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A VARIETY OF APPLICATIONS OF A THEOREM OF B.H. NEUMANN ON GROUPS

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Introduction

B.H. Neumann [N1, N2] proved the following theorem: Let G be a group, and $G_1, ..., G_r$ be subgroups of G. If G is a set union of a finite number of cosets of the G_i irredundantly then $G_1 \cap ... \cap G_r$ is of finite index in G.

It is an interesting fact that this simple looking theorem is extremely useful for Galois theory [S1, S3]. It appears to be useful for group rings also [P]*. In this note our main purpose is to point out its utility in a variety of subjects like Banach spaces, curves, division rings, projective geometry, Riemann surfaces and vector spaces.

Before starting let us observe that it is easy to derive from the above theorem the following statement: If a subgroup H of G is contained in a set union of finite number of cosets of $G_1, ..., G_r$ irredundantly, then H is contained in a set union of finite number of cosets of $G_1 \cap ... \cap G_r$ (since $H \cap a_i G_i = \emptyset$ or $b_i(H \cap G_i)$ for some b_i).

1. Applications to vector spaces and projective geometry

PROPOSITION 1.1. *Let F be an infinite field and V a vector space over F. If $V = \bigcup_1^n (V_i + v_i)$, V_i are subspaces (not necessarily distinct) and $v_i \in V$ then $V = V_i$ for some i.*

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PROOF. Suppose $V \neq V_i$ for any i. We may assume that the union is irredundant and for each i,

$$v_i \notin \bigcup_{j \neq i}(V_j + v_j).$$

Suppose all the V_i are equal to V_1. In this case we may assume $v_i \notin V_1$ for $i \geq 2$. Then there exist a, b in F with $a \neq b$ such that av_2, bv_2 both belong to $V_1 + v_i$ for the same i. Then $(a-b)v_2 \in V_1$ and hence $v_2 \in V_1$, a contradiction. Suppose among the V_i, there are $r > 1$ distinct ones. Then by the theorem of Neumann, if $U = \cap V_i$ then $(U, +)$ is of finite index in $(V, +)$ and so $V = \bigcup_1^m (U + w_i)$, which by the above argument leads to a contradiction. Hence the proposition follows.

PROPOSITION 1.2. *Let V be a vector space over an infinite field F and $v_1, ..., v_n$ be n distinct elements of V. Then there exists a linear functional f on V such that $f(v_i) \neq f(v_j)$ whenever $i \neq j$.*

PROOF. Let V^* be the dual space of V. Let for each pair $(i, j), i \neq j, V_{ij} = \{f \in V^* | \, f(v_i) = f(v_j)\}$. Then V_{ij} is a subspace of V^*. It is easy to show that $V_{ij} \neq V^*$. Hence by Proposition 1.1, $V^* \neq \bigcup_{(i,j)} V_{ij}$. Let $f \in V^* \setminus \bigcup V_{ij}$. Then f is the required element.

PROPOSITION 1.3. *Let V be a vector space over an infinite field and W another vector space over F and let $f_1, ..., f_n$ be distinct linear maps of V into W. Let (w_{ij}) be an $n \times n$ skew symmetric matrix from W. Then there exists an element $v \in V$ such that for all $i, j, w_{ij} \neq f_i(v) - f_j(v)$.*

PROOF. For each pair (i, j) such that $i \neq j$ we define $V_{ij} = \ker\,(f_i - f_j)$ and choose an element $v_{ij} \in V$ if such an element exists such that $(f_i - f_j)(v_{ij}) = w_{ij}$; otherwise we let $v_{ij} = 0$. Each V_{ij} is a proper subspace of V. By Proposition 1.1 we have $V \neq \bigcup(V_{ij} + v_{ij})$. Hence if $v \in V \setminus \bigcup(V_{ij} + v_{ij})$, then $f_i(v) - f_j(v) \neq w_{ij}$ for $i \neq j$.

