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ON THE INTEGRAL CONVOLUTION FOR INVERSE G–TRANSFORMS

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Abstract. We give a new method for constructing general integral convolutions by means of the theory of Mellin type G–transforms.

As introduced in [1] and described in [2,3] the G–transform of a function $f(x)$ is the following integral

$$(Gf)(x) \equiv G^{m,n}_{p,q}\begin{bmatrix}(\alpha_p)\\(\beta_q)\end{bmatrix}\cdot[f(u)](x) = \frac{1}{2\pi i}\int_\sigma \Phi(s)f^*(s)x^{-s}ds, \qquad x>0, \tag{1}$$

where

$$\Phi(s)=\frac{\prod\limits_{j=1}^{m}\Gamma(\beta_j+s)\prod\limits_{j=1}^{n}\Gamma(1-\alpha_j-s)}{\prod\limits_{j=n+1}^{p}\Gamma(\alpha_j+s)\prod\limits_{j=m+1}^{q}\Gamma(1-\beta_j-s)},$$

$f^*(s)=\mathfrak{M}\{f(x);s\}=\int_0^\infty f(x)x^{s-1}dx$ is the Mellin transform [4] of the function $f(x)$, $\sigma=\{s,\Re(s)=\frac{1}{2}\}$, and the component of the p- and q-dimensional vectors (α_p) and (β_q) are complex parameters with the properties

$$\begin{aligned}&\Re\beta_j>-\frac{1}{2}, j=1,...,m, \quad \Re\beta_j<\frac{1}{2}, j=m+1,...,q,\\ &\Re\alpha_j<\frac{1}{2}, j=1,...,n, \quad \Re\alpha_j>-\frac{1}{2}, j=n+1,...,p.\end{aligned} \tag{2}$$

These conditions guarantee that $\Phi(s)$ is holomorphic in a strip symmetric to the line $\Re(s)=\frac{1}{2}$. The notations $\Re$ and $\Im$ mean real part imaginary part, respectively.

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The well ordered pair (c^*, γ^*), where $c^* = m + n - \frac{(p+q)}{2}, \gamma^* = \Re\Big(\sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \beta_j\Big)$, is called *the characteristic of the G–transform* (1).

The present paper is devoted to the constructions of some operators, which are called the convolutions $(f * g)(x)$ of two functions $f(x)$ and $g(x)$ belonging to the special spaces $\mathfrak{M}^{-1}_{c,\gamma}(L)$ [1]. These spaces are very convenient for the G–transforms of type (1), and by the actions of these transforms on convolutions we can get, for example, the factorization equality

$$(G^{-1}(f * g))(x) = (G_1 f)(x)\,(G_2 g)(x), \tag{3}$$

where the operators G^{-1} (the inverse G–transform), G_1, G_2 are the G–transforms with the kernels $\frac{1}{\Phi(s)}$, $\Phi_1(s)$, $\Phi_2(s)$, respectively.

Note that the G–transform (1) includes all known integral transforms like the Laplace, Stieltjes, Hankel, Meijer transforms e.t.c. and their inversions.

Definition 1 [1]. Let $c, \gamma \in \Re$, and $2\,sign(c) + sign(\gamma) \geq 0$. Denote by $\mathfrak{M}^{-1}_{c,\gamma}(L)$ the space of functions $f(x)$, $x > 0$, representable in the form

$$f(x) = \frac{1}{2\pi i} \int_\sigma f^*(s) x^{-s} ds, \quad x > 0, \tag{4}$$

where $f^*(s)|s|^\gamma e^{\pi c|\Im s|} \in L(\sigma)$, the space of complex-valued functions Lebesgue-integrable on σ, $\sigma = \{s|\Re s = \frac{1}{2}\}$.

