ACTA MATHEMATICA VIETNAMICA
Volume 19, Number 1, 1994, pp. 77-96

# SEPARABLE CALIBRATIONS AND MINIMAL SURFACES

HOANG XUAN HUAN

## 1. Introduction

The calibration method was studied systematically by Dao Trong Thi in [D1, D2] and Harvey R., Lawson H.B. in [HL]. Various concrete calibrations were used by many authors: Federer [F1], Berger [B], Dao Trong Thi [D1, D2], Harvey-Lawson [HL], Dadok-Harvey-Morgan [DHM], Le Hong Van [L] etc... to find minimal surfaces. In all but simplest cases,the determination of the comass and the maximal directions of a $p$-covector is the main obstacle to apply this method.

In this paper we study the $p$-covectors in $R^n$, which can be expressed in the form

$$\Omega = \Omega_1 + e^*_{V_1} \wedge \Omega_2 + \cdots + e^*_{V_1} \wedge \cdots \wedge e^*_{V_k} \Omega_{k+1}$$

where $R^n = V_1 \oplus \cdots \oplus V_{k+1}$ is an orthogonal decomposition of $R^n$, $\dim V_t = p_t \geq 2$ for $t \leq k$, $\Omega_i \in \wedge^{q_i}(W_i)$, $q_i = p - \sum\limits_{t<i} p_t, e^*_{V_i}$ is the unit $p_i$-covector of $V_i, W_i = \bigoplus\limits_{t>i} V_t$ for $i \leq k$ and $W_{k+1} = V_{k+1}$. In this case the comass of $\Omega$ can be calculated by the formula

$$\|\Omega\|^* = \max \|\Omega_i\|^*$$

and the maximal directions of $\Omega$ can be defined by the maximal directions of the terms, whose comass equal $\Omega$. On the other hand Federer in [F2] and Harvey-Lawson in [HL] have proved the equality

$$\|\varphi \wedge \eta\|^* = \|\varphi\|^* \|\eta\|^*$$

Received September 9, 1993

when $\varphi$ is decomposable (where $R^n = V \oplus W$ is an orthogonal decomposition of $R^n$, $\varphi \in \wedge^r(V)$ and $\eta \in \wedge^s(W)$.) This equality still holds when $\varphi$ has the expressions considered in this paper.

The author expresses his gratitude to Prof. Dao Trong Thi for his scientific advice.

## 2. Forms and comass

In this section we recall some notions and facts of external algebra (for details see [F2] and [G]). Let $R^n$ be the $n$-dimensional Euclidian space, $\wedge_{k,n}$ and $\wedge^{k,n}$ the dual spaces of the $k$-vectors and the $k$-covectors respectively. Consider an orthonormal basis $e_1, \cdots, e_n$ of $R^n$. One can identify the dual basis $e_1^*, \cdots, e_n^*$ with $e_1, \cdots, e_n$ (up to this point, the symbol * only means that we are considering covectors) . An arbitrary $p$-form $\varphi$ in $R^n$ has a unique expression $\varphi = \sum a_I e_I^*$, where $I = (i_1, \cdots, i_p), 1 \leq i_1 < \cdots < i_p \leq n$ and $e_I^* = e_{i_1}^* \wedge \cdots \wedge e_{i_p}^*$ . The comass of a $p$-covector $\varphi$ is defined by

$$\|\varphi\|^* = \sup\{\varphi(\epsilon) : \epsilon \text{ is any unit simple } p-\text{vector}\}$$

and the mass of a $p$-vector $\epsilon$ is defined by

$$\|\epsilon\| = \sup\{\varphi(\epsilon) : \|\varphi\|^* = 1\}.$$

Denote by span$(\varphi)$ the minimal subspace $V \subset R^n$ such that $\varphi \in \wedge^p(V)$. Then span $(\varphi) = \{v \in R^n : i_v(\varphi) = 0\}^\perp$. The rank $\varphi$ is the dimension of span $(\varphi)$. Let $\varphi$ and $S$ be a differential $p$-form and a $p$ -current in a Riemannian manifolds $M$ respectively. The comass of $\varphi$ is defined by $\|\varphi\|^* = \sup\{\varphi_x^* : x \in M\}$ and the mass of $S$ is defined by

$$M(S) = \sup\{S(\varphi) : \|\varphi\|^* = 1\}.$$

If $S$ is a surface of $M$, then $M(S) =$ volume $(S)$. For each $p$-vector $\varphi$ in $R^n$ the set of maximal directions of $\Omega$ is defined by

$$G(\varphi) = \{\epsilon \in \wedge_{p,n} : \varphi(\epsilon) = \|\varphi\|^*, \ \epsilon \text{ is unit simple }\}$$

and called the set of $\varphi$-maximal directions. If $\varphi$ is a differential $p$-form in a Riemannian manifolds $M$, then the set of $\varphi$-maximal directions is given by

$$G(\varphi) = \cup\{G(\varphi_x) : \|\varphi_x\|^* = \|\varphi\|^*\}.$$

A $p$-covector in $R^n$ can be considered as a parallel differential $p$-form in $R^n$. Let $\varphi$ be a $p$-form in $V \subset R^n$. Clearly $\varphi$ can be considered as a $p$-form in $R^n$ by identifying $\varphi$ with $\pi^*\varphi$ where $\pi$ is the orthogonal projection of $R^n$ on $V$.

## 3. Separable forms

In this section we present a class of forms, which is the main object of this paper. One can determine easily the comass and the maximal directions of each form in this class. The main reference is the beautiful fundamental paper of Havey R. and Lawson H.B. [HL]. To begin this section we recall the Havey-Lawson's Lemma of the canonical form of a simple $p$-vector (see [HL] for the proof).

3.1. LEMMA. *Suppose $V$ is a linear subspace of $R^n$ and $\epsilon$ is a unit simple $p$-vector. Then there exist a set of orthonormal vectors $e_1, \cdots, e_r$ in $V$, a set of orthonormal vectors $g_1, \cdots, g_s$ in $V^\perp$ and angles $0 < \theta_i < \pi/2$ for $i = 1, \cdots, k$ (where $k \le r, s \le p$ and $r + s - k = p$) such that*

$$\begin{aligned} \epsilon = (\cos\theta_1 e_1 + \sin\theta_1 g_1) \wedge \cdots \wedge (\cos\theta_k e_k + \sin\theta_k g_k) \\ \wedge e_{k+1} \wedge \cdots \wedge e_r \wedge g_{k+1} \wedge \cdots \wedge g_s. \end{aligned} \tag{3.1}$$

REMARKS. a) For the case $\dim V = q \le p$, we can take $s = q$ and $0 \le \theta_i \le \pi/2$ for $i \le q$ such that

$$\epsilon = (\cos\theta_1 e_1 + \sin\theta_1 g_1) \wedge \cdots \wedge (\cos\theta_q e_q + \sin\theta_q g_q) \wedge g_{q+1} \wedge \cdots g_p. \tag{3.2}$$

(If $\theta_i = 0$ (or $\pi/2$), then $g_i$ (or $e_i$) is only a formal symbol). b) For the case $\dim V = q \ge p$, we can take $s = r = p$ and $0 \le \theta_i \le \pi/2$ for $i \le q$ such that

$$\epsilon = (\cos\theta_1 e_1 + \sin\theta_1 g_1) \wedge \cdots \wedge (\cos\theta_p e_p + \sin\theta_p g_p). \tag{3.3}$$

(If $\theta_i = 0$ (or $\pi/2$), then $g_i$ (or $e_i$) is only a formal symbol).

