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A REDUCTION OF THE GLOBALIZATION AND U(1)-COVERING

TRAN DAO DONG

Abstract. We suggest a reduction of the globalization and multidimensional quantization to the case of reductive Lie groups by lifting to U(1)-covering. Our construction is connected with the M. Duflo's third method for algebraic groups. From a reductive datum of the given real algebraic Lie group we firstly construct geometric complexes with respect to U(1)-covering by using the unipotent positive distributions. Then we discribe in terms of local cohomology the maximal globalization of Harish-Chandra modules which correspond to the geometric complexes.

Introduction

In order to find irreducible unitary representations of a connected and simply connected Lie group G, the Kirillov's orbit method furnishes a procedure of quantization, starting from linear bundles over a G-homogeneous symplectic manifold (see [5]). In [1] and [2], Do Ngoc Diep has proposed the procedure of multidimensional quantization for general case, starting from arbitrary irreducible bundles. This procedure could be viewed as a geometric version of the construction of M. Duflo [4]. In 1988, W. Schmid and J.A. Wolf [3] described in terms of local cohomology the maximal globalization of Harish-Chandra modules to realize the discrete series representations of semi-simple Lie groups by using the geometric quantization and the derived Zuckerman functor modules. In [9], we modified the construction suggested by W. Schmid and J.A. Wolf to the case of U(1)-covering by applying the technique of P.L. Robinson and J.H. Rawnsley [6]. Our purpose is to give an algebraic version of the multidimensional quantization with respect to U(1)-covering. In this paper, we reduce the same problem to the case of reductive Lie groups. Using

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the unipotent positive distributions we construct geometric complexes and their corresponding Harish-Chandra modules. Then we will describe the maximal globalization of Harish-Chandra modules in terms of local cohomology with respect to U(1)-covering.

1. Unipotent positive distributions

Let G be a real algebraic Lie group. Denote by $\mathcal{G}$ the Lie algebra of G and $\mathcal{G}^*$ its dual space. The group G acts in $\mathcal{G}^*$ by the coadjoint representation. Denote by G_F the stabilizer of $F \in \mathcal{G}^*$ and by $\mathcal{G}_F$ its Lie algebra. Let U_F be the unipotent radical of G_F and $\mathcal{U}_F$ be its Lie algebra. Denote by Q_F the reductive component of G_F in its Cartan-Levi's decomposition $G_F = U_F \cdot Q_F$.

Let $G_F^{U(1)}$ be the U(1)-covering of G_F and $U_F^{U(1)}$ be the inverse image of U_F under the projection $\sigma_j : G_F^{U(1)} \longrightarrow G_F$, where σ_j is the homomorphism defined in [8,§2]. Since

$$1 \longrightarrow U(1) \longrightarrow U_F^{U(1)} \xrightarrow{\sigma_j} U_F \longrightarrow 1$$

is a short exact sequence then we have the split short exact sequence of corresponding Lie algebras

$$0 \longrightarrow \mathcal{U}(1) \longrightarrow \text{ Lie } U_F^{U(1)} \longrightarrow \mathcal{U}_F \longrightarrow 0.$$

Thus $U_F^{U(1)}$ is the U(1)-covering of U_F and we have $\quad \text{Lie } U_F^{U(1)} \cong \mathcal{U}_F \oplus \mathcal{U}(1)$.

From the local triviality of the Q_F-principal bundle $Q_F \rightarrowtail U_F\backslash G \twoheadrightarrow G_F\backslash G$ there exists a connection on the bundle. Then the Kirillov 2-form B_Ω of K-orbit Ω passing F induces a nondegenerate closed G-invariant 2-form $\widetilde{B}_\Omega$ on the horizontal part $T_H(U_F \setminus G)$ defined by the formula

$$\widetilde{B}_\Omega(f)(\widetilde{X},\widetilde{Y}) = B_\Omega(F)(k_*\widetilde{X}, k_*\widetilde{Y}),$$

where $f \in U_F \setminus G$, $k(f) = F \in \Omega$, and k_* is the linear lifting isomorphism induced from k (see [7]). As in [8], the symplectic group $\mathcal{S}p(T_{(f)H}(U_F\backslash G); \widetilde{B}_\Omega(f))$

has an $U(1)$-connected covering $Mp^c(T_{(f)H}(U_F\backslash G))$ and we obtain the following isomorphisms

$$\mathcal{S}p(T_{(f)H}(U_F \setminus G)) \cong \mathcal{S}p(\mathcal{G}/\mathcal{G}_F) \text{ and } Mp^c(T_{(f)H}(U_F \setminus G)) \cong Mp^c(\mathcal{G}/\mathcal{G}_F).$$

Using these isomorphisms we can view $U_F^{U(1)}$ as a Lie subgroup of the Cartersian product of Lie groups $U_F \times Mp^c(T_{(f)H}(U_F \setminus G))$.

We do not assume that the orbit Ω passing $F \in \mathcal{G}^*$ is an integral orbit, i.e. there does not exist a unitary character χ_F of G_F.

DEFINITION 1.1. A point $F \in \mathcal{G}^*$ is called $(u, U(1))$-admissible (u for unipotent radical, $U(1)$ for $U(1)$-covering) iff there exists a unitary character $\theta_F^{U(1)}$: $U_F^{U(1)} \longrightarrow S^1$ such that

$$d\theta_F^{U(1)}(X,\varphi) = \frac{i}{\hbar}(F(X)+\varphi)$$

where $(X,\varphi) \in \mathcal{U}_F \oplus \mathcal{U}(1)$.