The space $\mathfrak{M}^{-1}_{c,\gamma}(L)$ is a Banach space with the norm

$$||f||_{\mathfrak{M}^{-1}_{c,\gamma}(L)} = \int_{-\infty}^{+\infty} e^{\pi c|\tau|} |\tau|^\gamma \big|f^*(\frac{1}{2} + i\tau) d\tau\big|. \tag{5}$$

Proposition 1. *If*

$$2\,sign(c' - c) + sign(\gamma' - \gamma) \geq 0, \tag{6}$$

then

$$\mathfrak{M}^{-1}_{c',\gamma'}(L) \subset \mathfrak{M}^{-1}_{c,\gamma}(L). \tag{7}$$

PROOF. Suppose that inequality (6) holds. Then from (5) we have

$$\begin{aligned}||f||_{\mathfrak{M}^{-1}_{c,\gamma}(L)} &= \int_\sigma e^{\pi c|\Im s|}|s^\gamma f^*(s)ds| \\ &= \int_\sigma e^{\pi c'|\Im s|}|s^{\gamma'} f^*(s)| \cdot |e^{\pi(c-c')|\Im s|}s^{\gamma-\gamma'}ds| \\ &\le C\int_\sigma e^{\pi c'|\Im s|}|s^{\gamma'} f^*(s)ds| = C||f||_{\mathfrak{M}^{-1}_{c',\gamma'}(L)},\end{aligned}$$

where the constant C is defined as

$$C = \sup_{s\in\sigma}\{e^{\pi(c-c)'|\Im s|}|s|^{\gamma-\gamma'}\} < +\infty.$$

The finiteness of C follows from (6).

PROPOSITION 2. *If* $f(x), g(x) \in \mathfrak{M}^{-1}_{c,\gamma}(L)$, *then* $x^{\frac{1}{2}}f(x)g(x) \in \mathfrak{M}^{-1}_{c,\min\{\gamma,2\gamma\}}(L)$.

PROOF. According to Definition 1 of the space $\mathfrak{M}^{-1}_{c,\gamma}(L)$ we can represent the function $h(x) = x^{\frac{1}{2}}f(x)g(x)$ in the form

$$h(x) = \frac{x^{\frac{1}{2}}}{(2\pi i)^2}\int_{\sigma_s}\int_{\sigma_t} f^*(s)g^*(t)x^{-s-t}ds\,dt, \tag{8}$$

where $\sigma_s \times \sigma_t = \{(s,t) \in \mathbb{C}^2 \mid \Re(s) = \Re(t) = \frac{1}{2}\}$.

By substituting $\tau = s + t - \frac{1}{2}$ we can write (8) as

$$h(x) = \frac{1}{2\pi i}\int_{\sigma_\tau} F(\tau)x^{-\tau}d\tau, \tag{9}$$

where $\sigma_\tau = \{\tau \in \mathbb{C} \mid \Re(\tau) = \frac{1}{2}\}$ and

$$F(\tau) = \frac{1}{2\pi i}\int_{\sigma_t} f^*(\tau - t + \frac{1}{2})g^*(t)dt \qquad \text{for} \quad \tau \in \sigma_\tau. \tag{10}$$

Consequently, according to Definition 1, we must show that $h(x) \in \mathfrak{M}^{-1}_{c,\min\{\gamma,2\gamma\}}(L)$, i.e.

$$F(\tau)|\tau|^\gamma e^{\pi c|\Im(\tau)|} \in L\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right) \qquad \text{if} \quad \gamma \ge 0,$$

$$F(\tau)|\tau|^{2\gamma} e^{\pi c|\Im(\tau)|} \in L\left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right) \qquad \text{if} \quad \gamma < 0.$$

We note that in the second case ($\gamma < 0$), from Definition 1 it follows that $c > 0$.

Let $\gamma \geq 0$. Using representation (9) we get the inequality

$$\int_{\sigma_\tau} e^{\pi c|\Im(\tau)|}|\tau|^\gamma |F(\tau)d\tau|$$
$$\leq \frac{1}{2\pi}\int_{\sigma_s}\int_{\sigma_t} e^{\pi c|\Im(s)+\Im(t)|}\left|s+t-\frac{1}{2}\right|^\gamma |f^*(s)g^*(t)ds\,dt|.$$

From $f(x)$, $g(x) \in \mathfrak{M}^{-1}_{c,\gamma}(L)$ we conclude that

$$e^{\pi c|\Im(s)|}|s|^\gamma f^*(s) \in L(\sigma_s) \qquad \text{and} \qquad e^{\pi c|\Im(t)|}|t|^\gamma g^*(t) \in L(\sigma_t).$$