The following theorem is the main result of this section.

3.2. THEOREM. *Let $R^n = V \oplus W$ ($\dim V = q \geq 2$) be an orthogonal decomposition of $R^n$ and $\Omega = \Omega_1 \oplus e_V^* \wedge \Omega_2$ a p-form such that $\Omega_1, \Omega_2$ are forms in $W$ (where $e_V^*$ is a unit simple q-covector of $V$). Then*

*a) $\|\Omega\|^* = \max(\|\Omega_1\|^*, \|\Omega_2\|^*)$*

*b) If $\|\Omega_1\|^* > \|\Omega_2\|^*$, then $G(\Omega) = G(\Omega_1)$*

*c) If $\|\Omega_1\|^* < \|\Omega_2\|^*$, then $G(\Omega) = \{e_V \wedge G(\Omega_2)\}$*

*d) If $\|\Omega_1\|^* = \|\Omega\|_2^*$ and $q \geq 3$, then $G(\Omega) = G(\Omega_1) \cup e_V \wedge G(\Omega_2)$*

*e) If $\|\Omega_1^*\| = \|\Omega_2^*\|$ and $q = 2$, then $G(\Omega) = G(\Omega_1) \cup \{e_V \wedge G(\Omega_2)\} \cup A(\Omega)$ where $A(\Omega) = \{(\cos\theta e_1 + \sin\theta g_1) \wedge (\cos\theta e_2 + \sin\theta g_2) \wedge \epsilon,\ \epsilon \in G(\Omega_2)$ and $g_1 \wedge g_2 \wedge \epsilon \in G(\Omega_1), 0 \leq \theta \leq \pi/2\}$ .*

PROOF. Let $\epsilon$ be any unit simple $p$-vector and put $\epsilon$ in canonical form (3.2) as in Lemma 3.1

$$\epsilon = (\cos\theta_1 e_1 + \sin\theta_1 g_1) \wedge \cdots \wedge (\cos\theta_q e_q + \sin\theta_q g_q) \wedge g_{q+1} \wedge \cdots \wedge g_p.$$

Then

$$\begin{aligned}\Omega(\epsilon) &= \cos\theta_1 \cdots \cos\theta_q \Omega_2(g_{q+1} \wedge \cdots \wedge g_p) + \sin\theta_1 \cdots \sin\theta_q \Omega_1(g_1 \wedge \cdots \wedge g_p)\\ &\leq (\cos\theta_1 \cdots \cos\theta_q + \sin\theta_1 \cdots \sin\theta_q) \max(\|\Omega_1\|^*, \|\Omega_2\|^*)\\ &\leq (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) \max(\|\Omega\|_1^*, \|\Omega_2\|^*)\\ &= \cos(\theta_1 - \theta_2) \max(\|\Omega_1\|^*, \|\Omega_2\|^*) \leq \max(\|\Omega\|_1^*, \|\Omega_2\|^*). \end{aligned} \tag{3.4}$$

Therefore, $\|\Omega\|^* \leq \max(\|\Omega_1\|^*, \|\Omega_2\|^*)$. The inverse inequalities are obvious. Thus

$$\|\Omega\|^* = \max(\|\Omega_1\|^*, \|\Omega_2\|^*).$$

*Case* $\|\Omega_1\|^* \leq \|\Omega_2\|^*$. Analogously, the inequalities in (3.4) are equalities iff

$$\epsilon = e_V \wedge \epsilon_1$$

where $\epsilon_1$ belongs to $G(\Omega_2)$.

*Case* $\|\Omega_1\|^* = \|\Omega_2\|^*$ *and* $q \geq 3$. The inequalities in (3.4) become equalities iff $\theta_1 = \cdots \theta_q = \pi/2$ or $\theta_1 = \cdots \theta_q = 0$ and $\epsilon = g_1 \wedge \cdots \wedge g_p \in G(\Omega_1)$ or $\epsilon = e_V \wedge \epsilon_1$ where $\epsilon_1 = g_{q+1} \wedge \cdots \wedge g_p \in G(\Omega_2)$. Thus,

$$G(\Omega) = G(\Omega_1) \cup \{e_V \wedge G(\Omega_2)\} \cup A(\Omega).$$

*Case* $\|\Omega_1\|^* = \|\Omega_2\|^*$ *and* $q = 2$ . The inequalities in (3.4) become equalities iff $\theta_1 = \theta_2 = \theta$ and $g_1 \wedge \cdots \wedge g_p \in G(\Omega_1), g_3 \wedge \cdots \wedge g_p \in G(\Omega_2)$ or $\epsilon \in G(\Omega_1) \cup \{e_V \cup G(\Omega_2)\}$. Thus

$$G(\Omega) = G(\Omega_1) \cup \{e_V \wedge G(\Omega_2)\} \cup A(\Omega).$$

The theorem is proved.

The following proposition is the second application of Lemma 3.1

3.3. Proposition. *Let $R^n = V \oplus W$ be an orthogonal decomposition of $R^n$ and $\Omega_1, \Omega_2$ two p-forms on V, W respectively ($p \geq 2$). Put $\Omega = \Omega_1 + \Omega_2$, then*

$$\|\Omega\|^* = \max(\|\Omega_1\|^*, \|\Omega_2\|^*).$$

*Moreover,*

*a) If $\|\Omega_1\|^* > \|\Omega_2\|^*$, then $G(\Omega) = G(\Omega_1)$*

*b) If $\|\Omega_1\|^* = \|\Omega_2\|^*, p \geq 3$ then $G(\Omega) = G(\Omega_1) \cup G(\Omega_2)$*

*c) If $\|\Omega_1\|^* = \|\Omega_2\|^*$ and $p = 2$, then $G(\Omega) = \{(\cos\theta e_1 + \sin\theta_1 g_1) \wedge (\cos\theta e_2 + \sin\theta_2 g_2)$ where $e_1 \wedge e_2 \in G(\Omega_1), g_1 \wedge g_2 \in G(\Omega_2)\}$.*

Proof. Let $\epsilon$ be any unit simple $p$-vector and put $\epsilon$ in canonical form (3.3) as in Lemma 3.1

$$\epsilon = (\cos\theta_1 e_1 + \sin\theta_1 g_1) \wedge \cdots \wedge (\cos\theta_p e_p + \sin\theta_p g_p)$$

where $0 \leq \theta_i \leq \pi/2$, $e_1, \cdots, e_p$ and $g_1, \cdots, g_2$ are orthonormal vectors in $V$ and $W$ respectivly. Then

$$\begin{aligned}\Omega(\epsilon) &= \cos\theta_1 \cdots \cos\theta_p \Omega_1(e_1 \wedge \cdots \wedge e_p) = \sin\theta_1 \cdots \sin\theta_p \Omega_2(g_1 \wedge \cdots \wedge g_p) \\ &\leq \cos(\theta_1 - \theta_2) \max(\|\Omega_1\|^*, \|\Omega_2\|^*) \leq \max(\|\Omega_1\|^*, \|\Omega_2\|^*). \qquad (3.5)\end{aligned}$$

Hence $\|\Omega\|^* \leq \max(\|\Omega_1\|^*, \|\Omega_2\|^*)$. The inverse inequality is clear . Therefore, $\|\Omega\|^* = \max(\|\Omega_1\|^*, \|\Omega_2\|^*)$.