We see that if F is U(1)-admissible (see [8]) then it is $(u, U(1))$-admissible, but the converse does not hold in general.

DEFINITION 1.2. A smooth complex tangent distribution $\widetilde{L} \subset (T(U_F \setminus G))_{\mathbb{C}}$ is called a unipotent positive distribution iff

(i) $\widetilde{L}$ is an integrable and G-invariant subbundle of $(T_H(U_F \setminus G))_{\mathbb{C}}$.

(ii) $\widetilde{L}$ is invariant under the action Ad of G_F.

(iii) $\forall f \in U_F \setminus G$, the fibre $\widetilde{L}_f$ is a positive polarization of the symplectic vector space $((T_{(f)H}(U_F \setminus G))_{\mathbb{C}}, \widetilde{B}_\Omega(f))$, i.e.

(α) $\dim \widetilde{L}_f = \frac{1}{2} \dim T_{(f)H}(U_F \setminus G)$,

(β) $\widetilde{B}_\Omega(f)(\widetilde{X},\widetilde{Y}) = 0$ for all $\widetilde{X},\widetilde{Y} \in \widetilde{L}_f$,

(γ) $i\widetilde{B}_\Omega(f)(\widetilde{X},\overline{\widetilde{X}}) \geq 0$ for all $\widetilde{X} \in \widetilde{L}_f$,

where $\overline{\widetilde{X}}$ is the conjugation of $\widetilde{X}$. We say that $\widetilde{L}$ is strictly positive iff the inequality (γ) is strict for nonzero $\widetilde{X} \in \widetilde{L}_f$.

We see that if $\widetilde{L}$ is a unipotent positive distribution then the inverse image $\mathcal{B}$ of $L_F = k_*\widetilde{L}_f$ under the natural projection $p : \mathcal{G}_{\mathbb{C}} \longrightarrow \mathcal{G}_{\mathbb{C}}/(\mathcal{G}_F)_{\mathbb{C}}$ is a positive polarization in $\mathcal{G}_{\mathbb{C}}$ (see [8]).

Let $\widetilde{L}$ be a unipotent positive distribution such that $\widetilde{L}\cap\overline{\widetilde{L}}$ and $\widetilde{L}+\overline{\widetilde{L}}$ are the complexifications of some real distributions. Then the corresponding complex subalgebra $\mathcal{B}=p^{-1}(k_*\widetilde{L}_f)$ satisfies the following conditions: $\mathcal{B}\cap\overline{\mathcal{B}}$ and $\mathcal{B}+\overline{\mathcal{B}}$ are the complexifications of the real Lie subalgebras $\mathcal{B}\cap\mathcal{G}$ and $(\mathcal{B}+\overline{\mathcal{B}})\cap\mathcal{G}$. Denote by B_0 and N_0 the corresponding analytic subgroups.

The unipotent positive distribution $\widetilde{L}$ is called *closed* iff all the subgroups B_0, N_0 and the semi-direct products $B=G_F\cdot B_0$ and $N=G_F\cdot N_0$ are closed in G. In what follows, we assume that $\widetilde{L}$ is closed. We know that B_0 is a normal subgroup in B and G_F has adjoint action on B_0. Moreover, $G_F^{U(1)}$ acts on B_0 and we can define the semi-direct product $G_F^{U(1)}\ltimes B_0$.

Then $B^{U(1)}=G_F^{U(1)}\ltimes B_0$ is the $U(1)$-covering of $B=G_F\cdot B_0$ and we have

$$\text{Lie } B^{U(1)}\cong\mathcal{B}\oplus\mathcal{U}(1).$$

Denote by $B_0^{U(1)}$ the inverse image of B_0 in $B^{U(1)}$ under the $U(1)$-covering projection. As in [8] we have

PROPOSITION 1.3. *In a small neighbourhood of the identity of $U_F^{U(1)}$ we obtain*

$$\theta_F^{U(1)}(g,(\lambda,\widetilde{Adg}^{-1}))=\exp\left(\frac{i}{\hbar}(F(X)+\varphi)\right),$$

where $\varphi\in\mathbb{R}$ satisfying the relation $\lambda^2 DetC_{\widetilde{Adg}^{-1}}=\exp(\frac{i}{\hbar}\varphi)$.

The integral kernel of $\theta_F^{U(1)}$ is given by the formula

$$u(z,w)=\exp\left(\frac{i}{\hbar}(F(X)+\varphi)+\frac{i}{2\hbar}\langle z,w\rangle-\frac{1}{4\hbar}\langle w,w\rangle\right),$$

where $z,w\in(\mathcal{B}+\overline{\mathcal{B}})/(\mathcal{B}\cap\overline{\mathcal{B}})$.

Denote by $Z_{irr}^{U(1)}(F)$ the set of all equivalent classes of irreducible unitary representations of G_F such that the restriction of the composition of σ_j and each of them to $U_F^{U(1)}$ is a multiple of the character $\theta_F^{U(1)}$. When F is $(u,U(1))$-admissible and $\tau\in Z_{irr}^{U(1)}(F)$, the pair (F,τ) is called a reductive datum. Let $\widetilde{\sigma}$ be some fixed irreductive unitary representation of G_F in a separable Hilbert $\widetilde{V}$ such that the restriction of $(\widetilde{\sigma}\circ\sigma_j)$ to $U_F^{U(1)}$ is a multiple of the character $\theta_F^{U(1)}$.

