}{d\tau} = 1$, which means that the imaginary part of $P(W(\tau))$ does not depend on τ. Similarly, if $Z(\tau)$ is a solution of $\frac{dZ(\tau)}{d\tau} = V(Z(\tau))$, then $Z(\tau)$ can be continued for any τ and the real part of $P(Z(\tau))$ does not depend on τ.

Let $W_a(\tau)$ be a solution of

$$\frac{dW(\tau)}{d\tau} = U(W(\tau)),$$

$$W(0) = a,$$

and $Z_b(\tau)$ be that of

$$\frac{dZ(\tau)}{d\tau} = V(Z(\tau)),$$

$$Z(0) = b.$$

We define the map

$$h : P^{-1}(t_0) \times D_{\delta_0} - B_{R_0} \to P^{-1}(D_{\delta_0}) - B_{R_0},$$

$$(x,t) \mapsto Z_{b(t)}(Imt),$$

where $b(t) = W_x(Ret)$. Then this map h defines a trivialization of the fibration

$$P : P^{-1}(D_{\delta_0}) - B_{R_0} \to D_{\delta_0}.$$

Hence Theorem 2.3 is proved.

3. Main Theorem

Let $V = P^{-1} \subset C^n$ and let R be a sufficiently large number. We consider the Milnor map at infinity

$$\Phi : S_R - V \to S^1,$$

$$\Phi(x) = \frac{P(x)}{||P(x)||}.$$

Generally, this map does not define a locally trivial fibration. Nevertheless, as it was shown in [8], the Milnor fibration at infinity exists in certain cases.

For a polynomial $P(x)$, we put

$$M(P) = \{x \in C^n | \exists \lambda \in C, gradP(x) = \lambda \bar{x}\}.$$

We say that a polynomial $P(x)$ is *semitame* if for every sequence $\{x^k\} \subseteq M(P)$ such that $\lim_{k\to\infty} ||x^k|| = \infty$ and $\lim_{k\to\infty} gradP(x^k) = 0$, we have $\lim_{k\to\infty} P(x^k) \notin C - \{0\}$ [8].

THEOREM 3.1([8]). *If P is semitame, then for R sufficiently large, the Milnor map at infinity is the projection map of a smooth fiber bundle.*

We say that a polynomial $P(x)$ is *tame* if there exist $R > 0, \delta > 0$ such that $||gradP(x)|| \geq \delta$ for $||x|| \geq R$, [1].

Evidently, if $P(x)$ is tame, then it is semitame. Thus, by Theorem 3.1 its Milnor's fibration at infinity exists.

DEFINITION 3.2. *Let $P(x)$ and $F(x)$ be tame polynomials. We say that they are **equivalent at infinity** if their Milnor's fibrations are equivalent.*

It is easy to see that a polynomial $P(x)$ is tame if and only if its Lojasiewicz's number at infinity is nonnegative.

MAIN THEOREM. *Let $P(x)$ be a tame polynomial of n complex variables and let $L_\infty(P)$ be its Lojasiewicz's number at infinity. Then for any polynomial $Q(x)$ of degree $\deg Q(x) < L_\infty(P)+1$, the polynomial $P(x)$ and $P(x)+Q(x)$ are equivalent at infinity.*

The proof of this theorem can be carried out by Milnor's technique as it was shown in [8,9].

We need the following lemmas.

LEMMA 3.3[8, THEOREM 11]. *Let $P(x)$ be a semitame polynomial and $D \subset C$ be a closed disc centered at 0 such that D contains all critical values of $P(x)$. Then for R sufficiently large, the fibration*

$$\Phi(P) : S_R - P^{-1}(D) \to S^1,$$

is a fiberbundle equivalent to the fiberbundle

$$P : P^{-1}(\partial D) \to \partial D.$$

LEMMA 3.4. *Let $P(x)$ be a tame polynomial and suppose that $\deg Q(x) < L_\infty(P) + 1$. Then $L_\infty(P) = L_\infty(P+Q)$. In particular, the polynomial $P(x)+Q(x)$ is tame.*

PROOF. Straightforward.

Let us consider the family $F_s(x) = P(x) + sQ(x), s \in [0,1]$. Let A_s be a set of critical values of $F_s(x)$.