*Case* $\|\Omega_1\|^* > \|\Omega_2\|^*$. The inequalities in (3.5) become equalities iff $e_1 \wedge \cdots \wedge e_p$ belongs to $G(\Omega_1)$ and $\theta_1 = \cdots = \theta_p = 0$. Therefore, $G(\Omega) = G(\Omega_1)$.

*Case* $\|\Omega_1\|^* = \|\Omega_2\|^*\|$ *and* $p \geq 3$. The inequalities in (3.5) are equalities iff

$$\theta_1 = \cdots = \theta_p = 0, \quad e_1 \wedge \cdots \wedge e_p \in G(\Omega_1)$$

or

$$\theta_1 = \cdots = \theta_p = \pi/2, \quad g_1 \wedge \cdots \wedge g_p \in G(\Omega_2).$$

It follows that $G(\Omega) = G(\Omega_1) \cup G(\Omega_2)$.

*Case* $\|\Omega_1\|^* = \|\Omega_2\|^*$ *and* $p = 2$. The inequalities in (3.5) are equalities iff $\theta_1 = \theta_2 = \theta$ and $e_1 \wedge e_2 \in G(\Omega_1)$, $g_1 \wedge g_2 \in G(\Omega_2)$. The proof is complete.

REMARK. For the case when $\Omega_1$ and $\Omega_2$ are decomposable forms, Proposition 3.3 is Harvey-Lawson's result (see [HL]).

3.4. DEFINITION. A given $p$-form $\Omega$ is said to be separable with respect to $V$ (or $V$-separable) iff $\Omega$ can be expressed as in Theorem 3.2.

3.5. THEOREM. *Let $\varphi$ and $\psi$ be $V$-separable. Then $*\varphi$ and $\varphi \wedge \psi$ are also $V$-separable (where $*\varphi$ is the * Horge of $\varphi$).*

PROOF. Since $\varphi = \varphi_1 + e_V^* \wedge \varphi_2$, we get that $*\varphi = *\varphi_2' + e_V^* \wedge (*\varphi_1')$ (where $*\varphi_i$ is * Horge of $\varphi_i$ in $V$ ). Therefore, $*\varphi$ is separable with respect to $V$. On the other hand, since $\psi = \psi_1 + e_V^* \wedge \psi_2$ we have

$$\varphi \wedge \psi = \varphi_1 \wedge \psi_1 + e_V^* \wedge (\varphi_2 \wedge \psi_1 + (-1)^r \varphi_1 \psi_2)$$

where $r =$ ( degree of$\varphi_1$).( dim $V$). It is evident that $\varphi_1 \wedge \psi_1$ and $\varphi_2 \wedge \psi_1 + (-1)^r \varphi_1 \wedge \psi_2$ are covectors in $V^\perp$ . Thus, $\varphi \wedge \psi$ is $V$-separable. The theorem is proved.

Now, we consider an orthogonal decomposition of $R^n$; $R^n = V_1 \oplus \cdots \oplus V_{k+1}$, dim $V_i \geq 2$ for $i \leq k$. Theorem 3.2 can be extended as follows.

3.6. THEOREM. *Let $\Omega$ be a $p$-form in $R^n$ which can be expressed in the form*

$$\Omega = \Omega_1 + e^*_{V_1} \wedge \Omega_2 + \cdots + e^*_{V_k} \wedge \Omega_{k+1}$$

*where* $\Omega_i \in \wedge q_i\ (W_i)$, $q_i = p - \sum_{t<i} \dim V_t$ *for* $i \leq k+1$, $W_i = \bigoplus_{t>i} V_t$ *for* $i \leq k$ *and* $W_{k+1} = V_{k+1}$ . *Then*

$$\|\Omega\|^* = \max(\|\Omega_i\|^*, i \leq k+1) \text{ and } G(\Omega) = G(\Omega')$$

*where* $\Omega'$ *is obtained from* $\Omega$ *by deleting those terms corresponding to* $\Omega_i$ *such that* $\|\Omega_i\|^* \leq \|\Omega\|^*$.

PROOF. With $k = 1$ this statement is just Theorem 3.2. Suppose theorem has been proved for $k = m-1$. Let us show that the statement is valid for $k = m$.

Indeed, we can express $\Omega$ as follows

$$\Omega = \Omega_1 + e^*_{V_1} \wedge (\Omega_2 + \cdots + e^*_{V_2} \wedge \cdots \wedge e^*_{V_m} \wedge \Omega_{m+1}) = \Omega_1 + e^*_{V_1} \wedge \overline{\Omega}$$

where $\overline{\Omega} = \Omega_2 + \cdots + e^*_{V_2} \wedge \cdots \wedge e^*_{V_m} \wedge \Omega_{m+1}$.

From Theorem 3.2 we have $\|\Omega\|^* = \max(\|\Omega_1\|^*, \|\overline{\Omega}'\|^*)$. But by induction

$$\|\overline{\Omega}\|^* = \max(\|\Omega_i\|^*, 2 \leq i \leq k+1) \quad \text{and} \quad G(\overline{\Omega}) = G(\overline{\Omega}').$$

Therefore,

$$\|\Omega\|^* = \max(\|\Omega_i\|^*, 1 \leq i \leq k+1) \text{ and } G(\Omega) = G(\Omega').$$

3.7. DEFINITION. A given $p$-form is said to be separable with respect to $(V_1, \cdots, V_k)$ iff $\Omega$ can be expressed as in Theorem 3.6. In this case, $\Omega$ can be called $(V_1, \cdots, V_k)$-separable.

3.8. THEOREM. *Let* $\Omega$ *be a* $(V_1, \cdots, V_k)$*-separable* $p$*-form in* $R^n$. *Then the following assertions hold.*

*a) For any* $q$*-form in* $R^m, \Omega \wedge \eta$ *is also* $(V_1, \cdots, V_k)$*-separable in* $R^{n+m}$.

*b) If* $\eta$ *is* $(W_1, \cdots, W_l)$*-separable in* $R^m$, *then* $\Omega \wedge \eta$ *is* $(V_1, \cdots, V_k, W_1, \cdots, W_l)$*-separable in* $R^{n+m}$.

*c) If $\eta$ is $(V_1, \cdots, V_k)$-separable in $R^m$, then $\Omega \wedge \eta$ is also $(V_1, \cdots, V_k)$-separable.*

PROOF. We can verify immediately by definition. Let us prove the assertion a). From

$$\Omega \wedge \eta = \Omega_1 \wedge + e^*_{V_1} \wedge \Omega_2 \wedge \eta + \cdots + e^*_{V_1} \wedge \cdots \wedge e^*_{V_k} \wedge \Omega_{k+1} \wedge \eta,$$

it follows that $\Omega \wedge \eta$ is $(V_1, \cdots, V_k)$-separable. The proofs of b) and c) are analogous.

Now, suppose that $R^n = V_1 \oplus \cdots \oplus V_k$ is an orthogonal decomposition of $R^n$. For any multi-index $I = (i_1, \cdots, i_q)$ we denote by $e^*_I$ the $p$-form $e^*_I = e^*_{V_{i_1}} \wedge \cdots \wedge e^*_{V_{i_q}}$ where $p = |I| = \sum_{j \in J} \dim V_j$.