DEFINITION 1.4. The triplet $(\widetilde{L}, \rho, \sigma_0)$ is called a $(\widetilde{\sigma}, \theta_F^{U(1)})$-unipotent positive polarization, and $\widetilde{L}$ is called a weakly Lagrangian distribution iff

(i) σ_0 is an irreducible representation of the subgroup B_o in a Hilbert space V' such that the point σ_0 in the dual $\widehat{B_0}$ is fixed under the natural action of G_F and

$$(\sigma_0 \circ \sigma_j)|_{G_F^{U(1)} \cap B_0^{U(1)}} = (\widetilde{\sigma} \circ \sigma_j)|_{G_F^{U(1)} \cap B_0^{U(1)}}$$

(ii) ρ is a representation of the complex Lie algebra $\mathcal{N} \oplus \mathcal{U}(1)_{\mathbb{C}}$ in V' which satisfies E. Nelson's condition and

$$d(\sigma_0 \circ \sigma_j) = \rho|_{\mathcal{B} \oplus \mathcal{U}(1)}.$$

By a similar way as in [8, §2] we obtain

PROPOSITION 1.5. *Let $F \in \Omega$ be $(u, U(1))$-admissible and suppose that $(\widetilde{L}, \rho, \sigma_o)$ is a $(\widetilde{\sigma}, \theta_F^{U(1)})$-unipotent positive polarization. Then there exists a unique irreducible representation σ of $B^{U(1)}$ in space $V = \widetilde{V} \otimes V'$ such that*

$$\sigma|_{G_F^{U(1)}} = \widetilde{\sigma} \circ \sigma_j \quad \text{and} \quad d\sigma = \rho|_{\mathcal{B} \oplus \mathcal{U}(1)}.$$

2. The construction of geometric complexes

Suppose that G is a connected real reductive Lie group. We fix a Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}_{\mathbb{C}}$ and consider $F \in \mathcal{G}^*$ such that $(\mathcal{G}_F)_{\mathbb{C}} = \mathcal{H}$. Then $H = G_F$ is a Cartan subgroup of G. Let $(F, \widetilde{\sigma})$ be a reductive datum and $\sigma : B^{U(1)} \longrightarrow U(V)$ be the representation obtained in Proposition 1.5. Denote by $\mathbb{E}^{U(1)}$ and $\mathcal{E}^{U(1)}$ the homogeneous vector bundles on $H \setminus G$ and $U_F \setminus G$ respectively associated with the restrictions of σ on $H^{U(1)}$ and $U_F^{U(1)}$. In the category of smooth vector bundles we have the bundles $k^*\mathbb{E}^{U(1)}$ and $\mathcal{E}^{U(1)}$ are equivalent. In the view of [3], we can say the bundles $\mathbb{E}^{U(1)}$ and $\mathcal{E}^{U(1)}$ associated to the basic datum $(H, \mathcal{B}, \widetilde{\sigma} \circ \sigma_j)$.

Suppose that $\dim \Omega_F = m$. Let $\mathcal{C}^q(\mathcal{E}^{U(1)})$ denote the sheaf of differential forms of type $(0, q)$ on $U_F \setminus G$ with values in $\mathcal{E}^{U(1)}$. We know that each differential form of this type is a section of the bundle $\mathcal{E}^{U(1)} \otimes \Lambda^q \mathbb{N}^*$ where $\mathbb{N}$

is the inverse image bundle $k^*\aleph$ of the homogeneous vector bundle $\aleph \longrightarrow H\setminus G$ with fibre $\mathcal{N} \cong \mathcal{B}/\mathcal{H}$ and $I\!N^*$ is its dual. Denote by $\mathcal{O}(\mathcal{E}^{U(1)})$ the sheaf of germs of partially holomorphic C^∞ sectons of $\mathcal{E}^{U(1)}$ that are annihilated by $\mathcal{N}$. Then as in [9, §1] we obtain a cochain complex

$$C^\infty(U_F\setminus G;\mathcal{E}^{U(1)}\otimes\Lambda^\cdot I\!N^*),\quad \overline{\partial}_E \tag{2.1}$$

Denote by

$$H^p(C^\infty(U_F\setminus G;\mathcal{E}^{U(1)}\otimes\Lambda^\cdot I\!N^*))$$

the p-th derived group of the cochain complex (2.2) and $H^p(U_F\setminus G;\mathcal{O}(\mathcal{E}^{U(1)}))$ the sheaf cohomology group of the space $U_F\setminus G$ of degree p with coefficients in $\mathcal{O}(\mathcal{E}^{U(1)})$. By a similar argument as in [9, §1] we obtain

PROPOSITION 2.1. *There exists a canonical isomorphism*

$$H^p(C^\infty(U_F\setminus G;\mathcal{E}^{U(1)}\otimes\Lambda^\cdot I\!N^*))\cong H^p(U_F\setminus G;\mathcal{O}(\mathcal{E}^{U(1)})),\quad p\geq 0 \tag{2.2}$$

We note that the differential $\overline{\partial}_E$ of (2.2) extends naturally to hyperfunction sections, so we obtain a complex

$$C^{-\omega}(U_F\setminus G;\mathcal{E}^{U(1)}\otimes\Lambda^\cdot I\!N^*)),\quad \overline{\partial}_E \tag{2.3}$$

