ACTA MATHEMATICA VIETNAMICA
Volume 19, Number 1, 1994, pp. 13–24

ON THE GROWTH RATE OF SLOWLY VARYING FUNCTIONS

ALLAN GUT

Abstract. A function L is slowly varying (at infinity) iff $\frac{L(tx)}{L(x)} \to 1$ as $x \to \infty$ for every $t > 0$. Motivated by two examples we investigate to what extent, if at all, the limit of the ratio equals 1 when t is replaced by some function of x growing to infinity.

1. Introduction

A positive, measurable function L on $(0, \infty)$ is *slowly varying* (at infinity) if, for all $t > 0$,

$$\frac{L(tx)}{L(x)} \to 1 \quad \text{as} \quad x \to \infty. \tag{1.1}$$

The "typical" slowly varying function is $\log x$ (or rather $\max\{\log x, 1\}$) and its iterates.

Slowly varying functions have several additional nice properties. There are,, however, properties enjoyed by $\log x$, which do not hold for arbitrary slowly varying function.

Recently I was confronted with the following question: Does there exist $c > 0$ and $\alpha > 1$ such that the relation

$$L(x^\alpha) < cL(x) \tag{1.2}$$

holds for all slowly varying functions L ?

For $L(x) = \log x$ the answer is trivially positive. However, the following example shows that the relation (1.2) does not hold for all slowly varying functions.

Received March 26, 1993

EXAMPLE 1.1. Define

$$f(x) = (\log x)^{\log\log x}, \quad x > e.$$

Then, for $t > 0$,

$$\begin{aligned} \frac{f(tx)}{f(x)} &= \left(\frac{\log t + \log x}{\log x}\right)^{\log\log x} \cdot (\log t + \log x)^{\log(\log tx \log x} \\ &= \left(1 + \frac{\log t}{\log x}\right)^{\log\log x} .(\log x + \log t)^{\log(1+\log t/\log x)} \to 1 \quad \text{as} \quad x \to \infty, \end{aligned}$$

since the first factor tends to 1 as $x \to \infty$ and $\log(1+\log t/\log x) \sim \log t/\log x \to 0$ as $x \to \infty$, so that the second factor also tends to 1 as $x \to \infty$. This proves that f is slowly varying.

However, for all $\alpha > 1$ we have

$$\begin{aligned} \frac{f(x^\alpha)}{f(x)} &= \frac{(\alpha \log x)^{\log\alpha + \log\log x}}{(\log x)^{\log\log x}} \\ &= \alpha^{\log\alpha + \log\log x} \cdot (\log x)^{\log\alpha} \to \infty \quad \text{as} \quad x \to \infty, \end{aligned}$$

that is, (1.2) does not hold.

Some time later, and independently, I came across a related question, namely the problem whether or not the property

$$\frac{L(x(L(x))^{1/r})}{L(x)} \to 1 \quad \text{as} \quad x \to \infty \quad (r > 1) \tag{1.3}$$

is universal for slowly varying functions; see Heyde (1968), p. 358.

In Bojanic and Seneta (1971) (see also Bingham et al. (1987), p. 78) it is shown that the function

$$f(x) = \exp\{(\log x)^\beta\}, \quad 0 < \beta < 1,$$

is slowly varying and that, for $\alpha > 0$,

$$\frac{f(x(f(x))^\alpha)}{f(x)} \to \begin{cases} 1 & \text{for} \quad 0 < \beta < 1/2 \\ \exp\{\alpha\beta\} & \text{for} \quad \beta = 1/2 \\ +\infty & \text{for} \quad 1/2 < \beta < 1. \end{cases} \tag{1.4}$$

In Rosalsky (1987), pp. 214–215 it is shown that the function $\exp\{\frac{\log x}{\log\log x}\}$ is slowly varying, but such that the ratio in (1.3) tends to $+\infty$ as $x \to \infty$.

REMARK 1.1. Note that the functions in these counterexamples can also be used to provide an answer to (1.2).

