ACTA MATHEMATICA VIETNAMICA
Volume 19, Number 1, 1994, pp. 61–66

A CHARACTERIZATION OF SOME CUBIC (m, n)-METACIRCULANT GRAPHS

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Abstract. It has been proved in [5] that if a graph G is isomorphic to a cubic (m, n)-metacirculant graph $MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ with $S_0 \neq \emptyset$, then G is isomorphic to either a union of finitely many disjoint copies of a circulant graph $C(2\ell, S)$, where $\ell > 1$ and $S = \{1, -1, \ell\}$ or a union of finitely many disjoint copies of a generalized Petersen graph $GP(d, k)$, where $d > 2$ and $k^2 \equiv \pm 1 \pmod d$. In this paper, we prove that the converse is also true.

1. Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then we denote the vertex-set and the edge-set of G by $V(G)$ and $E(G)$, respectively. For a positive integer n, we write Z_n for the ring of integers modulo n and Z_n^* for the multiplicative group of units in Z_n.

Let m and n be two positive integers, $\alpha \in Z_n^*, \mu = \lfloor m/2 \rfloor$ and $S_0, S_1, ..., S_\mu$ be subsets of Z_n satisfying the following conditions:

(1) $0 \notin S_0 = -S_0$;

(2) $\alpha^m S_r = S_r$ for $0 \leq r \leq \mu$;

(3) If m is even, then $\alpha^\mu S_\mu = -S_\mu$.

Then we define the (m, n)-metacirculant graph $G = MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ to be the graph with vertex-set $V(G) = \{v_j^i \mid i \in Z_m, j \in Z_n\}$ and edge-set $E(G) = \{v_j^i v_h^{i+r} \mid 0 \leq r \leq \mu; i \in Z_m; h, j \in Z_n; (h - j) \in \alpha^i S_r\}$, where superscripts and subscripts are always reduced modulo m and modulo n, respectively. The subset S_i is called the $(i+1)$-th symbol of $MC(m, n, \alpha, S_0, S_1,$

Received August 14, 1993
This research was supported in part by the Vietnamese National Basic Research Program in Natural Sciences

$..., S_\mu)$.

The class of (m, n)-metacirculant graphs was introduced in [1] as a natural generalization of the Petersen graph for the primary reason of providing a class of vertex-transitive graphs in which there might be some new nonhamiltonian connected vertex-transitive graphs. Among these graphs, cubic (m, n)-metacirculant graphs are especially attractive, being at the same time the simplest nontrivial (m, n)-metacirculant graphs and those most likely to be nonhamiltonian because of their small number of edges.

Since the past ten years there have been many papers dealing with problems of (m, n)-metacirculant graphs (see, for example, our References). In particular, a characterization of graphs which are isomorphic to cubic (m, n)-metacirculant graphs with first symbol $S_0 \neq \emptyset$ has been started in [5] and one of the results obtained there can be described as follows.

Let n be a positive integer and S be a subset of Z_n satisfying $0 \notin S = -S$ (mod n). Then we define the circulant graph $G = C(n, S)$ to be the graph with vertex-set $V(G) = \{v_i \mid i \in Z_n\}$ and edge-set $E(G) = \{v_i v_j \mid i, j \in Z_n; (j - i) \in S\}$, where subscripts are always reduced modulo n.

For integers n and k with $n \geq 2$ and $1 \leq k \leq n-1$ we define the generalized Petersen graph $G = GP(n, k)$ to be the graph with vertex-set $V(G) = \{u_i, v_i \mid i \in Z_n\}$ and edge-set $E(G) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \mid i \in Z_n\}$, where subscripts are always reduced modulo n.

The following theorem has been proved in [5].

THEOREM 1 [5]. *Let $G = MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ be a cubic (m, n)–metacirculant graph with first symbol $S_0 \neq \emptyset$. Then its components are isomorphic to each other and to some of the following graphs:*

1) a circulant graph $C(2\ell, S)$, where $\ell > 1$ and $S = \{1, -1, \ell\}$;

2) a generalized Petersen graph $GP(d, k)$, where $d > 2$ and $k^2 \equiv \pm 1$ (mod d).

Thus, if a graph G is isomorphic to a cubic (m,n)-metacirculant graph $MC(m,n,\alpha,S_0,S_1,...,S_\mu)$ with first symbol $S_0 \neq \emptyset$, then G is isomorphic to either a union of finitely many disjoint copies of $C(2\ell,S)$, where $\ell > 1$ and $S = \{1,-1,\ell\}$ or a union of finitely many disjoint copies of $GP(d,k)$, where $d > 2$ and $k^2 \equiv \pm 1 \pmod{d}$.

This paper is a sequel to [5]. We will prove here that the converse of Theorem 1 is also true. Thus, with this result we will complete the characterization of graphs isomorphic to cubic (m,n)-metacirculant graphs with first symbol $S_0 \neq \emptyset$, started in [5]. More precisely, we will prove the following result.

THEOREM 2. *A graph G is isomorphic to a cubic (m,n)-metacirculant graph $F = MC(m,n,\alpha,S_0,S_1,...,S_\mu)$ with first symbol $S_0 \neq \emptyset$ if and only if G is isomorphic to either a union of finitely many disjoint copies of $C(2\ell,S)$, where $\ell > 1$ and $S = \{1,-1,\ell\}$ or a union of finitely many disjoint copies of $GP(d,k)$, where $d > 2$ and $k^2 \equiv \pm 1 \pmod{d}$.*

We note that the above characterization has been used in [8] to classify all cubic (m,n)-metacirculant graphs which are not Cayley graphs.

2. Proof of Theorem 2

The necessity is clear by Theorem 1. We prove now the sufficiency.