LEMMA 3.5. *Let $P(x)$ be a tame polynomial and let $degQ(x) < L_\infty(P)+1$. Then exists a closed disc D centered at 0 such that $A_s \subset D$ for any $s \in [0,1]$.*

PROOF. It is enough to show that the set of all critical points of polynomials $F_s(x), s \in [0,1]$, is bounded. Let $x(s)$ be a critical point of $F_s(x)$. In a neighbourhood of infinity we have

$$||gradP(x(s))|| \geq c||x(s)||^{L_\infty(P)},$$

and

$$||gradQ(x(s))|| \leq \frac{c}{2}||x(s)||^{L_\infty(P)},$$

for some $c > 0$. Since $gradF_s(x(s)) = 0$,

$$||gradP(x(s))|| = s||gradQ(x(s))|| \leq ||gradQ(x(s))||.$$

This is impossible if $x(s)$ is unbounded. Thus Lemma 3.5 is proved.

LEMMA 3.6. *Let D be a closed disc as in Lemma 3.5. Then for every R sufficiently large and for every $c \in C, s \in [0,1]$, the hypersurface $F_s^{-1}(c)$ intersects S_R transversally.*

PROOF. By contradiction, suppose that there exists $x^k \in C^n, s_k \in [0,1], \lambda_k \in C$, such that $||x^k|| \to \infty$ as $k \to \infty$ and

$$gradF_{s_k}(x^k) = \lambda_k \bar{x}^k. \tag{2}$$

The set consisting of all (s_k, x^k) for which (2) holds is an unbounded semi-algebraic set in $[0,1] \times C^n$. Thus, by a version at infinity of Curve Selection Lemma, there exists a meromorphic curve $(s(\tau), x(\tau)), \tau \in (o, \epsilon]$, such that

$$gradF_{s(\tau)}(x(\tau)) = \lambda(\tau)\bar{x}(\tau),$$

and $||x(\tau)|| \to \infty$ as $\tau \to 0$. We have

$$\frac{dF_{s(\tau)}(x(\tau))}{d\tau} = \lambda(\tau) < x(\tau), \frac{dx(\tau)}{d\tau} > + \frac{ds(\tau)}{d\tau}Q(x(\tau)).$$

We can then show

$$2|\frac{dF_{s(\tau)}(x(\tau))}{d\tau}| \geq \frac{gradF_{s(\tau)}(x(\tau))}{||x(\tau)||}\frac{d||x(\tau)||^2}{d\tau} - 2|\frac{ds(\tau)}{d\tau}||Q(x(\tau))|. \quad (3)$$

Since $degQ(x) < L_\infty(P)+1$, by Lemma 3.4

$$L_\infty(F_s) = L_\infty(P).$$

Then there exists $c > 0$ such that

$$||gradF_{s(\tau)}(x(\tau)|| \geq c||x(\tau)^{L_\infty(P)}.$$

Continuing (3) we get

$$2|\frac{F_{s(\tau)}(x(\tau))}{d\tau}| \geq c||x(\tau)||^{L_\infty(P)-1}\frac{d||x(\tau)||^2}{d\tau} - 2|\frac{ds(\tau)}{d\tau}||Q(x(\tau))|. \quad (4)$$

Let

$$||x(\tau)|| = a\tau^\rho + o(\tau^\rho),$$

$$|s(\tau)| = s_0 + b\tau^m + o(\tau^m),$$

$$|F_{s(\tau)}(x(\tau))| = t_0 + c\tau^q + o(\tau^q).$$

Since $F_s(\tau)(x(\tau)) \in D$ and $s(\tau) \in [0,1]$, we have $t_0 \in D, q > 0, m > 0$. Since $||x(\tau)|| \to \infty$ as $\tau \to 0, \rho < 0$. Since $degQ(x) < L_\infty(P)+1, |Q(x(\tau)| \in o(||x(\tau)||^{L_\infty(P)+1})$.

From (4) it follows that

$$q \leq \rho(L_\infty(P)+1).$$

This inequality is impossible because $q > 0, \rho < 0, L_\infty(P) \geq 0$. In this way, Lemma 3.6 is proved.