3.9. DENIFITION.

a) A $p$-form $\Omega$ is said to be simply separable with respect to $(V_1, \cdots, V_k)$ if $\Omega$ can be expressed as follows

$$\Omega = \sum a_I e^*_I, \text{ where } \dim V_j \geq 2 \text{ for } j \leq k.$$

b) A $(V_1, \cdots, V_m)$-separable $p$-form $(m < k)$:

$$\Omega = \Omega_1 + \cdots + e^*_{V_1} \wedge \cdots \wedge e^*_{v_m} \wedge \Omega_{m+1}$$

is called $E$-separable if every $\Omega_i$ is simply separable with respect to a suitable decomposition of $W_i$ (where $W_i$ is defined as in Theorem 3.6). This expression is called an canonical form of $\Omega$.

For an arbitrary $E$-separable (or simply separable) $p$-form with a canonical form, we denote by $\overline{\Omega}$ the $p$-form which is formed from $\Omega$ by deleting the terms corresponding to $\Omega_i$ such that $\|\Omega_i\|^* < \|\Omega\|^*$( or $|a_I| < \|\Omega\|^*$).

3.10. THEOREM. *Let $\Omega$ be a simply separable $p$-form with respect to $(V_1, \cdots, V_k)$ in $R^n$. The following assertions are valid*

*a) $\|\Omega\|^* = \max(|a_I|)$ and $G(\Omega) = G(\overline{\Omega})$.*

*b) For any $q$-form $\eta$ in $R^m$, the comass and maximal directions of $(p+q)$-form $\Omega \wedge \eta$ in $R^n \times R^m$ are defined as follows*

$$\|\Omega \wedge \eta\|^* = \|\Omega\|^* \|\eta\|^*, \; G(\Omega \wedge \eta) = G(\Omega) \wedge G(\eta).$$

*c) If $p$ is even, then $\Omega^q$ is also simply separable for any $q$.*

*d) If $\operatorname{rank}\Omega = n$, then $*\Omega$ is also simply separable with respect to $(V_1, \cdots, V_k)$.*

PROOF. Since the proofs of assertions c) and d) are straightforward, we shall omit the details here. We will prove a) and b) by induction.

With $k = 1$, then $\Omega$ is decomposable and the assertions are trivial. Suppose the statement has been proved for $k = m - 1$. Let us show that the statement is valid for $k = m$.

In fact, $\Omega = e^*_{V_1} \wedge \sum_{1 \in I} a_I e^*_{I'} + \sum_{1 \notin I} a_I e^*_I$ where $I' = (i_2, \cdots, i_q)$ when $I = (1, i_2, \cdots, i_q)$. Put $\Omega_1 = \sum_{1 \notin I} a_I e^*_I$ and $\Omega_2 = \sum_{1 \in I} a_I e^*_{I'}$ . Then $\Omega_1$ and $\Omega_2$ are simply separable with respect to $(V_2, \cdots, V_k)$. By induction

$$\begin{aligned} \|\Omega_1\|^* &= \max(|a_I|, 1 \in I) \quad \text{and} \quad G(\Omega_1) = G(\overline{\Omega}_1), \\ \|\Omega_2\|^* &= \max(|a_I|, 1 \in I) \quad \text{and} \quad G(\Omega_2) = G(\overline{\Omega}_2), \\ \|\Omega_i \wedge \eta\|^* &= \|\Omega_i\|^* \|\eta\|^* \quad \text{and} \quad G(\Omega_i \wedge \eta) = G(\Omega_i) \wedge G(\eta), \ i = 1, 2. \end{aligned}$$

It follows from Theorem 3.2 that

$$\begin{aligned} \|\Omega\|^* &= \max(a_I) \quad \text{and} \quad G(\Omega) = G(\overline{\Omega}), \\ \|\Omega \wedge \eta\|^* &= \|\Omega\|^* \|\eta\|^* \quad \text{and} \quad G(\Omega \wedge \eta) = G(\Omega) \wedge G(\eta). \end{aligned}$$

The theorem is proved.

REMARK. Using Theorem 3.6 and assertion a) we can define the comass and the maximal directions of an $E$-separable form.

Now, let $R^n = V \oplus W$ be an orthogonal decomposition of $R^n$, $\Omega$ an $E$-separable $p$-form in $V$ (with respect to $(V_1, \cdots, V_k)$ and $\eta$ an arbitrary $q$-form in $W$ we will prove the following theorem is valid

3.11. THEOREM. *The following assertions hold:*

*a) $\|\Omega\| \wedge \eta\|^* = \|\Omega\|^* \|\eta\|^*$ and $G(\Omega \wedge \eta) = G(\Omega) \wedge G(\eta)$.*

*b) If $\eta$ is $E$-separable then $\Omega \wedge \eta$ is also $E$-separable* *c) If $\varphi$ is a $(p+q)$-form in $W$, then $\|\Omega \wedge \eta + \varphi\|^* = \max(\|\Omega\|^*\|\eta\|^*, \|\varphi\|^*)$ and $G(\Omega \wedge \eta) \cup G(\varphi) \subset G(\Omega \wedge \eta + \varphi)$.*

PROOF. Since the proofs of assertions b) and c) are straightforward from definition and Theorem 3.2, we shall omit the details here. We will prove a).

*Case* $k = 1$. We have $\Omega = \Omega_1 + e^*_{V_1} \wedge \Omega_2$, where $V = V_1 + V_2$ and $\Omega_1, \Omega_2$ are simply separable in $V_2$. It follows from Theorem 3.10 that $\|\Omega_i \wedge \eta\|^* = \|\Omega_i\|^*\|\eta\|^*$ and $G(\Omega_i \wedge \eta) = G(\Omega_i) \wedge G(\eta)$ for $i = 1, 2$. By Theorem 3.2 we have

$$\|\Omega \wedge \eta\|^* = \|\Omega\|^*\|\eta\|^* \text{ and } G(\Omega \wedge \eta) = G(\Omega) \wedge G(\eta).$$

Suppose the statement has been proved for $k = m - 1$. Let us show that the statement is valid for $k = m$. Indeed,

$$\Omega \wedge \eta = \Omega_1 \wedge \eta + e^*_{V_1} \wedge (\Omega_2 + \cdots + e^*_{V_2} \wedge \cdots e^*_{V_m} \wedge \Omega_{m+1}) \wedge \eta$$

Put $\Omega' = \Omega_2 + \cdots + e^*_{V_2} \wedge \cdots \wedge e^*_{V_m} \wedge \Omega_{m+1}$. By induction, we have

$$\|\Omega_1 \wedge \eta\|^* = \|\Omega_1\|^*\|\eta\|^* \text{ and } G(\Omega_1 \wedge \eta) = G(\Omega_1) \wedge G(\eta).$$

$$\|\Omega' \wedge \eta\|^* = \|\Omega'\|^*\|\eta\|^* \text{ and } G(\Omega' \wedge \eta) = G(\Omega') \wedge G(\eta).$$

It follows from Theorem 3.2 that

$$\begin{aligned}\|\Omega \wedge \eta\|^* &= \max(\|\Omega_1\|^*\|\eta\|^*, \|\Omega'\|^*\|\eta\|^*)\\ &= \|\eta\|^* \max(\|\Omega_1\|^*, \|\Omega'\|^*) = \|\Omega\|^*\|\eta\|^*\end{aligned}$$

and $G(\Omega \wedge \eta)$ is determined by $G(\Omega_1 \wedge \eta)$ and $G(e^*_{V_1} \wedge \Omega' \wedge \eta)$ as mentioned above. It is straightforward that $G(\Omega \wedge \eta) = G(\Omega) \wedge G(\eta)$. This completes the proof.