Hence it is evident that the last double integral converges if

$$\sup_{(s,t)\in\sigma_s\times\sigma_t} \exp(\pi c(|\Im(s)+\Im(t)| - |\Im(s)| - |\Im(t)|)) \cdot \left|s+t-\frac{1}{2}\right|^\gamma |s|^{-\gamma}|t|^{-\gamma} < \infty. \tag{11}$$

It is not difficult to see that (11) is equivalent to

$$\sup_{(s,t)\in\sigma_s\times\sigma_t} \exp(\pi c(|\Im(s)+\Im(t)| - |\Im(s)| - |\Im(t)|)) \cdot |s+t|^\gamma |s|^{-\gamma}|t|^{-\gamma} < \infty. \tag{12}$$

This inequality now is true since the assumption $\gamma \geq 0$ and Definition 1 imply $c \geq 0$. Take into account, furthermore, that

$$|\Im(s)+\Im(t)| \leq |\Im(s)| + |\Im(t)|, \quad \frac{|s+t|^\gamma}{|s|^\gamma|t|^\gamma} = \left|\frac{1}{2}+\frac{1}{t}\right|^\gamma < \infty, \ (s,t) \in \sigma_s \times \sigma_t.$$

If $\gamma < 0$, then instead of inequality (12) we must have

$$\sup_{(s,t)\in\sigma_s\times\sigma_t} \exp(\pi c(|\Im(s)+\Im(t)| - |\Im(s)| - |\Im(t)|)) \cdot |s+t|^{2\gamma} |s|^{-\gamma}|t|^{-\gamma} < \infty. \tag{13}$$

In this case from Definition 1 it follows that $c > 0$. Further, for $(s,t) \in \sigma_s \times \sigma_t$,

$$|s+t|^{2\gamma}|s|^{-\gamma}|t|^{-\gamma} = \left|1+\frac{t}{s}\right|^{2\gamma}\left|\frac{s}{t}\right|^\gamma < \infty, \ |s| > |t|,$$
$$|s+t|^{2\gamma}|s|^{-\gamma}|t|^{-\gamma} = \left|1+\frac{s}{t}\right|^{2\gamma}\left|\frac{t}{s}\right|^\gamma < \infty, \ |t| > |s|,$$
$$|s+t|^{2\gamma}|s|^{-\gamma}|t|^{-\gamma} = 2^{2\gamma}, \ |s| = |t|.$$

Thus inequality (13) holds if $\gamma < 0$.

The following Theorem 1 is obtained in [1].

THEOREM 1. *The operator G in (1) with the characteristics (c^*, γ^*) is defined in the space $\mathfrak{M}^{-1}_{c,\gamma}(L)$ if and only if*

$$2\,sign(c+c^*) + sign(\gamma+\gamma^*) \geq 0, \tag{14}$$

and acts then as an isomorphism from $\mathfrak{M}^{-1}_{c,\gamma}(L)$ onto $\mathfrak{M}^{-1}_{c+c^*,\gamma+\gamma^*}(L)$.

Note that in case $\Phi(s) \equiv 1$ the representation inverse to (1) takes the form (4). Thus the identical transform is also a G–transform.

DEFINITION 2 [5]. We call the double integral

$$(f * g)(x) = \frac{1}{(2\pi i)^2} \int_{\sigma_s} \int_{\sigma_t} \Phi(s+t)\Phi_1(s)\Phi_2(t) f^*(s) g^*(t) x^{-s-t} ds\, dt, \tag{15}$$

where σ_s, σ_t are the contours $\Re(s) = \frac{1}{2}$, $\Re(t) = \frac{1}{2}$, the G–convolution of the functions $f(x)$ and $g(x)$.

Let (c_j^*, γ_j^*), $j = 1,2$, be the characteristic of the G–transforms with the kernels $\Phi_j(\tau)$, $j = 1,2$, and the corresponding complex parameters of vectors $(a^j_{p_j})$, $(b^j_{q_j})$, $j = 1,2$.