Under the fibration $U_F\setminus G\longrightarrow H\setminus G$, the bundle $\mathcal{E}^{U(1)}\longrightarrow U_F\setminus G$ pushes down to the bundle $\mathcal{E}^{U(1)}\longrightarrow H\setminus G$ and the sheaf $\mathcal{O}(\mathcal{E}^{U(1)})\longrightarrow U_F\setminus G$ pushes down to the sheaf $\mathcal{O}(\mathcal{E}^{U(1)})\longrightarrow H\setminus G$ of germs of partially holomorphic C^∞ sections over $H\setminus G$. Then we have

$$C^{-\omega}(H\setminus G;\mathcal{E}^{U(1)}\otimes\Lambda^\cdot\aleph^*),\quad \overline{\partial}_E \tag{2.4}$$

Denote by $C^{-\omega}_{Q_F}(U_F\setminus G;\mathcal{E}^{U(1)}\otimes\Lambda^p I\!N^*)$ the space of Q_F-equivariant partially holomorphic C^∞ sections of the space $C^{-\omega}(U_F\setminus G;\mathcal{E}^{U(1)}\otimes\Lambda^p I\!N^*)$, we have

PROPOSITION 2.2. *There exists a canonical isomorphism of vector spaces*

$$C^{-\omega}_{Q_F}(U_F\setminus G;\mathcal{E}^{U(1)}\otimes\Lambda^p I\!N^*)\cong C^{-\omega}(H\setminus G;\mathcal{E}^{U(1)}\otimes\Lambda^p\aleph^*).$$

PROOF. Pull back the complex (2.4) to G as was done in [9, §1], we see that (2.4) is isomorphic to the complex

$$[C^{-\omega}(G) \otimes V \otimes \Lambda^{\cdot}\mathcal{N}^*]^H, \quad \overline{\partial}_E \tag{2.5}$$

Then our assertion follows from the definition of Q_F-equivariant sections (see [7, §3]). □

Let X denote the flag variety of Borel subalgebras of $\mathcal{G}_{\mathbb{C}}$. Since H normalizes $\mathcal{B}$, there exists a natural G-invariant fibration $H \setminus G \longrightarrow S$, where $S = G \cdot \mathcal{B}$ is the G-orbit passing $\mathcal{B}$ in X. Then, as in [9], we obtain the Cauchy-Riemann complex

$$C^{-\omega}(S; \mathcal{E}^{U(1)} \otimes \Lambda^{\cdot} I\!N_S^*), \quad \overline{\partial}_S \tag{2.6}$$

where $I\!N_S = I\!\Gamma^{0,1}(S)$ is the G-homogeneous vector bundle based on $\mathcal{N}/\mathcal{N} \cap \overline{\mathcal{N}}$ and $\overline{\partial}_E$ is the Cauchy-Riemann operator (see [3, §4]).

Denote by $X^{U(1)}$ the flag variety of U(1)-invariant Borel subalgebras of Lie algebra $\mathcal{G}_{\mathbb{C}} \oplus \mathcal{U}(1)_{\mathbb{C}}$ and $\pi_X : X^{U(1)} \longrightarrow X$ is the natural projection. Using the Gauss' decomposition $G = K \cdot B$, where K ia a fixed maximal compact subgroup in G, we obtain $B \setminus G \cong B^{U(1)} \setminus K \cdot B^{U(1)}$. Let

$$S^{U(1)} = (K \cdot B^{U(1)}) \cdot (\mathcal{B} \oplus \mathcal{U}(1)_{\mathbb{C}})$$

be the orbit passing $(\mathcal{B} \oplus \mathcal{U}(1)_{\mathbb{C}})$ in $X^{U(1)}$, we see that $B^{U(1)}$ is the stabilizer of $\mathcal{B} \oplus \mathcal{U}(1)_{\mathbb{C}}$ and there exists a diffeomorphism of $S^{U(1)}$ onto S. Then we have the complex

$$C^{-\omega}(S^{U(1)}; \pi_X^* \mathcal{E}^{U(1)} \otimes \Lambda^{\cdot}\aleph_S^*), \quad \overline{\partial}_E \tag{2.7}$$

where $\aleph_S = \pi_X^* I\!\Gamma^{0,1}(S)$ and $\aleph_S^*$ is its dual.

PROPOSITION 2.3. *There are canonical isomorphisms*

$$H^p(C^{-\omega}_{Q_F}(U_F \setminus G; \mathcal{E}^{U(1)} \otimes \Lambda^{\cdot} I\!N^*)) \cong H^p(C^{-\omega}(S^{U(1)}; \pi_X^* \otimes \Lambda^{\cdot}\aleph_S^*))$$
$$\cong H^p([C^{-\omega}(G) \otimes V \otimes \Lambda^{\cdot}(\mathcal{N}/\mathcal{N} \cap \overline{\mathcal{N}})^*]^{\mathcal{N} \cap \overline{\mathcal{N}}, H})$$

PROOF. Applying the Poincaré Lemma to the fibres of $H \setminus G \longrightarrow S$ we see that the inclusion of (2.7) in the complex (2.4) induces an isomorphism of cohomology. Then the proposition follows from Proposition 2.2. □

Let $\widetilde{S}$ denote the germ of neighbourhoods of S in X, we see that $\mathcal{E}^{U(1)} \longrightarrow S$ has a unique holomorphic $\mathcal{G}$-equivariant extension $\widetilde{\mathcal{E}}^{U(1)} \longrightarrow \widetilde{S}$. Then as in [9] we obtain the Dolbeault complex

$$C^{-\omega}(\widetilde{S}^{U(1)}; \pi_X^* \widetilde{\mathcal{E}}^{U(1)} \otimes \Lambda^{\cdot} \aleph_X^*)), \quad \overline{\partial} \tag{2.8}$$

where $\aleph_X = \pi_X^* \varPi_X^{0,1}$ and coefficients are hyperfunctions on $\widetilde{S}$ with support in $S^{U(1)}$.