3.12. THEOREM. *Let $\varphi$ be $V$-separable. Then $V \subset \operatorname{span}\varphi$. In particular if $\varphi$ is simply separable with respect to $(V_1, \cdots, V_k)$ then $\operatorname{span}\varphi = V_1 \oplus \cdots \oplus V_k$.*

PROOF. Let $e$ be any normal vector of $R^n$ such that $i_e(\varphi) = 0$. Put $e$ in the canonical form as in Lemma 3.1 with respect to $V$: $e = \cos\theta e' + \sin\theta g$ where $e' \in V$ and $g \in V^\perp$. When $\varphi = e^*_V \wedge \Omega_1 + \Omega_2$, then $i_e(\varphi) = \cos\theta i_{e'}(e^*_v) \wedge \Omega_1 - \sin\theta e^*_V \wedge i_g(\Omega_1) + \sin\theta i_g(\Omega_2)$. Since $\dim V \geq 2$, it follows that when $i_e(\Omega) = 0$

then $\cos\theta = 0$ and $e \in V^{\perp}$. Thus $V \subset$ span $\varphi$. Obviously, it implies that if $\varphi$ is simply separable with respect to $(V_1, \cdots, V_k)$, then $\operatorname{span}\varphi = V_1 \oplus \cdots \oplus V_k$.

## 4. Other properties of separable forms

Let $\Omega$ be a non-zero $p$-form in $R^n$. The action of the group $SO_n$ over $R^n$ induces an action over $p$-form $\Omega$ as follows:

$$A(\Omega) = A^*\Omega \text{ for any } A \text{ of } SO_n.$$

We denote by $T(\Omega)$ the isotopy group of $\Omega$ and observe the following fact:

4.1. THEOREM. *$T(\Omega) = SO_n$ if and only if $p = n$.*

PROOF. If $p = n$, then it is clear that $T(\Omega) = SO_n$. We suppose that $p$ is smaller then $n$. Consider an arbitrary $p$-vector $\epsilon$ of $G(\Omega)$. Thus, $\Omega(\epsilon) = \|\Omega\|^*$ and $\epsilon = e_1 \wedge \cdots \wedge e_p$ where $e_1, \cdots, e_p$ are orthonormal. Since $p < n$, one can take a normal vector $e$ such that $e, e_1, \cdots, e_p$ are orthonormal and define the transformation $A \in SO_n$ as follows: $A(e) = e_1$ and $A$ is identical action on $(\operatorname{span}(e, e_1))^{\perp}$. Put $h = e \wedge e_2 \wedge \cdots \wedge e_p$, then

$$\Omega(h) = \Omega(A(\epsilon)) = A^*\Omega(\epsilon) = \|\Omega\|^*.$$

Hence, $h \in G(\Omega)$. Now, we take

$$\epsilon' = \frac{\sqrt{2}}{2}(\epsilon + h) = \frac{\sqrt{2}}{2}(e_1 + e) \wedge e_2 \wedge \cdots \wedge e_p.$$

Thus, $\epsilon'$ is unit simple and $\Omega(\epsilon') = \frac{\sqrt{2}}{2}(\Omega(\epsilon) + \Omega(h)) = \sqrt{2}\Omega(\epsilon) = \sqrt{2}\|\Omega\|^*$. But $\Omega(\epsilon') \leq \|\Omega\|^*$, then $\|\Omega\|^* = 0$ and $\Omega = 0$. This is impossible. Thus $p = n$.

Now, let $V$ be a $q$-dimensional subspace of $R^n, \varphi \in SO_n$ a transformation on $V$. One can consider $\varphi$ as a transformation on $R^n$ by extending it identically on $V^{\perp}$.

4.2. THEOREM. *A $p$-form $\Omega$ is $V$-separable if and only if $\Omega$ is $SO_q$-invariant on $V$.*

PROOF. Let $\Omega$ be separable with respect to $V$. Then $\Omega = \Omega_1 + e_V^* \wedge \Omega_2$, where $\Omega_1$ and $\Omega_2$ are the forms in $W = V^{\perp}$. It is obvious that $\Omega_1$ and $\Omega_2$ are

$SO_q$-invariant; Therefore, $\Omega$ is $SO_q$-invariant on $V$. Conversely, suppose that $\Omega$ is $SO_q$-invariant on $V$ and $e_1, \cdots, e_q$ are an orthonormal basis of $V$. We choose vectors $e_{q+1}, \cdots, e_n$ such that $e_1, \cdots, e_n$ form an orthonormal basis of $R^n$. Then $\Omega$ can be expressed as follows

$$\Omega = \sum_{t \leq p} e^*_{j_1} \wedge \cdots \wedge e^*_{j_t} \wedge \Omega_{j_1, \cdots, j_t}$$

where $q + 1 \leq j_i \leq n$ for $i \leq t$ and $\Omega_{j_1, \cdots, j_t} \in \wedge^*(V)$.

Since $\Omega$ is $SO_q$ invariant on $V$, then $\Omega_{j_1, \cdots j_t}$ is also $SO_q$-invariant on $V$ for every $(j_1, \cdots, j_t)$. It follows from Theorem 4.1 that $\Omega_{j_1, \cdots, j_t} = 0$ for every $(j_1, \cdots, j_t)$ such that $0 < p - t < q$. Therefore, $\Omega$ can be expressed in the form

$$\Omega = \Omega_1 + e^*_V \wedge \Omega_2.$$

The theorem is proved.

Now, we consider an orthogonal decomposition of $R^n$:

$$R^n = V_1 \oplus \cdots \oplus V_k, \ \dim V_i = q_i \text{ for } 1 \leq i \leq k.$$

4.3. THEOREM. *A p-form $\Omega$ is simply separable with respect to $(V_1, \cdots, V_k)$ if and only if $\Omega$ is $SO_{q_1} \times \cdots \times SO_{q_k}$-invariant.*

PROOF. If $\Omega$ is simply separable, then obviously $\Omega$ is $SO_{q_1} \times \cdots \times SO_{q_k}$-invariant. Assume that $\Omega$ is $SO_{q_1} \times \cdots \times SO_{q_k}$-invariant. We will prove that $\Omega$ is simply separable with respect to $(V_1, \cdots, V_k)$. With $k = 1$ this statement is just Theorem 4.1. Suppose the statement has been proved for $k = m - 1$. Let us show that the statement is valid for $k = m$.

Indeed, since $\Omega$ is $SO_{q_1}$-invariant on $V_1$, it follows that $\Omega = \Omega_1 + e^*_{V_1} \wedge \Omega_2$ by Theorem 4.2. But $\Omega$ is $SO_{q_2} \times \cdots \times SO_{q_m}$-invariant on $V_2 + \cdots + V_m$, then $\Omega_1$ and $\Omega_2$ are also $SO_{q_2} \times \cdots \times SO_{q_m}$-invariant. By induction $\Omega_1$ and $\Omega_2$ are simply separable with respect to $(V_2, \cdots, V_m)$. Thus, $\Omega$ is simply separable with respect to $V_1, \cdots, V_m$. This completes the proof.