Now to prove the equality (3) we have to understand its left part described by the following definition of the G–transform (1) of a function $f(x)$ with

$$x^\lambda f(x) \in \mathfrak{M}^{-1}(L) = \mathfrak{M}^{-1}_{0,0}, \lambda \in \Re \quad \text{(similarly for subspaces} \quad \mathfrak{M}^{-1}_{c,\gamma}(L)).$$

DEFINITION 3. Let λ be a real constant and $x^\lambda f(x) \in \mathfrak{M}^{-1}(L)$, i.e.

$$x^\lambda (f(x) = \frac{1}{2\pi i} \int_{\sigma_\tau} f^*(\tau+\lambda) x^{-\tau} d\tau, \tag{16}$$

where $f^*(\tau+\lambda) \in L(\sigma_\tau)$. Then the G–transform with the kernel $H(\tau)$ of the function $f(x)$ is interpreted as follows

$$\begin{aligned}(Gf)(x) &= \frac{x^{-\lambda}}{2\pi i} \int_{\sigma_\tau} H(\tau+\lambda) f^*(\tau+\lambda) x^{-\tau} d\tau \\ &= \frac{1}{2\pi i} \int_{\Re(\tau)=\lambda+\frac{1}{2}} H(\tau) f^*(\tau) x^{-\tau} d\tau.\end{aligned} \tag{17}$$

We note that, if $H(\tau)f^*(\tau)$ is analytic in the strip

$$-\epsilon+\min\left\{\frac{1}{2},\frac{1}{2}+\lambda\right\}<\Re(\tau)<\epsilon+\max\left\{\frac{1}{2},\frac{1}{2}+\lambda\right\},\quad \epsilon>0,$$

and $H(\tau)f^*(\tau)\to 0$ uniformly as $|\Im(\tau)|\to\infty$ in the strip, then by the Cauchy theorem G–transform (17) coincides with the G–transform defined by (1), whose integral contour is the line $\sigma_\tau=\{\tau\in\mathbb{C}\,\|\Re(\tau)=\frac{1}{2}\}$.

THEOREM 2. *Let $f(x),g(x)\in\mathfrak{M}^{-1}_{c,\gamma}(L)$. Let the kernels $\Phi_1(s),\Phi_2(t)$ satisfy condition (2), and the kernel $\Phi(s)$ satisfy the conditions*

$$\begin{cases}\Re(b_j)>-\frac{1}{2}, & j=1,2,...,m,\\ \Re(a_j)<0, & j=1,2,...,n,\\ \Re(a_j)>-\frac{1}{2}, & j=n+1,...,p,\\ \Re(b_j)<0, & j=m+1,...,q.\end{cases}\tag{18}$$

Let, further, the following inequalities hold:

$$\begin{aligned}2\,sign(c+c_j^*)+sign(\gamma+\gamma_j^*)\ge 0,\quad j=1,2,\\ 2\,sign(c+\tilde{c}^*)+sign(\tilde{\gamma}^*)\ge 0,\end{aligned}\tag{19}$$

where

$$\begin{aligned}\tilde{c}^*&=\min\{c_1^*,c_2^*\}+c^*;\\ \tilde{\gamma}^*&=\min\{\gamma+\min\{\gamma_1^*,\gamma_2^*\},2(\gamma+\min\{\gamma_1^*,\gamma_2^*\})\}+\gamma^*+\frac{p-q}{2}.\end{aligned}\tag{20}$$

Then G–convolution integral (15) exists and $x^{\frac{1}{2}}(f*g)(x)$ belongs to $\mathfrak{M}^{-1}_{c+\tilde{c}^*,\tilde{\gamma}^*}(L)$. Moreover, we have factorization property (3) for $(f*g)(x)$.