By a similar way as in [9, §1], we have

PROPOSITION 2.4. *There is a canonical isomorphism*

$$H^p(C^{-\omega}(\widetilde{S}^{U(1)}; \pi_X^* \widetilde{\mathcal{E}}^{U(1)} \otimes \Lambda^{\cdot} \aleph_X^*)) \cong H^p(\widetilde{S}^{U(1)}; \mathcal{O}(\widetilde{\mathcal{E}}^{U(1)})) \tag{2.9}$$

where the right hand side of (2.9) is local cohomology along $\widetilde{S}$.

3. G-Modules and their induced topologies

We fix a basic datum $(H, \mathcal{B}, \widetilde{\sigma} \circ \sigma_j)$ and consider the G-orbit $S = G \cdot \mathcal{B} \subset X$. Denote by Y the variety of ordered Cartan subalgebras and $G_{\mathbb{C}}$ the adjoint group of $\mathcal{G}_{\mathbb{C}}$. Let $S_Y := G \cdot \mathcal{H} \subset Y$ be the G-orbit through the base point in Y.

PROPOSITION 3.1. *There are canonical isomorphisms of G-modules*

$$\begin{aligned} H^p(C^{-\omega}_{Q_F}(U_F \setminus G; \mathcal{E}^{U(1)} \otimes \Lambda^{\cdot} I\!N^*)) &\cong H^p(C^{-\omega}(S^{U(1)}; \pi_X^* \mathcal{E}^{U(1)} \otimes \Lambda^{\cdot} \aleph_S^*)) \\ &\cong H^{p+u}(\widetilde{S}^{U(1)}; \mathcal{O}(\widetilde{\mathcal{E}}^{U(1)})) \end{aligned}$$

where $u = codim_{I\!R}(S \subset X)$.

PROOF. The first isomorphism follows from Proposition 2.3. We only need to show that the complexes (2.4) and (2.8) have naturally isomorphic cohomologies with a sheaf of degree by $u = \operatorname{codim}_{I\!R}(S)$.

Let $\varPi_{Y|X}$ denote the complexified relative tangent bundle of the fibration p, and $\varPi^{1,0}_{Y|X}$, $\varPi^{0,1}_{Y|X}$ the subbundle of holomorphic, respectively antiholomorphic, relative tangent vectors. Denote by $Y^{U(1)}$ the variety of U(1)-invariant ordered

Cartan subalgebras of $\mathcal{G}_{\mathbb{C}} \oplus \mathcal{U}(1)_{\mathbb{C}}$. We have the natural projection $\pi_Y : Y^{U(1)} \longrightarrow Y$. Suppose that

$$S_Y^{U(1)} = (K \cdot B^{U(1)})(\mathcal{H} \oplus \mathcal{U}(1)_{\mathbb{C}}) \subset Y^{U(1)} \tag{3.1}$$

is the orbit passing the base point $\mathcal{H} \oplus \mathcal{U}(1)_{\mathbb{C}}$ in $Y^{U(1)}$, we have $S_Y^{U(1)} \approx S_Y$.

Let $\mathcal{C}^{-\omega}(X^{U(1)})$ be the sheaf of hyperfunctions on $X^{U(1)}$ with support in $S^{U(1)}$ and $\mathcal{C}^{-\omega}(Y^{U(1)}; \Lambda^p \aleph^*_{Y|X})$ the sheaf of hyperfunction sections of $\Lambda^p \aleph^*_{Y|X}$ on $Y^{U(1)}$ with support in $S_Y^{U(1)}$, where $\aleph_{Y|X} = \pi_Y^*(\mathit{\Pi}_{Y|X})$. As in [9, §2], we obtain the complex

$$C^{-\omega}(S_Y^{U(1)}; (\pi_X \circ p^{U(1)})^* \widetilde{\mathcal{E}}^{U(1)} \otimes \Lambda^{\cdot}(\aleph^{1,0}_{Y|X})^*) \tag{3.2}$$

which coincides with the complex (2.4). Combining this with Proposition 2.2. we obtain desired isomorphisms. □