4.4. LEMMA. *Let $\Omega = \Omega_1 + e_V^* \wedge \Omega_2$ be a separable form with respect to $V$. Then for any normal vector $e$ of $V$, we have $\Omega_1 = i_e(e^* \wedge \Omega)$.*

PROOF. We choose an orthonormal basis of $V$: $e_1, \cdots e_q$ such that $e_1 = e$. Then

$$\Omega = \Omega_1 + e^* \wedge e_2^* \cdots \wedge e_q^* \wedge \Omega_2 \quad \text{and} \quad e^* \wedge \Omega = e^* \wedge \Omega_1.$$

Hence, $i_e(e^* \wedge \Omega_1) = \Omega_1$.

4.5. THEOREM. *Let $\Omega$ be separable with respect to both $V_1$ and $V_2$, where $V_1 \cap V_2 \neq \{0\}$. Then $\Omega$ is separable with respect to $W = V_1 + V_2$.*

PROOF. Let $\Omega = \Omega_1 + e_{V_1}^* \wedge \Omega_2 = \Omega_1' + e_{V_2}^* \wedge \Omega_2'$, where $\Omega_1, \Omega_2$ are two forms in $V_1^\perp$ and $\Omega_1', \Omega_2'$ are two forms in $V_2^\perp$. Taking a normal vector $e$ of $V_1 \cap V_2$ , we get $\Omega_1 = \Omega_1' = i_e(e^* \wedge \Omega)$. Then $\Omega_1$ is a form in $V_1^\perp \cap V_2^\perp = (V_1 + V_2)^\perp = W^\perp$. On the other hand, put $\Omega' = \Omega - \Omega_1$, then

$$\Omega' = e_{V_1}^* \wedge \Omega_2 = e_{V_2}^* \wedge \Omega_2'.$$

we will show that $\Omega' = e_W^* \wedge \eta$, where $\eta$ is a form in $W^\perp$. The statement will be proved by induction on $\operatorname{codim} V_1$ in $W$.

Suppose that $\operatorname{codim} V_1 = 1$. Let $e$ be a normal vector of $W$ such that $e \perp V_1$. Then $\Omega'$ can be expressed as follows

$$\Omega' = e_{V_1}^* \wedge \Omega_2 = e_{V_1}^* \wedge (e^* \wedge \eta_1 + \eta_2),$$

where $\eta_1$ and $\eta_2$ are two forms in $W^\perp$. Let us show that $\eta_2 = 0$. In fact, $e = f_1 + f_2$ where $f_1 \in V_1$ and $f_2 \in V_2$. On the other hand

$$f_1^* \wedge \Omega' = f_1 \wedge e_{V_1}^* \wedge \Omega_2 = 0,$$
$$f^* \wedge \Omega' = f_2 \wedge e^* \wedge \Omega_2 = 0.$$

Thus,

$$e^* \wedge \Omega' = e^* \wedge e_{V_1}^* \wedge \eta_2 = f_1 \wedge \Omega' + f_2 \wedge \Omega' = 0.$$

This follows that $\eta_2 = 0$.

Now, suppose the statement has been proved for $\operatorname{codim} V_1 = k$ in $W$. Let us show that the statement is valid for $\operatorname{codim} V_1 = k + 1$ in $W$.

Indeed, for an arbitrary vector $e$ of $V_2$ such that $e \notin V_1$ we denote by $V_2' = \{g \in V_2, g \perp e\}$ and $W' = V_1 + V_2'$. Then $\Omega$ is $V_2'$-separable and codim $V_1 = k$ in $W'$. By induction $\Omega$ is $W'$-separable. But codim $W' = 1$ in $W = W' + V_2$. Then $\Omega$ is $W$-separable. The theorem is proved.

4.6. LEMMA. *Let $\Omega$ be a non-zero $k$-form in $V$ and $V = V_1 + V_2$, dim $V_1 = q \geq 3$, $k < m =$ dim $V$. If $\Omega$ is separable with respect to both $V_1$ and $V_2$, then $V_1 \perp V_2$.*

PROOF. Without loss of generality, we can suppose dim $V_1 \geq \dim V_2$. Since $\Omega$ is $V_1$-separable, we get $q \leq k < m \leq 2q$ and $\Omega = \Omega_1 + e^*_{V_1} \wedge \Omega_2$ where $\Omega_1$ and $\Omega_2$ are forms in $V_1^\perp$.

Assume that $\Omega_1 \neq 0$. As $q \leq k =$ degree of $\Omega_2 \leq \dim V_1^\perp = \dim V_2 \leq q$, we have dim $V_2 = k = q$ and $\Omega = \mu e^*_{V_1} + \lambda e^*_{V_1^\perp}$. But dim $V_1 \geq 3$, it follows from Theorem 3.2 that

$$G(\Omega) \subset \{e_{V_1}, e_{V_1^\perp}\}. \tag{4.1}$$

Analogously, we get $\Omega = \mu' e^*_{V_2} + \lambda' e^*_{V_2^\perp}$ and

$$G(\Omega) \subset \{e_{V_2}, e_{V_2^\perp}\}. \tag{4.2}$$

Since $V_1 \cap V_2 = 0$, it follows from (4.1) and (4.2) that $V_1 \perp V_2$.

Assume that $\Omega_1 = 0$. We have

$$\Omega = e^*_{V_1} \wedge \Omega_2 = \Omega_1' + e^*_{V-2} \wedge \Omega_2',$$

where $\Omega_1'$ and $\Omega_2'$ are forms in $V_2^\perp$. It follows from Theorem 3.2 that $G(\Omega) = e_{V_1} \wedge G(\Omega_2)$ and $G(\Omega_1') \subset G(\Omega)$ or $e_{V_2} \wedge G(\Omega_2') \subset G(\Omega)$. Let $G(\Omega_1') \subset G(\Omega) = e_{V_1} \wedge G(\Omega_2)$, then $V_1 \subset \operatorname{span}(\Omega_1') \subset V_2^\perp$. This implies $V_1 \perp V_2$. On the other hand, when $e_{V_2} \wedge G(\Omega_2') \subset G(\Omega)$ there exists a unit simple $k$-vector $\epsilon$ of $G(\Omega)$ such that $\epsilon = e_{V_1} \wedge \epsilon_1 = e_{V_2} \wedge \epsilon_2$, $\epsilon_1 \in G(\Omega_2)$, $\epsilon_2 \in G(\Omega_2)$. Since $k < m$ there exists a normal vector $e$ of $V$ such that $\epsilon \wedge e \neq 0$. But $V = V_1 + V_2$ then there exist two vectors $f_1 \in V_1$, $f_2 \in V_2$ such that $e = f_1 + f_2$. Moreover, $e \wedge \epsilon = f_1 \wedge \epsilon + f_2 \wedge \epsilon = f_1 \wedge e_{V_1} \wedge \epsilon_1 = f_2 \wedge e_{V_2} \wedge \epsilon_2 = 0$. This is a contradiction. The proof is complete.

4.7. THEOREM. . *Let a p-form $\Omega$ be separable with respect to both $V_1$ and $V_2$. If $\dim V_1 \geq 3$, then $V_1 \perp V_2$ or $\Omega$ is separable with respect to $V = V_1 + V_2$.*

PROOF. Put $W = V^{\perp}$, $\dim W = m$ and take an orthonormal basis $e_1, \cdots, e_m$ of $W$. Then $\Omega$ can be expressed as follows

$$\Omega = \Omega_1 + \sum_{t<p} e^*_{j_1} \wedge \cdots e^*_{j_t} \wedge \Omega_{j_1,\cdots,j_t}$$

where $\Omega_1, \Omega_{j_1,;j_t}$ are forms in $V$ and separable with respect to both $V_1, V_2$ for any $(j_1, \cdots, j_t)$. By Lemma 4.6 it follows that $V_1 \perp V_2$ or $\Omega_1$ and $\Omega_{j_1,\cdots,j_t}$ are separable with respect to $V$ for any $(j_1, \cdots, j_t)$. Thus, $V_1 \perp V_2$ or $\Omega$ is separable with respect to $V$. The theorem is proved.