PROOF. Conditions (19) imply that G_j–transforms, $j=1,2$, exist and

$$(G_1f)(x)\in\mathfrak{M}^{-1}_{c+c_1^*,\gamma+\gamma_1^*}(L),\quad (G_2g)(x)\in\mathfrak{M}^{-1}_{c+c_2^*,\gamma+\gamma_2^*}(L).$$

By Proposition 1 we have

$$(G_1f)(x),\ (G_2g)(x)\in\mathfrak{M}^{-1}_{c+\min\{c_1^*,c_2^*\},\gamma+\min\{\gamma_1^*,\gamma_2^*\}}(L).$$

Further, by arguments similar to those of the proof of Proposition 2, we conclude from the representation

$$x^{\frac{1}{2}}(f*g)(x)=\frac{1}{2\pi i}\int_{\sigma_\tau}\Phi(\tau+\frac{1}{2})\hat{F}(\tau)x^{-\tau}d\tau, \tag{21}$$

with

$$\hat{F}(x)=\frac{1}{2\pi i}\int_{\sigma}\Phi_1\left(\tau-t+\frac{1}{2}\right)\Phi_2(t)f^*\left(\tau-t+\frac{1}{2}\right)g^*(t)dt \quad \text{for} \quad \tau\in\sigma_\tau, \tag{22}$$

that

$$\frac{1}{2\pi i}\int_{\sigma_\tau}\hat{F}(\tau)x^{-\tau}d\tau=x^{\frac{1}{2}}(G_1f)(x)\,(G_2g)(x) \tag{23}$$

and

$$x^{\frac{1}{2}}(G_1f)(x)(G_2g)(x)\in\mathfrak{M}^{-1}_{c+\min\{c_1^*,c_2^*\},\min\{\gamma+\min\{\gamma_1^*,\gamma_2^*\},2(\gamma+\min\{\gamma_1^*,\gamma_2^*\}\}\}}(L).$$

From representation (21) it follows that $x^{\frac{1}{2}}(f*g)(x)$ is the G–transform (1) of the function $x^{\frac{1}{2}}(G_1f)(x)(G_2g)(x)$ with the kernel $\Phi(\tau+\frac{1}{2})$ and the characteristic pair $(c^*,\gamma^*+\frac{p-q}{2})$. Thus, by Theorem 1 and by conditions (19) and (20) integral (21) converges absolutely and $x^{\frac{1}{2}}(f*g)(x)\in\mathfrak{M}^{-1}_{c+\tilde{c}^*,\tilde{\gamma}^*}(L)$.

Further, in accordance with Definition 3, we have

$$\begin{aligned}(G^{-1}(f*g))&=\frac{x^{-\frac{1}{2}}}{2\pi i}\int_{\sigma_\tau}\frac{1}{\Phi\left(\tau+\frac{1}{2}\right)}\Phi(\tau+\frac{1}{2})\hat{F}(\tau)x^{-\tau}d\tau\\&=\frac{x^{-\frac{1}{2}}}{2\pi i}\int_{\sigma_\tau}\hat{F}(\tau)x^{-\tau}d\tau=(G_1f)(x)\,(G_2g)(x).\end{aligned} \tag{24}$$

Theorem 2 is completely proved.

THEOREM 3. *Let the conditions of Theorem 2 be fulfilled and let $\Phi(s)$ be the Mellin transform of some function $\varphi(t)\in L(\mathbf{R}_+)$. Then for convolution (15) we have the Parseval equality*

$$(f*g)(x)=\int_0^\infty\varphi(t)(G_1f)\left(\frac{x}{t}\right)(G_2g)\left(\frac{x}{t}\right)\frac{dt}{t},\quad x>0. \tag{25}$$

Moreover, if for the characteristics (c_j^*,γ_j^*), $j=1,2$, the inequality

$$2\,sign(c_j^*)+sign(\gamma_j^*-1)>0,\quad j=1,2, \tag{26}$$

holds and if $f(x), g(x)$ belong to $\mathfrak{M}^{-1}_{c,\gamma}(L) \cap L(\mathrm{R}_+, x^{-\frac{1}{2}})$ $\left(L(\mathrm{R}_+, x^{-\frac{1}{2}})\right.$ is the space of functions summable with the weight $\left.x^{-\frac{1}{2}}\right)$, then convolution (15) can be represented in the form

$$(f * g) = \int_0^\infty \int_0^\infty S\left(\frac{x}{u}, \frac{x}{v}\right) f(u) g(v) \frac{du\, dv}{uv}, \quad x > 0, \tag{27}$$

where

$$S(x,y) = \frac{1}{(2\pi i)^2} \int_{\sigma_s} \int_{\sigma_t} \Phi(s+t)\Phi_1(s)\Phi_2(t) x^{-s} y^{-t} ds\, dt.$$

PROOF. Representations (25) and (27) are easily obtained from the Fubini theorem, which is applicable by the conditions of Theorem 3.