The purpose of this note is to study the more general question that these examples raise, namely:

Does there exist a positive function g, such that the relation

$$\frac{L(x \cdot g(x))}{L(x)} \to 1 \quad \text{as} \quad x \to \infty \tag{1.5}$$

holds for all slowly varying functions?

Since the answer is trivially yes for slowly varying functions which have a finite limit as $x \to \infty$ we exclude this case in the following. We thus assume from now on that $L(x) \to +\infty$ as $x \to \infty$. We shall also restrict ourselves to consider (ultimately) nondecreasing slowly varying functions. Thus, widely oscillating slowly varying functions, such as $L(x) = \exp\{(\log x)^{\frac{1}{3}} \cos((\log x)^{\frac{1}{3}})\}$ (cf. Bingham et al. (1987), p. 16) are excluded from our investigation.

After some preliminary considerations in Section 2 we show, in Section 3, that for the class $g(x) = \log x$ or an iterate of $\log x$ the answer is negative (even with the assumption of ultimate monotonicity). In Section 4 we show that the answer is negative also in the general case. More precisely we show that for any (ultimately) nondecreasing function g that tends to $+\infty$ as $x \to \infty$ the following holds:

(i) $f(x) = \exp\{\frac{\log x}{\log\log g(x)}\}$ is slowly varying and

$$\limsup_{x\to\infty} \frac{f(xg(x))}{f(x)} = +\infty.$$

(ii) $f(x) = \exp\{\frac{\log x}{\log g(x)}\}$ is slowly varying and there exists a sequence $\{x_k, k \geq 1\}$ of real numbers tending to $+\infty$ as $k \to \infty$, such that

$$\lim_{k\to\infty} \frac{f(x_k g(x_k))}{f(x_k)} = e.$$

As a comparison we also show that

(iii) $f(x) = \exp\{\frac{\log x}{g(x)}\}$ is slowly varying and

$$\frac{f(xg(x))}{f(x)} \to 1 \quad \text{as} \quad x \to \infty.$$

For a given function g we can thus, by choosing f suitably, obtain an infinite limit superior and a finite limit point which is different from 1 (it follows easily that any limit different from 1 could have been obtained in (ii)). That the limit of the ratio can be 1 is obvious; (iii) is there merely for the comparison. It follows, in particular, that the trichotomy in (1.4) also holds in the general case.

If the function g is suitably well behaved the ratio in (1.5) actually converges to the quantities given in the right hand sides of (i)–(iii), respectively. This is, for example, the case when g is an iterated logarithm as will be seen in Section 3.

2. Preliminaries

A positive measurable function U on $(0, \infty)$ is *regular varying* (at infinity) with exponent $\rho(-\infty < \rho < \infty)$ if, for all $t > 0$,

$$\frac{U(tx)}{U(x)} \to t^\rho \quad \text{as} \quad x \to \infty. \tag{2.1}$$

Moreover, if U varies regularly with exponent ρ, then $U(x) = x^\rho L(x)$, where L is slowly varying.

It is easy to see that if L is slowly varying and $h_1(x)$ and $h_2(x)$ are real valued functions tending to ∞ as $x \to \infty$, then

$$\frac{h_1(x)}{h_2(x)} \to c \ (0 < c < \infty) \quad \text{as} \quad x \to \infty \Longrightarrow \frac{L(h_1(x))}{L(h_2(x))} \to 1 \quad \text{as} \quad x \to \infty. \tag{2.2}$$

This means that the ratio between the arguments need not be constant for (1.1) to hold; approximately constant is enough.

Where $c = 1$ (2.2) also holds for regularly varying functions (if $c \neq 1$ the limit equals c^ρ). Moreover, if U is regularly varying with exponent $\rho \neq 0$, then

$$\frac{h_1(x)}{h_2(x)} \to 1 \quad \text{as} \quad x \to \infty \Longleftrightarrow \frac{U(h_1(x))}{U(h_2(x))} \to 1 \quad \text{as} \quad x \to \infty. \tag{2.3}$$

see e.g. Gut (1988), Lemma B.2.2.