Evidently, if $\Omega$ is simply separable with respect to $(V_1, \cdots, V_k)$, then $\Omega$ is also separable with respect to $V_t$ for any $t \leq k$.

4.8. DEFINITION. Let $\Omega$ be simply separable with respect to $(V_1, \cdots, V_k)$ where $\dim V_1 \geq \cdots \geq \dim V_k$. The expression of $\Omega$ with respect to $(V_1, \cdots, V_k)$ is said to be extremal if for every $t \leq k$ there is not any subspace $V \neq V_t, V_t \subset V$ such that $\Omega$ is separable with respect to $V$.

4.9. THEOREM. *Suppose that $\Omega$ has two extremal expressions with respect to $(V_1, \cdots, V_k)$ and $(W_1, \cdots, W_k)$. Then $k = k'$ and $\dim V_t = \dim W_t$ for every $t \leq k$. Moreover, we can reorder $(W_1, \cdots, W_k)$ such that $W_t = V_t$ for every $t$ with $\dim V_t \geq 3$.*

PROOF. By Theorem 3.12, if $\dim V_t = \dim W_s = 2$ for every $t \leq k$ and $s \leq k'$, then $k = k' = (\operatorname{rank} \Omega)/2$. The statement is evident.

Let $\dim V_t \geq 3$. We will prove that there exists a $W_s$ such that $V_t = W_s$.

Indeed, it follows from Theorem 4.7 that $V_t \perp W_i$ for any $i \leq k'$ or there exists a $W_s$ such that $\Omega$ is $(V_t + W_s)$-separable. But $V_t \subset W_1 + \cdots + W_{k'}$, then the first case is impossible. On the other hand, the expressions of $\Omega$ with respect to $(V_1, \cdots, V_k)$ and $(W_1, \cdots, W_{k'})$ are extremal, then $V_t = W_s$. By the same argument we get that if $\dim W_i \geq 3$, then there exists a $V_j$ such that $W_i = V_j$. But $V_1 + \cdots + V_k = W_1 + \cdots + W_{k'} = \operatorname{span} \Omega$, then $k = k'$ and $\dim V_t = \dim W_t$ for every $t \leq k$. Moreover, by reordering $(W_1, \cdots, W_k)$ we get $V_t = W_t$ for $\dim V_t \geq 3$. This completes the proof.

## 5. Some special cases

In order to illustrate the set of separable forms, we consider some examples.

### 5.1. Complex case.

Let $C^n = R^{2n}$ be a $n$-dimensional complex space with the complex structure J. Any $(p,q)$-form $\Omega$ in $C^n$ can be expressed as follows

$$\Omega = \sum a_{I,K} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge \overline{dz_{k_1}} \wedge \cdots \wedge \overline{dz_{k_q}} = \operatorname{Re}\Omega + i \operatorname{Im}\Omega.$$

**Proposition.** *Suppose that $\Omega$ has an expression of the form*

$$\Omega = e_1^* \wedge Je_1^* \wedge \Omega_1 + \Omega_2$$

*where $\Omega_1$ and $\Omega_2$ are forms in $C^{n-1} = (e_1, Je_1)^{\perp}$ . Then Re $\Omega$ and Im $\Omega$ are separable with respect to $V =$ span $(e_1, Je_1)$. In particular, if*

$$\Omega = \sum a_I dz_{i_1} \wedge \overline{dz_{i_1}} \wedge \cdots \wedge dz_{i_p} \wedge \overline{dz_{i_p}},$$

*then Re $\Omega$ and Im $\Omega$ are simply separable with respect to $(V_1, \cdots, V_n)$ where $V_j =$ span $(e_j, Je_j), z_i = e_j + iJe_j$.*

**Proof.** Since $\Omega = e_1^* \wedge Je_1^* \wedge \Omega_1 + \Omega_2$, then evidently $\Omega$ is $SO_2$-invariant on $V$ and so are Re $\Omega$, Im $\Omega$. It follows from Theorem 4.2 that Re $\Omega$ and Im $\Omega$ are separable with respect to $V$.

Let

$$\Omega = \sum a_I dz_{i_1} \wedge \overline{dz_{i_p}} \wedge \cdots \wedge dz_{i_p} \wedge \overline{dz_{i_p}}.$$

Obviously, $\Omega$ is $SO_2 \times \cdots \times SO_2$-invariant on $V_1 + \cdots + V_n$ and so are Re $\Omega$, Im $\Omega$. In view of Theorem 4.3 we get that Re $\Omega$ and Im $\Omega$ are simply separable with respect to $(V_1, ..., V_n)$.

**Remark.** For the case $\Omega$ is a $(p,p)$-form in this theorem, the comass of $\Omega$ has been determined by Le Hong Van in [L].

5.2. CASE $p=2$ OR $p=n-2$.

PROPOSITION. *i) Let $\Omega$ be a 2-form on $R^n$. Then $\Omega^m$ is simply separable for $m \leq n$.*
*ii) Let $\Omega$ be a $(n-2)$-form on $R^n$. Then $\Omega$ is simply separable or $\Omega = e^* \wedge \Omega_1$ where $\Omega_1$ is simply separable and $e \perp span(\Omega_1)$.*

PROOF. i) Let be a 2-form. Evidently there exists an orthonormal basis of $R^n : e_1, \cdots, e_n$ such that

$$\Omega = \lambda_1 e_1^* \wedge e_2^* + \cdots + \lambda_k e_{2k-1}^* \wedge e_{2k}^*, \quad k \leq n.$$

Thus, $\Omega$ is simply separable. By Theorem 3.10 we get that $\Omega^m$ is simply separable for $m \leq n$.

ii) Let $\Omega$ be a $(n-2)$-form. Then $*\Omega$ is a 2-form and can be expression in the form

$$*\Omega = \lambda_1 e_1^* \wedge e_2^* + \cdots + \lambda_k e_{2k-1}^* \wedge e_{2k}^*, \quad k \leq n.$$

for an orthonormal basis $e_1, \cdots, e_n$ of $R^n$. Put $V_t =$ span $(e_{2t-1}, e_{2t})$ for $t \leq k$ and $W = (V_1 + \cdots + V_k)^{\perp}$. If dim $W \geq 2$, then $\Omega = *(*\Omega)$ is simply separable with respect to $(V_1, \cdots, V_k, W)$. If dim $W = 1$, then $\Omega = e^* \wedge \Omega_1$ where $e$ is a normal vector of $W$, $\Omega_1$ is the *Horge of $*\Omega$ restricted on $W^{\perp}$. Evidently, $\Omega_1$ is simply separable with resspect to $(V_1, \cdots, V_k)$. This completes the proof.