The G–transform (1) includes many particular cases of integral transforms of convolution type treated in [3], for example:

1. The modified operators of fractional calculus:

$$\begin{aligned}
\left(x^\beta I^\alpha_{0+} x^{-\alpha-\beta} f\right)(x) =& G^{0,1}_{1,1}\begin{pmatrix} \alpha+\beta \\ \beta \end{pmatrix} \cdot [f(u)](x) \\
=& \frac{x^\beta}{\Gamma(\alpha)} \int_0^x \frac{f(t) t^{-\alpha-\beta}}{(x-t)^{1-\alpha}} dt, \quad \Re\alpha > 0, \\
\left(x^\beta I^\alpha_{0+} x^{-\alpha-\beta} f\right)(x) =& G^{0,1}_{1,1}\begin{pmatrix} \alpha+\beta \\ \beta \end{pmatrix} \cdot [f(u)](x) \\
=& \frac{x^\beta}{\Gamma(\alpha+n)\left(\frac{d}{dx}\right)^n} \int_0^x \frac{f(t) t^{-\alpha-\beta}}{(x-t)^{1-\alpha-n}} dt, \\
& -n < \Re\alpha \le 0, \quad n = [-\Re\alpha] + 1, \\
\left(x^\beta I^\alpha_{-} x^{-\alpha-\beta} f\right)(x) =& G^{1,0}_{1,1}\begin{pmatrix} \alpha+\beta \\ \beta \end{pmatrix} \cdot [f(u)](x) \\
=& \frac{x^\beta}{\Gamma(\alpha)} \int_x^\infty \frac{f(t) t^{-\alpha-\beta}}{(t-x)^{1-\alpha}} dt, \quad \Re\alpha > 0.
\end{aligned}$$

2. The operators of the modified Laplace transform and their inverses:

$$\begin{aligned}
\left(x^\alpha \Lambda_+ x^{-\alpha} f\right)(x) =& G^{1,0}_{0,1}\begin{pmatrix} - \\ \alpha \end{pmatrix} \cdot [f(u)](x) = x^\alpha \int_0^\infty e^{-\left(\frac{x}{t}\right)} f(t) t^{-\alpha-1} dt, \\
\left(x^\alpha \Lambda_- x^{-\alpha} f\right)(x) =& G^{0,1}_{1,0}\begin{pmatrix} 1+\alpha \\ - \end{pmatrix} \cdot [f(u)](x) = x^\alpha \int_0^\infty e^{-\left(\frac{t}{x}\right)} f(t) t^{-\alpha-1} dt,
\end{aligned}$$

$$\left(x^\alpha \Lambda_+^{-1} x^{-\alpha} f\right)(x) = G_{1,0}^{0,0}\binom{\alpha}{-} \cdot [f(u)](x),$$
$$\left(x^\alpha \Lambda_-^{-1} x^{-\alpha} f\right)(x) = G_{0,1}^{0,0}\binom{-}{1+\alpha} \cdot [f(u)](x).$$

3. The operator of the generalized Stieltjes transform:

$$\{\Gamma(\varrho)(1+x)^{-\varrho}\} \cdot [f(u)](x) = G_{1,1}^{1,1}\binom{1-\varrho}{0} \cdot [f(u)] = \Gamma(\varrho)\int_0^\infty \frac{u^\varrho f(u)}{(x+u)^\varrho}\frac{du}{u}.$$

4. The operator of the ${}_1F_1$ –transform

$$G_{1,2}^{1,1}\binom{1-a}{0, 1-b} \cdot [f(u)](x) = \left\{\Gamma\begin{bmatrix} a \\ b \end{bmatrix} {}_1F_1(a; b; -x)\right\} [f(u)]$$
$$= \Gamma\begin{bmatrix} a \\ b \end{bmatrix} \int_0^\infty {}_1F_1\left(a; b; -\frac{x}{u}\right) f(u)\frac{du}{u}.$$