PROPOSITION 1.4. *Let X be a Banach space and $v_1, ..., v_n$ be distinct elements of X. Then there exists a $f \in X^*$, the dual of X such that $f(v_i) \neq f(v_j)$ if $i \neq j$. As a consequence if $f_1, ..., f_n$ are distinct continuous functions on $[0,1]$ there exists a function g of bounded variation such that $\int f_i dg \neq \int f_j dg$ if $i \neq j$.*

PROOF. This follows straightaway from the proof of Proposition 1.2 and Riesz representation theorem [RN, p.110].

PROPOSITION 1.5. *Let $\mathsf{P}^n(F)$ be the $n-$dimensional projective space over an infinite field F. Then $\mathsf{P}^n(F)$ cannot be a set union of a finite number of proper subspaces. Further if $s_1, ..., s_n$ are a finite number of projective collineations of $\mathsf{P}^n(F)$ then there exists a point p such that $s_i(p) \neq s_j(p)$ if $i \neq j$.*

PROOF. $\mathsf{P}^n(F) = \frac{F^{n+1}\setminus(0)}{\sim}$ and each subspace of $\mathsf{P}^n(F)$ is image of a subspace of F^{n+1}. Hence the first assertion follows from Proposition 1.1. Whenever $i \neq j$ define $A_{ij} = \{p \mid s_i(p) = s_j(p)\}$. Then $A_{ij} = \{p | s_i^{-1} s_j(p) = p\}$. We claim now that A_{ij} is a finite union of subspaces of $\mathsf{P}^n(F)$. Let σ be any projective collineation of $\mathsf{P}^n(F)$. Let $A = \{p \mid \sigma(p) = p\}$. In A we define relation $\sim$ as follows: $x \sim y$ if $x = y$ or the line xy is contained in A. The relation $\sim$ is reflexive symmetric. If $x \sim y$ and $y \sim z$, then we claim $x \sim z$. If two of them coincide or they are collinear the claim follows. Otherwise they are distinct and non-collinear. Hence the line xy and the line yz are inside A. Since σ is a collineation, we easily get that the line xz is contained in A and so $x \sim z$. This equivalence relation yields equivalence classes. If V is an equivalence class, then it is a subspace. We assert now that if $V_1, V_2, ..., V_k$ are equivalence classes, then $V_k \cap (V_1 + ... + V_{k-1}) = \emptyset$ where $V_1 + ... + V_{k-1}$ is the subspace generated by $V_1, ..., V_{k-1}$. First $V_k \cap V_1 = \emptyset$. Suppose $V_k \cap (V_1 + ... + V_r) = \emptyset$. We claim $V_k \cap (V_1 + ... + V_{r+1}) = \emptyset$. Let if possible $a \in V_k \cap (V_1 + ... + V_{r+1})$. Then a belong to the line bc, where a, b, c are distinct, $b \in V_{r+1}, c \in V_1 + ... + V_r$. Since σ fixes a and $b, \sigma(c) = c'$ is a point on the line $ab =$ the line bc. But $c \in V_1 + ... + V_r$. Hence $c' \in V_1 + ... + V_r$.

If $c = c'$, then σ fixes a, b, c and hence every point on the line abc (since σ is projective). Then $a \sim b$ and so $a \in V_{r+1}$ yielding $V_{r+1} \cap V_k \neq \emptyset$, a contradiction. If $c \neq c'$, then the line cc' is contained in $V_1 + ... + V_r$ and hence $a \in V_1 + ... + V_r$, a contradiction since $V_k \cap = (V_1 + ... + V_r) = \emptyset$. Hence the assertion follows. Now $\mathsf{P}^n(F)$ can have at most $n+2$ such equivalence classes. Thus A is a finite union of subspaces. Hence each A_{ij} is a finite union of subspaces. By the first statement $\mathsf{P}^n(F) \neq \bigcup A_{ij}$. Hence if $p \notin \bigcup A_{ij}$ then p is the required point.