Now we fix a Cartan involution θ of G with $\theta H = H$. It defines the maximal compactly embedded subgroup $K = \{x \in G : \theta x = x\}$ of G. Then $H = T \times A$ with $T = H \cap K$ and $A = \exp(\mathcal{A} \cap \mathcal{G})$, where $\mathcal{H} = \mathcal{T} + \mathcal{A}$ are the (± 1)-eigenspaces of $\theta|_H$. Consider the orbit $S = G \cdot \mathcal{B} \subset X, \mathcal{H} \subset \mathcal{B}$. Proposition 7.1 in [3] follows that there exists a relative orbit $S_{max} = G \cdot \mathcal{B}_{max}$, where $\mathcal{H} \subset \mathcal{B}_{max}$ and $\mathcal{B}_{max}$ is maximally real for that condition. Then, as in [3, §2], G has a cuspidal parabolic subgroup $P = MAN_H$, where $Z_G(\mathcal{A}) = M \times A$, $\theta M = M$ and $\mathcal{B}_{max} \subset \mathcal{P}$, with $\mathcal{P} = \mathrm{Lie}P$. Moreover, the fibrations $S \longrightarrow S_{max}$ and $S_{max} \longrightarrow P \setminus G$ induce a fibration $S \longrightarrow P \setminus G$. Then, as in [9, §2], we obtain a complex of sheaves

$$\mathcal{C}^{-\omega}_{P\setminus G}(S^{U(1)}; \pi_X^* \mathcal{E}^{U(1)} \otimes \Lambda^p \aleph^*_S) \tag{3.3}$$

consist of germs of sections of the bundles $\pi_X^* \mathcal{E}^{U(1)} \otimes \Lambda^p \aleph^*_S \longrightarrow S^{U(1)}$, coefficients in $C^{-\omega}_{P\setminus G}(S^{U(1)})$.

Taking global sections, we arrive at a subcomplex of the complex (2.7)

$$C^{-\omega}_{P\setminus G}(S^{U(1)}; \pi_X^* \mathcal{E}^{U(1)} \otimes \Lambda^{\cdot} \aleph^*_S), \quad \overline{\partial}_S \tag{3.4}$$

By a similar argument in [9, §2], we have

PROPOSITION 3.2. *The inclusion of (3.4) in the Cauchy-Riemann complex (2.7) induces isomorphisms of cohomology.*

PROPOSITION 3.3. *The vector spaces*

$$C^{-\omega}_{P\backslash G}(S^{U(1)}; \pi_X^* \mathcal{E}^{U(1)} \otimes \Lambda^p \aleph_S^*)$$

have natural Fréchet topologies. In those topologies, $\overline{\partial}_S$ is continuous and the actions of G are Fréchet representations.

4. A reduction of the globalization

We recall some notions from [3, §3]: An admissible Fréchet G-module has property (MG) if it is the maximal globalization of its underlying Harish-Chandra module. A complex $(C^\cdot, d)$ of Fréchet G-modules has property (MG) if d has closed range, the cohomologies $H^p(C^\cdot, d)$ are admissible and of finite length, and each $H^p(C^\cdot, d)$ has property (MG).

Given a basis datum $(H, \mathcal{B}, \widetilde{\sigma} \circ \sigma_j)$, the corresponding homogeneous vector bundle $\mathcal{E}^{U(1)} \longrightarrow S^{U(1)}$ has property (MG) if the partially smooth Cauchy-Riemann complex (2.9) has property (MG). Denote

$$H^p(S^{U(1)}; \mathcal{E}^{U(1)}) = H^p(C^{-\omega}(S^{U(1)}; \pi_X^* \mathcal{E}^{U(1)} \otimes \Lambda^\cdot \aleph_S^*)) \tag{4.1}$$

Proposition 3.2 shows that $H^p(S^{U(1)}; \mathcal{E}^{U(1)})$ is calculated by a Fréchet complex. Then $H^p(S^{U(1)}; \mathcal{E}^{U(1)})_{(K)}$ is calculated by the subcomplex of K-finite forms in that Fréchet complex and these forms are smooth. Then we can define morphisms

$$H^p(S^{U(1)}; \mathcal{E}^{U(1)})_{(K)} \longrightarrow A^p(G, H, \mathcal{B}, \widetilde{\sigma} \circ \sigma_j) \tag{4.2}$$

where $A^p(G, H, \mathcal{B}, \widetilde{\sigma} \circ \sigma_j) \cong H^p(C^{for}_{Q_F}(U_F \setminus G; \mathcal{E}^{U(1)} \otimes \Lambda^\cdot I\!N^*)_{(K)})$ are Harish-Chandra modules for G (see [3, §3]).

We recall as in [3] that the bundle $\mathcal{E}^{U(1)} \longrightarrow S^{U(1)}$ has property (Z) if the maps (4.2) are isomorphisms. In other words, $\mathcal{E}^{U(1)} \longrightarrow S^{U(1)}$ has property (Z) if $H^p(S^{U(1)}; \mathcal{E}^{U(1)})$ is the globalization of the Harish-Chandra module $A^p(G, H, \mathcal{B}, \widetilde{\sigma} \circ \sigma_j)$.

We consider the following condition of a pair $(F, \widetilde{\sigma})$:

$$\begin{cases} \textit{There exist a positive root system } \Phi^+ \textit{ and a number } C > 0 \\ \textit{such that : if } \mathcal{E}^{U(1)} \longrightarrow S^{U(1)} \textit{ is irreductive, } \lambda = d(\widetilde{\sigma} \circ \sigma_j)|_{\mathcal{H}} \in \mathcal{H}^*, \\ \lambda_{\mathbb{R}} \textit{ is the restriction of } \lambda \textit{ to the real form } \mathcal{H}_{\mathbb{R}} \textit{ on which roots take} \\ \textit{real value, and } \langle \lambda_{\mathbb{R}}, \alpha \rangle > 0 \textit{ for all } \alpha \in \Phi^+, \textit{ then } \mathcal{E}^{U(1)} \longrightarrow S^{U(1)} \\ \textit{has properties } (MG) \textit{ and } (Z). \end{cases} \tag{4.3}$$

As in [9, §2] we have

PROPOSITION 4.1. *We fix $(F, \widetilde{\sigma})$ and suppose that (4.3) is true. Then for arbitrary basic data of the form $(H, \mathcal{B}, \widetilde{\sigma} \circ \sigma_j)$, the bundle $\mathcal{E}^{U(1)} \longrightarrow S^{U(1)}$ has both properties (MG) and (Z).*

We fix a basic datum $(H, \mathcal{B}, \widetilde{\sigma} \circ \sigma_j)$. Let $S = G \cdot \mathcal{B} \in X$ and $u = \mathrm{codim}_{\mathbb{R}}(S)$. Recall as in [3] that the polarization $\mathcal{B}$ is maximally real if it maximizes the dimension of $\mathcal{B} \cap \overline{\mathcal{B}}$.