Assume first that G is the union of t disjoint copies of $C(2\ell,S)$, where $\ell > 1$ and $S = \{1,-1,\ell\}$. Set $m = t, n = 2\ell, \alpha = 1, S_0 = S = \{1,-1,\ell\}$ and $S_1 = ... = S_\mu = \emptyset$ ($\mu = \lfloor m/2 \rfloor$). Now it is easy to verify that $m,n,\alpha,S_0,...,S_\mu$ satisfy conditions (1)-(3) in the definition of (m,n)-metacirculant graphs. Therefore, we can construct the (m,n)-metacirculant graph $F = MC(m,n,1,S_0,S_1,$ $...,S_\mu)$ with the parameters chosen as above. It is clear that F is a cubic (m,n)-metacirculant graph with first symbol $S_0 \neq \emptyset$. Now let $V(C(2\ell,S)) = \{v_j \mid j \in Z_{2\ell}\}$,

$$G_0, G_1, ..., G_{t-1}$$

be t disjoint copies of $C(2\ell, S)$ and f_i be an isomorphism of $C(2\ell, S)$ onto G_i. Let $\varphi : V(G) \to V(F)$ be the following mapping :

$$\varphi(f_i(v_j)) = v_j^i,$$

where $i = 0, ..., m-1$ and $j = 0, ..., n-1$. It is not difficult to see that φ is an isomorphism between G and F.

Assume now that G is the union of t disjoint copies of $GP(d,k)$, where $d > 2$ and $k^2 \equiv \pm 1 \pmod d$. Let $t = 2^a b$ with b odd. Set $m = 2b$ and $n = 2^a d$. Choose subsets $S_0, S_1, ..., S_b$ of Z_n as follows: $S_0 = \{2^a, n - 2^a\}, S_1 = ... = S_{b-1} = \emptyset$ and $S_b = \{0\}$. In order to construct an (m,n)-metacirculant graph $F = MC(m, n, \alpha, S_0, ..., S_b)$ we must choose an appropriate element $\alpha \in Z_n^*$. We distinguish the following two cases.

(i) $a = 0$ or $a > 0$ but k is odd.

In this case, we take $\alpha = k$. Since $k^2 \equiv \pm 1 \pmod d$, we have $\alpha^b \equiv \pm k \pmod d$. Moreover, by definition $gcd(\alpha, d) = gcd(k, d) = 1$. If $a = 0$, then $n = 2^a d = d$. Therefore, $\alpha \in Z_n^*$. If $a > 0$ but k is odd, then $gcd(\alpha, 2^a) = gcd(k, 2^a) = 1$. Therefore, we again have $\alpha \in Z_n^*$. We show now that $m, n, \alpha, S_0, S_1, ..., S_b$ satisfy conditions (1)-(3) in the definition of (m,n)-metacirculant graphs. Conditions (1) and (3) are trivially satisfied. Let $\alpha^b = id \pm k$ and $k^2 = jd \pm 1$. We have

$$\begin{aligned}
\alpha^m S_0 &= (\alpha^b)^2 S_0 = (id \pm k)^2 S_0 \\
&= ((i^2 d \pm 2ik + j)d \pm 1)S_0 \\
&\equiv \{\pm(i^2 d \pm 2ik + j)n + 2^a, \mp(i^2 d \pm 2ik + j)n - 2^a\} \\
&\equiv \{2^a, n - 2^a\} \pmod n.
\end{aligned}$$

So, $\alpha^m S_0 = S_0$. It is also trivial that $\alpha^m S_j = S_j$ for all $j = 1, 2, ..., b$. Thus, condition (2) is also satisfied.

(ii) $a > 0$ and k is even.

Since $k \in Z_d^*$, this case happens only if d is odd. Take $\alpha = d + k$. Then $gcd(\alpha, d) = 1$. Since d is odd and k is even, α is odd. Therefore, $gcd(\alpha, 2^a) = 1$. Thus, $\alpha \in Z_n^*$. Moreover, from $k^2 \equiv \pm 1 \pmod d$ it follows that $\alpha^b \equiv \pm k$ (mod

d). As in (i) we can show that $m, n, \alpha, S_0, ..., S_b$ satisfy conditions (1)-(3) in the definition of (m, n)-metacirculant graphs.

Thus, in both cases we can construct the (m, n)-metacirculant graph $F = MC(m, n, \alpha, S_0, ..., S_b)$ with the parameters chosen correspondingly in each case. Moreover, it is clear that F is a cubic (m, n)-metacirculant graph with first symbol $S_0 \neq \emptyset$.

Now let $V(GP(d,k)) = \{u_x, v_x \mid x \in Z_d\}$,

$$G_0^0, G_1^0, ..., G_{2^a-1}^0,$$
$$G_0^1, G_1^1, ..., G_{2^a-1}^1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$G_0^{b-1}, G_1^{b-1}, ..., G_{2^a-1}^{b-1}$$

be t disjoint copies of $GP(d,k)$ and f_j^i be an isomorphism of $GP(d,k)$ onto G_j^i. Let $F = MC(m, n, \alpha, S_0, ..., S_b)$ be the cubic (m, n)-metacirculant graph we have just constructed in the preceding paragraphs. Let $\varphi : V(G) \to V(F)$ be the following mapping :

$$\varphi(f_j^i(u_x)) = v^i_{((j+2^a x)\alpha^i)},$$
$$\varphi(f_j^i(v_x)) = v^{b+i}_{((j+2^a x)\alpha^i)},$$

where $i = 0, 1, ..., b-1; j = 0, 1, ..., 2^a - 1; x = 0, 1, ..., d-1$ and α is chosen as in (i) and (ii), respectively. It is not difficult to see that φ is an isomorphism between G and F.

The proof of Theorem 2 is complete.

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