Since, for example,

$$\frac{\log(x\log x)}{\log x} = \frac{\log x + \log\log x}{\log x} \to 1 \quad \text{as} \quad x \to \infty, \tag{2.4}$$

it is clear that the converse of (2.2) cannot hold for slowly varying functions in general.

We also note that the ratio between the arguments in (2.4) tends to $+\infty$ and, yet, the ratio of the values of the function tends to 1. Furthermore

$$\frac{\log(x^\alpha)}{\log x} = \alpha \quad \text{and} \quad \frac{\log(e^x)}{\log x} = \frac{x}{\log x} \to \infty \quad \text{as} \quad x \to \infty, \tag{2.5}$$

that is, if the ratio between the arguments grows somewhat faster than logarithmic the ratio limit exists, but differs from 1 and if it grows exponentially fast then the ratio limit is $+\infty$.

With this in mind it is natural to consider the question posed in (1.5).

We close this section by recalling some formulas from analysis. The first one is

$$\log(1+y) \le y \quad \text{for} \quad 0 < y < 1 \quad \text{and} \quad \log(1+y) \sim y \quad \text{as} \quad y \to 0,$$

from which it, for example, follows that, for $t > 0$,

$$\log\log tx - \log\log x = \log\left(1 + \frac{\log t}{\log x}\right) \sim \frac{\log t}{\log x} \quad \text{as} \quad x \to \infty,$$

$$\log_3 tx - \log_3 x = \log\left(1 + \frac{\log(1 + (\log x)^{-1}\log t)}{\log\log x}\right)$$

$$\sim \frac{\log t}{\log x \cdot \log\log x} \quad \text{as} \quad x \to \infty,$$

$$\log_m tx - \log_m x \sim \left(\prod_{k=1}^{m-1} \log_k x\right)^{-1} \cdot \log t \quad \text{as} \quad x \to \infty,$$

where $\log_1 x = \max\{\log x, 1\}$ and $\log_m x = \log_1(\log_{m-1} x)$ for $m \ge 2$.

These formulas will be used below without specific reference. They will also be used in cases where t is replaced by $\log x$ or some function of x which grows like $o(\log x)$ as $x \to \infty$.

Finally, we also mention the relation

$$(1+y)^\beta \sim 1+\beta y \quad \text{as} \quad y \to 0 \quad (\beta > 0).$$

3. The case $g = \log_m x$

In this section we thus do not make any monotonicity assumptions on the slowly varying functions.

Consider the following weakening of (1.2): Does it hold universally for slowly varying functions L that there exists $c > 0$, such that

$$L(x \log x) < cL(x)? \tag{3.1}$$

To see that the answer is negative we consider the following generalization of Rosalsky's example.

EXAMPLE 3.1. Define, for $\gamma > 1$,

$$f(x) = \exp\left\{\frac{\log x}{(\log_2 x)^{\gamma-1}}\right\}, \quad x > 0. \tag{3.2}$$

For $\gamma = 2$ we thus rediscover Rosalsky's example.

Let $t > 0$ and $x > e^e$. Then, if $t > 1$, we have

$$\begin{aligned} 1 \le \frac{f(tx)}{f(x)} &= \exp\left\{\frac{\log t + \log x}{(\log_2 tx)^{\gamma-1}} - \frac{\log x}{(\log_2 x)^{\gamma-1}}\right\} \\ &\le \exp\left\{\frac{\log t}{(\log_2 x)^{\gamma-1}}\right\} \to 1 \quad \text{as} \quad x \to \infty, \end{aligned}$$

and, if $0 < t < 1$, we have

$$1 \ge \frac{f(tx)}{f(x)} \ge \exp\left\{\frac{\log t}{(\log_2 tx)^{\gamma-1}}\right\} \to 1 \quad \text{as} \quad x \to \infty.$$

This proves that f is slowly varying.