THEOREM 1. *For any maximally real polarization $(H, \mathcal{B}, \widetilde{\sigma} \circ \sigma_j)$ there are topological isomorphisms between Fréchet G-modules*

$$\begin{aligned} H^p(C^{-\omega}_{Q_F}(U_F \setminus G; \mathcal{E}^{U(1)} \otimes \Lambda^{\cdot}\aleph^*)) &\cong H^p(C^{-\omega}(S^{U(1)}; \pi^*_X \mathcal{E}^{U(1)} \otimes \Lambda^{\cdot}\aleph^*_S)) \\ &\cong H^{p+u}(\widetilde{S}; \mathcal{O}(\widetilde{\mathcal{E}}^{U(1)})) \end{aligned}$$

which are canonically and topologically isomorphic to the action of G on the maximal globalization of $A^p(G, H, \mathcal{B}, \widetilde{\sigma} \circ \sigma_j)$.

PROOF. Suppose that $\mathcal{B}$ is maximally real polarization. Then G has a cuspidal parabolic subgroup $P = M \cdot A \cdot N_H$ such that $\mathcal{B} \subset \mathcal{P}$, $\mathcal{P} = \mathrm{Lie}P$, where $H = T \times A$ with $T = H \cap K$ and $A = \exp(\mathcal{A} \cap \mathcal{G})$. We note that $S^{U(1)} \cong (H \cdot N_H) \setminus G$ and $S^{U(1)}$ fibres over $P \setminus G$ with holomorphic fibres $T \setminus M$. Let $\mathcal{E}^{U(1)} \longrightarrow S^{U(1)}$ be irreducible, $\lambda = d(\widetilde{\sigma} \circ \sigma_j)|_{\mathcal{H}} \in \mathcal{H}^*$.

By a similar argument as in [9, §3] we see that the condition (4.3) is true. Thus, Proposition 4.1 follows that $\mathcal{E}^{U(1)} \longrightarrow S^{U(1)}$ satisfies both (MG) and (Z). Combining this with Propositions 3.1, 3.2 and 3.3 we obtain desired isomorphisms. □

Now we will extend the indicated results to arbitrary polarizations.

Fix a reductive datum $(F, \widetilde{\sigma})$ as in subsection 2.1. Suppose that $\mathcal{B} \subset \mathcal{G}_{\mathbf{C}}$ is a polarization such that $\mathcal{H} \subset \mathcal{B}$ and $\mathcal{B}$ is not maximal real. Applying Lemma 7.2 in [3] we have a complex simple root system for $(\mathcal{G}, \mathcal{H})$. Denote by S_α the Weil reflection and let

$$\Phi_0^+ = S_\alpha \Phi^+, \quad \mathcal{B}_0 = S_\alpha \mathcal{B} \quad \text{and} \quad S_0 = G \cdot \mathcal{B}_0 \tag{4.4}$$

Given $\gamma \in \Phi(\mathcal{G}_{\mathbf{C}}, \mathcal{H})$, we can view γ as an element of $(\mathcal{H} \oplus \mathcal{U}(1)_{\mathbf{C}})^*$. Since $\mathcal{H}$ is the Cartan subalgebra of $\mathcal{G}_{\mathbf{C}}$, we have a representation $e^\gamma : H^{U(1)} \longrightarrow \mathbf{C}^*$. Thus, the bundle $L_\gamma \longrightarrow H \setminus G$ associated to e^γ pushes down separately to line bundles $L_\gamma \longrightarrow S^{U(1)}$ and $L_\gamma \longrightarrow S_0^{U(1)}$. Applying Lemma 10.6 in [3] with $V = C^{-\omega}(G)$ we have G-equivariant morphisms of complexes

$$C^{-\omega}(H\backslash G; \pi_X^* \mathcal{E}^{U(1)} \otimes \Lambda^p \aleph_{S_0}^*) \longrightarrow C^{-\omega}(H\backslash G; \pi_X^* \mathcal{E}^{U(1)} \otimes L_{-\alpha} \otimes \Lambda^{p+1} \aleph_S^*) \tag{4.5}$$

which induce morphisms of subcomplexes

$$C^{-\omega}(S_0^{U(1)}; \pi_X^* \mathcal{E}^{U(1)} \otimes \Lambda^p \aleph_{S_0}^*) \longrightarrow C^{-\omega}(S^{U(1)}; \pi_X^* \mathcal{E}^{U(1)} \otimes L_{-\alpha} \otimes \Lambda^{p+1} \aleph_S^*) \tag{4.6}$$