LEMMA 3.7. *Let D and $F_s(x)$ be as in Lemmas 3.5 and 3.6. Then, for R sufficiently large, the fibrations*

$$F_i : F_i^{-1}(\partial D) \cap B_R \to \partial D,$$

$$x \mapsto F_i(x), \quad (M_i)$$

where $i = 0$ or 1, are equivalent.

PROOF. This follows from Lemmas 3.5 and 3.6 by using standard Milnor's technique [7] (see [9] or [8]).

Now, Main Theorem follows from Theorem 3.1 and Lemma 3.7. In fact, let ψ be the isomorphism of fibration $(M_i), i = 0, 1$, and let $\psi_i, i = 0, 1$, be the isomorphisms of fibrations

$$F_i : F_i^{-1}(\partial D) \to \partial D,$$

$$x \mapsto F_i(x),$$

and

$$\Phi_i : S_R - F_i^{-1}(0) \to S^1,$$
$$\Phi_i(x) = \frac{F_i(x)}{||F_i(x)||},$$

with $i = 0, 1$. Then $\varphi = \psi_1 \circ \psi \circ \psi_0^{-1}$ will be the isomorphism of the Milnor fibrations at infinity of $F_0 = P$ and $F_1 = P+Q$. Thus, Main Theorem is proved.

REMARK 3.8. In [8] it was shown that if a polynomial is convenient and nondegenerated with respect to its Newton's diagram at infinity, then its Milnor's fibration at infinity is equivalent to the Milnor fibration at infinity of its principal part. Since one can show that if a polynomial $P(x)$ is convenient and nondegerated with respect to its Newton's diagram at infinity, then it is tame. Moreover, one can show that in this case one has the following inequality for the Lojasiewicz number at infinity

$$L_\infty(P) \leq \min degQ(x) - 1,$$

where $Q(x)$ runs over all the monomials of the principal part of $P(x)$. Thus, in this case, the mentioned result of [8] is stronger than our Main Theorem.

REMARK 3.9. One would like to know if the converse of Main Theorem is true. Namely, if $P(x)$ is a tame polynomial and if $P(x)$ and $P(x) + Q(x)$ are equivalent at infinity for any polynomial $Q(x)$ of degrees $\leq r$, then does it follows that $r \leq L_\infty(P) + 1$? This could be considered as a version at infinity of the Bochnak-Lojasiewicz theorem [2].

REFERENCES

[1] Broughton S.A., *Milnor numbers and the topology of polynomial hypersurfaces*, Invent. Math. **92**(1988), 217-242.
[2] Bochnak J., *Lojasiewicz S., A converse of the Kuiper-Kuo theorem*, Proc. Liverpool Singularities Symposium 1, LNM 192, 1972, 254-246
[3] Ha Huy Vui, *Nombres de Lojasiewicz et singularités à l'infini des polynomes de deux variables complexes*, C.R. Acad. Sci. Paris. série I **311**(1990), 429-432.
[4] Ha Huy Vui, *On the irregularity at infinity of algebraic plane curves*, Preprint 91/4, Institute of Mathematics, Hanoi.
[5] Kuiper N.H., *C^1-equivalence of function near isolated critical points*, Proc. Sym. in Infinite Dimensional Topology (1967).
[6] Kuo T.C., *On C^0-sufficiency of jets of potential functions*, Topology **8** (1969), 167-171.
[7] Milnor J., *Singular points of Complex Hypersurfaces*, Ann. of Math. Studies 61, Princeton Uni. Press, 1968.
[8] Nemethi A., Zaharia A., *Milnor fibration at infinity*, Indag. Math. new series **3**(1992), 332-335.
[9] Oka M., *On the topology of the Newton boundary III*, J. Math. Soc. Japan **34**(1982), 541-549.
[10] Thom R., *Ensembles et morfismes stratifiés*, Bull. Amer. Math. Soc. **75**(1969), 249-312.
[11] Verdier J.L., *Stratifications de Whitney et théorème de Bertini-Sard*, Invent. Math. **36**(1976), 295-312.

INSTITUTE OF MATHEMATICS
P.O. BOX 631 BOHO, HANOI, VIETNAM