To check whether or not (3.1) holds we find that, as $x \to \infty$,

$$\begin{aligned} \frac{f(x \log x)}{f(x)} &= \exp\left\{\frac{\log x + \log_2 x}{(\log_2(x \log x))^{\gamma-1}} - f \log x (\log_2 x)^{\gamma-1}\right\} \\ &\sim \exp\left\{\frac{\log x + \log_2 x}{(\log_2 x + \frac{\log_2 x}{\log x})^{\gamma-1}} - \frac{\log x}{(\log_2 x))^{\gamma-1}}\right\} \end{aligned}$$

$$
\begin{aligned}
&= \exp\left\{\frac{(\log_2 x)^{2-\gamma}}{(1+(\log x)^{-1})^{\gamma-1}} - \frac{\log x \cdot ((1+(\log x)^{-1})^{\gamma-1} - 1}{(\log_2 x^{\gamma-1}(1+(\log x)^{-1})^{\gamma-1})}\right\} \\
&\sim \exp\left\{\frac{(\log_2 x)^{2-\gamma}}{1+(\gamma-1)/\log x} - \frac{\gamma-1}{(\log_2 x)^{\gamma-1}(1+(\gamma-1)/\log x)}\right\} \\
&\sim \exp\left\{(\log_2 x)^{2-\gamma} - \frac{\gamma-1}{(\log_2 x)^{\gamma-1}}\right\} \\
&\sim \exp\left\{(\log_2 x)^{2-\gamma}\right\},
\end{aligned}
$$

(where the last $\sim$ are treated with caution when $\gamma = 2$). It follows that, with f defined by (3.2), we have, as $x \to \infty$,

$$
\frac{f(x \log x)}{f(x)} \to \begin{cases} 1 & \text{for} \quad \gamma > 2 \\ e & \text{for} \quad \gamma = 2 \\ +\infty & \text{for} \quad 1 < \gamma < 2. \end{cases} \tag{3.3}
$$

This establishes that the three possibilities for the limit of the ratio may occur in this example.

The corresponding weakening of (1.3) is to ask whether or not, for every slowly varying function L, there exists $c > 0$, such that the relation

$$
L(x \log L(x)) < cL(x) \tag{3.4}
$$

holds.

The relevant ratio to investigate thus is $\frac{L(x \log L(x))}{L(x)}$. An analysis like that of Example 3.1, with f as defined in (3.2), shows that the conclusion in (3.3) also holds for $\frac{f(x \log f(x))}{f(x)}$ as $x \to \infty$. We omit the details.

As a final, and, in fact, fairly general, example we show that the generalization of (3.1) to iterated logarithms of an arbitrary order follows the same pattern.

EXAMPLE 3.2. Consider the function

$$
f_m(x) = \exp\left\{\frac{\log x}{\log_m x}\right\}, \quad x > 0, \tag{3.5}
$$

where m is an integer ≥ 3.

By arguing as in Example 3.1 it follows that f_m is slowly varying. Now, for $m \geq 3$, we have, for large x,

$$\begin{aligned}
\frac{f_{m+2}(x\log_m x)}{f_{m+2}(x)} &= \exp\left\{\frac{\log x + \log_{m+1} x}{\log_{m+2}(x\log_m x)} - \frac{\log x}{\log_{m+2} x}\right\} \\
&\sim \exp\left\{\frac{\log x + \log_{m+1} x}{\log_{n+2} x + (\prod_{k=1}^{m+1} \log_k x)^{-1} \cdot \log_{m+1} x} - \frac{\log x}{\log_{m+2} x}\right\} \\
&= \exp\left\{\frac{\log_{m+1} x}{\log_{m+2} x + (\prod_{k=1}^{m} \log_k x)^{-1}}\right. \\
&\qquad \left. - \frac{(\prod_{k=2}^{m} \log_k x)^{-1}}{\log_{m+2} x + (\prod_{k=1}^{m+1} \log_k x)^{-1}}\right\} \\
&\sim \exp\left\{\frac{\log_{m+1} x}{\log_{m+2} x}\right\} \to \infty \quad \text{as} \quad x \to \infty.
\end{aligned}$$