By a similar argument as in [9, §4], we obtain

PROPOSITION 4.2. *Suppose that $\widetilde{\sigma} \circ \sigma_j$ is irreducible, so $d(\widetilde{\sigma} \circ \sigma_j)|_{\mathcal{H}} = \lambda \in \mathcal{H}^*$, and suppose further that $2\langle \lambda + \rho - \alpha, \alpha \rangle / \langle \alpha, \alpha \rangle$ is not a positive integer. Then (4.6) induces an isomorphism of cohomology groups.*

THEOREM 2. *We fix $(H, \mathcal{B})$ and suppose that $\mathcal{B}$ is not maximal real. Then, for arbitrary basic data of the form $(H, \mathcal{B}, \widetilde{\sigma} \circ \sigma_j)$, the bundle $\mathcal{E}^{U(1)} \longrightarrow S^{U(1)}$ has both properties (MG) and (Z). In other words, Theorem 1 holds for arbitrary basic data of the form $(H, \mathcal{B}, \widetilde{\sigma} \circ \sigma_j)$.*

PROOF. According to Theorem 1, every $\mathcal{E}^{U(1)} \longrightarrow S_{max}^{U(1)}$ has both (MG) and (Z). Thus we may assume by induction on $\dim S^{U(1)} - \dim S_{max}^{U(1)}$ that every $\mathcal{E}^{U(1)} \longrightarrow S_0^{U(1)}$ has both (MG) and (Z). On the other hand, Corollary 8.12 and Lemma 8.13 in [3] show that we need only prove (MG) and (Z) for irreducible $\mathcal{E}^{U(1)} \longrightarrow S^{U(1)}$. Since the cohomologies and maps that occur in Theorem 1 all are compatible with coherent continuation, we may assume that $2\langle\lambda+\rho-\alpha,\alpha\rangle/\langle\alpha,\alpha\rangle$ is not a positive integer, where $\lambda = d(\widetilde{\sigma}\circ\sigma_j)|_{\mathcal{H}} \in \mathcal{H}^*$.

We know that (4.8) restricts to a morphism of subcomplexes

$$C^{-\omega}_{P\backslash G}(S_0^{U(1)};\pi_X^*\mathcal{E}^{U(1)}\otimes\Lambda^p\aleph^*_{S_0}) \longrightarrow C^{-\omega}_{P\backslash G}(S^{U(1)};\pi_X^*\mathcal{E}^{U(1)}\otimes L_{-\alpha}\otimes\Lambda^{p+1}\aleph^*_S) \tag{4.7}$$

Applying Propositions 3.2 and 4.2 we see that (4.9) induces an isomorphism in cohomology. By induction on $\dim S^{U(1)} - \dim S^{U(1)}_{max}$, the complex

$$C^{-\omega}_{P\backslash G}(S_0^{U(1)};\pi_X^*\mathcal{E}^{U(1)}\otimes L_{-\alpha}\otimes\Lambda^{\cdot}\aleph^*_{S_0}) \tag{4.8}$$

has property (MG). Then, as in [3], the complex

$$C^{-\omega}_{P\backslash G}(S^{U(1)};\pi_X^*\mathcal{E}^{U(1)}\otimes L_{-\alpha}\otimes\Lambda^{\cdot}\aleph^*_S) \tag{4.9}$$

has property (MG). Similarly, applying Lemma 10.6 in [3] with $V = C^{for}(G)$ we obtain a morphism of complexes

$$C^{for}_{Q_F}(U_F\setminus G;\mathcal{E}^{U(1)}\otimes\Lambda^p\aleph^*_{S_0}) \to C^{for}_{Q_F}(U_F\setminus G;\mathcal{E}^{U(1)}\otimes L_{-\alpha}\otimes\Lambda^{p+1}\aleph^*_S) \tag{4.10}$$

Then we have the following commutative diagram

$$\begin{array}{ccc}
C^{-\omega}_{P\backslash G}(S_0^{U(1)};\mathcal{E}^{U(1)};\otimes\Lambda^p\aleph^*_{S_0})_{(K)} & \to & C^{-\omega}_{P\backslash G}(S^{U(1)};\mathcal{E}^{U(1)}\otimes L_{-\alpha}\otimes\Lambda^{p+1}\aleph^*_S)_{(K)} \\
\downarrow & & \downarrow \\
C^{for}_{Q_F}(U_F\setminus G;\mathcal{E}^{U(1)};\otimes\Lambda^p\aleph^*_{S_0})_{(K)} & \to & C^{for}_{Q_F}(U_F\setminus G;\mathcal{E}^{U(1)}\otimes L_{-\alpha}\otimes\Lambda^{p+1}\aleph^*_S)_{(K)}
\end{array}$$

of morphisms of K-finite subcomplexes.

We note that the first horizontal arrow in the diagram induces an isomorphism of cohomology (see [3, §10]). Applying Proposition 4.2 and passage to the K-finite subcomplex, we see that the second horizontal arrow in the diagram induces an isomorphism of cohomology. Also, by induction on $\dim S - \dim S_{max}$, the first vertical arrow in the diagram is an isomorphism on cohomology. Then the second vertical arrow in the diagram is a cohomology isomorphism. In other words, the bundle $\mathcal{E}^{U(1)} \longrightarrow S^{U(1)}$ has property (Z). This completes the proof of Theorem 2. □

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DEPARTMENT OF MATHEMATICS
HUE COLLEGE OF TEACHER EDUCATION
32 LE LOI STREET, HUE, VIETNAM