5.3. CASE $n=5, n=7$.

PROPOSITION. *An arbitrary form in $R^5$ is simply separable and an arbitrary separable form in $R^7$ is E-separable.*

PROOF. In view of Proposition 5.2, the case $\Omega$ is form in $R^5$ is clear. Let $\Omega$ be separable in $R^7$. Then $\Omega = e_V^* \wedge \Omega_1 + \Omega_2$, $\dim V \geq 2$, $\Omega_1$ and $\Omega_2$ are the forms in $V^{\perp}$. But $\dim V^{\perp} \leq 5$ then $\Omega_1$ and $\Omega_2$ are simply separable. The proposition is proved.

## 6. Minimal surfaces in almost-product manifolds.

The prior calibrations were used by H. Federer [F2] and M. Berger [B]. The calibration method was introduced first by Dao Trong Thi [D1, D2] and later studied in deep by Havey-Lawson [HL] in order to study globally minimal currents and surfaces on Riemannian manifolds. The principle of this method can be described as follows. A current $S$ in a Riemannian manifolds $M$ is homologically mimimal if and only if there exists a closed form $\Omega$ such that the tangent space $S_x$ of $S$ belongs to $G(\Omega)$ almost every where. In this case we say that $S$ is calibrated by $\Omega$ or $\Omega$ calibrates $S$ on $M$. A differential closed $p$-form is said to be a calibration. Thus, a $p$-form in $R^n$ is also a parallel calibration in $R^n$. If a surface $S$ is a minimal current, then it is also a minimal surface.

To use this method, the main obstacle is the computation of the comass of $\Omega$ and the determination of the set $G(\Omega)$. The results in previous sections enable us to study minimal currents and surfaces in almost-product manifolds. In particular, we obtain the following results.

6.1. THEOREM. *Let $\varphi$ and $\psi$ be two differential forms on $M$ such that span $(\varphi_x) \perp$ span $(\psi_x)$ and $\varphi_x$ or $\psi_x$ is $E$-separable at every $x \in M$. Then*

$$\|\varphi \wedge \psi\|^* \leq \|\varphi\|^* \|\psi\|^*.$$

*In particular, if $\varphi$ and $\psi$ are two calibrations on $M$ and $N$ respectively such that $\varphi_x$ or $\psi_y$ is $E$-separable at every $x \in M$ or every $y \in N$, then $\varphi \wedge \psi$ is a calibration in $M \times N$ and*

$$\|\varphi \wedge \psi\|^* = \|\varphi\|^* \|\psi\|^*, \quad G(\varphi \wedge \psi) = G(\varphi) \wedge G(\psi).$$

PROOF. In view of Theorem 3.11 we get that

$$\|\varphi_x \wedge \psi_x\|^* = \|\varphi_x\|^* \|\psi_x\|^* \text{ for } x \in M$$

Thus,

$$\begin{aligned}\|\varphi \wedge \psi\|^* &= \sup \|\varphi_x\| \wedge \|\psi_x\|^* = \sup \|\varphi_x\|^* \|\psi_x\|^* \\ &\leq \sup \|\varphi_x\|^* \sup \|\psi_x\|^* = \|\varphi\|^* \|\psi\|^*.\end{aligned}$$

On the other hand, if $\varphi$ and $\psi$ are calibrations on $M$ and $N$ respectively as mentioned above, then $\varphi \wedge \psi$ is a calibration on $M \times N$. Moreover, $\|\varphi_x \wedge \psi_y\|^* = \|\varphi_x\|^* \|\psi_y\|^*$ and $G(\varphi_x \wedge \psi_y) = G(\varphi_x) \wedge G(\psi_y)$ at every $(x, y)$ such that $\|\varphi_x\|^* = \|\varphi\|^*$ and $\|\psi_y\|^* = \|\psi\|^*$. Hence

$$\|\varphi \wedge \psi\|^* = \|\varphi\|^* \|\psi\|^*, \quad G(\varphi \wedge \psi) = G(\varphi) \wedge G(\psi).$$

6.2. THEOREM. *Let $S$ and $T$ be two minimal currents in $M$ and $N$ respectively. If $S$ is calibrated by a calibration $\varphi$ such that $\varphi_x$ is $E$-separable for every $x \in M$, then $S \times T$ is minimal in $M \times N$.*

PROOF. Since $T$ is minimal in $N$, then there exists a calibration $\psi$ which calibrates $T$ on $N$ (see [D.2, Theorem 3.6]). It follows from Theorem 6.1 that $G(\varphi \wedge \psi) = G(\varphi) \wedge G(\psi)$. On the other hand, $\vec{S}_x \in G(\varphi)$ and $\vec{S}_y \in G(\psi)$ at every $(x, y) \in S \times T$. But $\overrightarrow{(S \times T)}_{(x,y)} = \vec{S}_x \times \vec{T}_y$. Then $\overrightarrow{(S \times T)}_{(x,y)}$ belongs to $G(\varphi \wedge \psi)$ for every $(x, y) \in S \times T$. Hence $S \times T$ is minimal in $M \times N$. The theorem is proved.

6.3. EXAMPLES.

*Example 1.* Let $S$ be a $p$-complex surface in $R^{2n} \approx C^n$ and $T$ any minimal current in $N$. Then $S \times T$ is minimal in $R^{2n} \times N$. Indeed, $S$ is calibrated by $\Omega^p$ where $\Omega$ is Kahler form. By Proposition 5.1, $\Omega$ is simply separable. In view of Theorem 6.2 $S \times T$ is minimal.

*Example 2.* Let $S$ be a $(n-2)$-dimensional current of a $n$-dimensional manifolds $M, T$ an arbitrary minimal current of $N$. By Proposition 5.2 and Theorem 6.2, $S \times T$ is minimal in $M \times N$.

*Example 3.* Let $S$ be a minimal current in $R^5$, $T$ an arbitrary minimal current in $N$. By Proposition 5.3 and Theorem 6.2, $S \times T$ is minimal in $R^5 \times N$.

REMARK. The above examples are also valid for the case that $S$ is a surface and $T$ is a minimal surface calibrated by an arbitrary calibration $\psi$.

## REFERENCES

[B] M. Berger, *Du côte chez Pu*, Ann. Scient Ec. Norm. Sup. **4** (1972), 1-44.

[D1] Dao Trong Thi, *Minimal currents on compact manifolds*, Izv. Akad. Nauk SSSR Ser. Math. **41** (1977), 853-867 (in Russian).

[D2] Dao Trong Thi, *Globally minimal currents and surfaces in Riemannian manifolds*, Acta Math. Vietnam. **10** (1985), 296-333.

[DHM] J. Dadok, R. Harvey and F. Morgan, *Calibrations on $R^8$*, Trans. Amer. Math. Soc. **305** (1988), 1-39.

[F1] H. Federer, *Some theorem on integral currents*, Trans. Amer. Math. Soc. **117** (1965), 43-67.

[F2] H. Federer, "Geometric Measure Theory," Berlin Springer, 1969.

[FF] H. Ferderer and W. H. Fleming, *Normal and integral currents*, Annals of Math. **72** (1966), 488-520.

[G] C. Godbillon, "Differential Geometry and Analytical Mechanics," Paris, Hermann 1969.

[H] Hoang Xuan Huan, *A class of calibrated forms on f-manifolds*, Acta Math. Vietnam. **16** (1991), 155-170.

[HL] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47-157.

[L] Le Hong Van, *Minimal surfaces on homogenous spaces*, Izv. Akad USSR, Ser. Math. **52** (1988), 1-39.

FACULTY OF MATHEMATICS, MECHANICS AND INFORMATICS
UNIVERSITY OF HANOI, VIETNAM