The same computations applied to f_{m+1} yield

$$\frac{f_{m+1}(x\log_m x)}{f_{m+1}(x)} \sim \exp\left\{\frac{\log_{m+1} x}{\log_{m+1} x}\right\} = e \quad \text{as} \quad x \to \infty.$$

With f_m one finally obtains

$$\begin{aligned}
1 \leq \frac{f_m(x\log_m x)}{f_m(x)} &= \exp\left\{\frac{\log x + \log_{m+1} x}{\log_m(x\log_m x)} - \frac{\log x}{\log_m x}\right\} \\
&\leq \exp\left\{\frac{\log_{m+1} x}{\log_m x}\right\} \to 1 \quad \text{as} \quad x \to \infty.
\end{aligned}$$

In the following, final, section we consider the general case.

4. The general case

The general problem thus is the following:

Does there exist a positive, nondecreasing function g tending to $+\infty$ as $x \to \infty$, such that the relation

$$\frac{L(x \cdot g(x))}{L(x)} \to 1 \quad \text{as} \quad x \to \infty \tag{4.1}$$

holds for all (ultimately) nondecreasing slowly varying functions L, such that $L(x) \to \infty$ as $x \to \infty$?

A somewhat weaker question is whether or not the ratio remains bounded as $x \to \infty$.

We now show that the answer to the question is as outlined at the end of Section 1.

First, let h be any nonnegative, nondecreasing function tending to $+\infty$ as $x \to \infty$ and set

$$L(x) = x^{1/h(x)} = \exp\left\{\frac{\log x}{h(x)}\right\}, \quad x > e \tag{4.2}$$

(and, for example, $L(x) = 1$, for $0 < x < e$ and assume that L is (ultimately) nondecreasing and tending to ∞ as $x \to \infty$). By performing computations like those of Example 3.1 it follows that L is slowly varying. Although we do not need it, we mention in passing, that it follows from Bingham et al. (1987), Proposition 1.3.6, that h is slowly varying.

To show that the limit of the ratio in (4.1) may be infinite we define

$$f(x) = \exp\left\{\frac{\log x}{h(x)}\right\}, \quad \text{where } h(x) = \log_2 g(x) \tag{4.3}$$

and g is a candidate in the problem. The function f thus defined is of type (4.2) and, hence, slowly varying.

In view of the results in Section 3 we suppose that $g(x) = o(\log x)$ as $x \to \infty$. Consequently, there exists a sequence $S = \left\{x_k,\ k \geq 1\right\}$ of real numbers tending to $+\infty$ as $k \to \infty$, such that, for $k = 1, 2, \ldots,$

$$\frac{g(x_k g(x_k))}{g(x_k)} \leq \frac{\log(x_k g(x_k))}{\log x_k} = 1 + \frac{\log g(x_k)}{\log x_k}. \tag{4.4}$$

Suppose that $x \in S$. It follows from (4.4) that

$$\begin{aligned} h(xg(x)) - h(x) &= \log_2 g(xg(x)) - \log_2 g(x) \\ &= \log(1 + (\log g(x))^{-1} \cdot \log \frac{g(xg(x))}{g(x)}) \\ &\leq \log(1 + (\log g(x))^{-1} \cdot \log(1 + \frac{\log g(x)}{\log x})) \\ &\leq \log(1 + \frac{1}{\log x}) \leq \frac{1}{\log x}. \end{aligned} \tag{4.5}$$

We are now ready to estimate the ratio; recall that we consider points $x \in S$.

$$\frac{f(xg(x))}{f(x)} = \exp\left\{\frac{\log x + \log g(x)}{h(xg(x))} - \frac{\log x}{h(x)}\right\}$$

$$
\begin{aligned}
&= \exp\left\{\frac{\log g(x)}{h(xg(x))} - \frac{\log x\cdot(h(xg(x)) - h(x))}{h(x)h(xg(x))}\right\} \\
&\geq \exp\left\{\frac{\log g(x)}{h(x)+\frac{1}{\log x}} - \frac{1}{h(x)(h(x)+\frac{1}{\log x})}\right\} \qquad (4.6)\\
&\geq \exp\left\{\frac{\log g(x)}{\log_2 g(x)+o(1)} - \frac{1}{(\log_2 g(x))^2}\right\} \to \infty \quad \text{as} \quad x\to\infty.
\end{aligned}
$$

We have thus proved that

$$
\limsup_{x\to\infty} \frac{f(xg(x))}{f(x)} = +\infty, \tag{4.7}
$$

and the proof for this case is complete.