2. Applications to skew fields, curves and Riemann surfaces

We first recall an already known application of the basic theorem.

PROPOSITION 2.1. [S2] *Let D be a skew field, S a subset closed under $+$ and.. Suppose $S \subset F_1 \cup ... \cup F_n$, a finite union of proper sub-skew fields of D. Then $S \subset F_i$ for some i. In particular, no skew field can be the set union of a finite number of proper sub-skew fields.*

PROPOSITION 2.2. *Let K be a sub-skew field of the skew field D and suppose $K \subset \bigcup_1^n a_i F_i b_i + c_i$, with $a_i, b_i, c_i \in D$ and F_i are sub-skew fields, $b_i \neq 0$. Then either K is finite or $K \subset b_i^{-1} F_i b_i$ for some i. If $K \subset \bigcup_1^n a_i F_i b_i + c_i$ with no term irredundant and K infinite, then $K \subset b_i^{-1} F_i b_i + c_i$ for each i.*

PROOF. Let us put $A_i = a_i b_i$ and $E_i = b_i^{-1} F_i b_i$. Then $K \subset \bigcup_1^n (A_i E_i + c_i), A_i \in D, c_i \in D, E_i$ sub-skew fields. We can assume that this union is irredundant (by successively omitting superfluous terms if necessary). From this we get that $K \cap A_1 E_1 \cap ... \cap A_n E_n$ is of finite index in $(K, +)$. If $K \cap A_1 E_1 \cap ... \cap A_n E_n$ is finite, we get that K is finite and we are through.

Let $K \cap A_1 E_1 \cap ... \cap A_n E_n$ be infinite. Let $s \in K \cap A_1 E_1 \cap ... \cap A_n E_n, s \neq 0$. We claim now that $K \cap A_1 E_1 \cap ... \cap A_n E_n = s(K \cap E_1 \cap ... \cap E_n)$. For, if $x \in K \cap E_1 \cap ... \cap E_n$, then $sx \in K$ since K is a sub-skew field, and $s \in A_i E_i, x \in E_i$ imply $sx \in A_i E_i$ since E_i is a sub-skew field. Hence $R.H.S. \subset L.H.S.$ Let $s' \in L.H.S.$ Then $s^{-1}s' \in K$ since K is a skew field, and $s = A_i e_i, s' = A_i f_i$ imply $s^{-1}s' \in E_i$ since E_i is a skew field. Hence $s^{-1}s' \in K \cap E_1 \cap ... \cap E_n$. Thus $s' \in R.H.S.$ If we put $K' = K \cap E_1 \cap ... \cap E_n$, we get $K \cap A_1 E_1 \cap ... \cap A_n E_n = sK'$. We claim that $K = sK'$. We now have (since sK' is of finite index in $(K, +)$), $K = \bigcup_1^m (sK' + d_i)$ for some $d_i \in K, d_1 = 0, d_i \notin sK'$ if $i \geq 2$. Let $m > 1$. Since $s \in K$ and $s \neq 0$, we get $s^{-1}K = K$ and so $K = \bigcup_1^m (K' + h_i), h_i = s^{-1} d_i$ and for $i \geq 2, h_i \notin K'$. Since $K \cap A_1 E_1 \cap ... \cap A_n E_n$ is infinite, K' is infinite. Hence there exist $a, b \in K', a \neq b$ and an i such that ah_2, bh_2 belong to $K' + h_i$. From this $(a-b)h_2 \in K'$. Now K' is also a sub-skew field. Hence $h_2 \in K'$.

This is a contradiction. Hence $K = sK'$ and thus $K = K' = K \cap E_1 \cap ... \cap E_n$. This gives $K \subset E_i$ for each i. Hence the theorem follows.

The next proposition is a slight generalization of Cartan-Brauer-Hua Theorem.