Now we give some examples of the convolutions (15) and their factorization properties. If, for example, $f(x), g(x) \in L(\mathsf{R}_+, x^{-\frac{1}{2}})$, we can write the following convolution (28) in the form (27) and obtain its factorization property

$$(f * g)_+)(x) = 2\int_0^\infty \int_0^\infty K_0\left(2\sqrt{x\frac{u+v}{uv}}\right) f(u)g(u)\frac{du\,dv}{uv} \tag{28}$$
$$(\Lambda_+^{-1}(f * g)_+)(x) = (\Lambda_+ f)(x)\,(\Lambda_+ g)(x).$$

Here $K_0(z)$ is the Macdonald function [2].

More general convolutions may be reduced by using two-dimensional integral operators (27) with Appel functions F_1, F_2. These functions have the integral representations

$$\Gamma\begin{bmatrix} \alpha, \beta, \beta' \\ \gamma_1 \end{bmatrix} F_1(\alpha, \beta, \beta'; \gamma_1; -x, -y)$$
$$= \frac{1}{(2\pi i)^2}\int_{\sigma_s}\int_{\sigma_t} \Gamma\begin{bmatrix} \alpha - s - t, \beta - s, \beta' - t, s, t \\ \gamma_1 - s - t \end{bmatrix} x^{-s} y^{-t} ds\,dt,$$
$$\Gamma\begin{bmatrix} \alpha, \beta, \beta' \\ \gamma_1, \gamma_1' \end{bmatrix} F_2(\alpha, \beta, \beta'; \gamma_1, \gamma_1'; -x, -y)$$
$$= \frac{1}{(2\pi i)^2}\int_{\sigma_s}\int_{\sigma_t} \Gamma\begin{bmatrix} \alpha - s - t, \beta - s, \beta' - t, s, t \\ \gamma_1 - s, \gamma_1' - t \end{bmatrix} x^{-s} y^{-t} ds\,dt.$$

For convenience we will use Slater's notation [3]

$$\Gamma\begin{bmatrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_q\end{bmatrix}=\frac{\Gamma(\alpha_1)\dots\Gamma(\alpha_p)}{\Gamma(\beta_1)\dots\Gamma(\beta_q)}.$$

The following convolutions (29) and (30) with their factorization properties are meaningful under appropriate conditions on the parameters and functions (see Theorem 2 and Theorem 3).

$$(f*g)(x)=\Gamma\begin{bmatrix}\alpha,\beta,\beta'\\ \gamma_1\end{bmatrix}\int_0^\infty\int_0^\infty F_1\left(\alpha,\beta,\beta',\gamma_1;-\frac{x}{u},-\frac{x}{v}\right)f(u)g(v)\frac{du\,dv}{uv}, \tag{29}$$

$$(x^{1-\alpha}I_+^{\alpha-\gamma}x^{\gamma-1})((f*g)(x))=\{\Gamma(\beta)(1+x)^{-\beta}\}(f(x))\{\Gamma(\beta')(1+x)^{-\beta'}\}(g(x)).$$

$$(f*g)=\Gamma\begin{bmatrix}\alpha,\beta,\beta'\\ \gamma_1,\gamma_1'\end{bmatrix}\int_0^\infty\int_0^\infty F_2(\alpha,\beta,\beta',\gamma_1,\gamma_1';-\frac{x}{u},-\frac{x}{v})f(u)g(v)\frac{du\,dv}{uv}, \tag{30}$$

$$\begin{aligned}&(x^{-\alpha}\Lambda_-^{-1}x^{\alpha}((f*g)(x))\\ &=\left\{\Gamma\begin{bmatrix}\beta\\ \gamma\end{bmatrix}{}_1F_1(\beta;\gamma;-x)\right\}(f(x))\cdot\left\{\Gamma\begin{bmatrix}\beta'\\ \gamma'\end{bmatrix}{}_1F_1(\beta';\gamma';-x)\right\}(g(x)).\end{aligned}$$

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