PROPOSITION 2.3. *Let D be a skew field with center Z. Let K_1 be a sub-skew field and $K_2, K_3, ..., K_n$ are proper sub-skew fields not contained in Z. If $xK_1x^{-1} \subset K_1$ for all $x \notin K_1 \cup ... \cup K_n$ then either $K_1 \subset Z$ or $K_1 = D$.*

PROOF. Suppose $K_1 \not\subset Z$ and $K_1 \neq D$. Let $k_1 \in K_1 \backslash Z$. Let $V_D(k_1) = \{x \in D : xk_1 = k_1x\}$. Then $V_D(k_1)$ is a proper sub-skew field of D. By Proposition 2.1, $D \neq K_1 \cup ... \cup K_n \cup V_D(k_1)$. Let $x \in D \backslash (K_1 \cup ... \cup K_n \cup V_D(k_1))$. Then $xk_1x^{-1} = k_2 \neq k_1$. We have $(1+x)k_1 = k_1 + k_2x$. Also $1+x \notin K_1 \cup ... \cup K_n \cup V_D(k_1)$. Hence $(1+x)k_1 = k_3(1+x), k_3 \in K_1$. Thus $k_3(1+x) = k_1 + k_2x, k_3 - k_1 = (k_2 - k_3)x$. This yields $k_1 = k_2 = k_3$, a contradiction. Hence the proposition follows.

PROPOSITION 2.4. *Let D be an infinite skew field. Let $a_{11}, ..., a_{1r_1}; a_{21}, ..., a_{2r_2}; ...a_{n1}, ..., a_{nr_n};$ be a finite collection of distinct elements such that a_{ij} is a conjugate of a_{i1} and $a_{ij} \notin Z$, the center of D. Then there exists an $x \in D \setminus (0)$ such that for each $i, xa_{i1}x^{-1} \neq a_{i1}, ..., a_{ir_i}$.*

PROOF. Let us put $K_{i1} = V_D(a_{i1})$. Then K_{i1} is a proper sub-skew field of D (since $a_{i1} \notin Z$). For i, j with $i \neq j$ let x_{ij} be an element of D such that $x_{ij}a_{i1}x_{ij}^{-1} = a_{ij}$ and $x_{ij} = 1$ if $j = 1$. By Proposition 2.2 we have $D \neq \bigcup_{i,j} x_{ij}K_{i1}$. Hence there exists $x \in D \setminus \bigcup_{i,j} x_{ij}K_{i1}$. Therefore $xa_{i1}x^{-1} \neq a_{ij}$ since $x \notin x_{ij}K_{i1}$. This proves the result.

As an easy consequence of the above we have

PROPOSITION 2.5. *Let D be a skew field with center Z. If $a \notin Z$ then a has infinitely many conjugates. If f(x) is a polynomial over Z of degree n with n+1 roots in D, then f(x) has infinitely many roots in D.*

PROPOSITION 2.6. *Let C be an irreducible curve. Let $C_1, ..., C_n$ be irreducible curves which are rational images of C. If every rational function on C is induced by a rational function on some C_i then C is birationally isomorphic to some C_i.*

PROOF. Let $f_i : C \to C_i$ be the rational maps. Let F_i be the function field of C_i, and F the field of functions of C. Each F_i is a subfield of F and the hypothesis implies $F = UF_i$. Now Proposition 2.1 yields $F = F_i$. Hence the result follows.

PROPOSITION 2.7. *Let S be a noncompact Riemann surface. Let $S_1, ..., S_n$ be noncompact Riemann surfaces which are holomorphic images of S. If each meromorphic function on S is induced by a meromorphic function on some S_i and every meromorphic function on S_i induces a meromorphic function on S, then S is biholomorphic to S_i for some i.*

PROOF. Let $f_i : S \to S_i$ be the holomorphic maps and let F_i be the meromorphic function field of S_i, and F the meromorphic function field of S. From the hypothesis we get $F = \bigcup F_i$. This yields $F = F_i$ for some i. From this it follows that S is biholomorphic to S_i.

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