By defining f as in (4.2) with $h(x) = \log g(x)$ we have the trivial upper bound (recall that h is nondecreasing)

$$
\frac{f(xg(x))}{f(x)} \leq \exp\left\{\frac{\log x + \log g(x)}{h(x)} - \frac{\log x}{h(x)}\right\} = e.
$$

Now, let $x \in S$. Then

$$
\begin{aligned}
h(xg(x)) - h(x) = \log g(xg(x)) - \log g(x) = \log\frac{g(xg(x))}{g(x)} \leq \\
\leq \log\left(1+\frac{\log g(x)}{\log x}\right) \leq \frac{\log g(x)}{\log x}. \qquad (4.8)
\end{aligned}
$$

By modifying the computations above for the lower bound in the obvious manner, it now follows that

$$
\frac{f(xg(x))}{f(x)} \geq \exp\left\{\frac{\log g(x) - 1}{\log g(x)(1+\frac{1}{\log x})}\right\} \to e \quad \text{as} \quad x\to\infty.
$$

We have thus show that

$$
\lim_{k\to\infty} \frac{f(x_k g(x_k))}{f(x_k)} = e, \tag{4.9}
$$

that is, we have obtained a limit point of the ratio which is a finite number different from 1.

Finally, for $h(x) = g(x)$ the analogous upper bound is $\exp\Big\{\log g(x)/g(x)\Big\}$ $\to 1$ as $x\to\infty$, in which case the ratio thus converges to 1 as $x \to \infty$.

This establishes the claims made at the end of Section 1 and we are done.

REMARK 4.1. If g satisfies some additional regularity condition one can obtain actual convergence of the ratios under investigation. One such case is when g is an iterated logarithm; recall Section 3. Another condition that is sufficient is that, for all sufficiently large x, we have

$$\frac{g(2x)}{g(x)} \leq 1 + \frac{c}{\log x}, \tag{4.10}$$

where c is some positive constant; with this assumption it is easily seen that the above estimates actually hold for all large x.

Again, the iterated logarithms enjoy this property.

REMARK 4.2. It is also possible to depart from the representation formula for slowly varying functions, according to which a function L is slowly varying iff it is of the form

$$L(x) = c(x) \cdot \exp\Big\{ \int_1^x \frac{\epsilon(y)}{y} dy \Big\},$$

where $c(x) \to c, (0 < c < \infty)$ and $\epsilon(x) \to 0$ as $x \to \infty$ (see E.g. Bingham et al. (1987), Theorem 1.3.1).

Now, let g be given. It is sufficient to find a solution of the problem within the class of slowly varying functions, for which $c(x) = c$ for all x and $\epsilon(x)$ is decreasing to 0 as $x \to \infty$. Under these assumptions we have

$$\frac{L(x \cdot g(x))}{L(x)} = \exp\Big\{ \int_x^{xg(x)} \frac{\epsilon(y)}{y} dy \Big\} \geq \exp\Big\{ \epsilon(xg(x)) \cdot \log g(x) \Big\}. \tag{4.11}$$

The ratio thus tends to ∞ as $x \to \infty$ for choices of $\epsilon(x)$, such that

$$\epsilon(xg(x)) \cdot \log g(x) \to \infty \quad \text{as} \quad x \to \infty. \tag{4.12}$$

Towards this end, let $\epsilon(x) = \frac{1}{\log_2 g(x)}$. The expression in (4.12) then becomes

$$\frac{\log g(x)}{\log_2 g(xg(x))}, \tag{4.13}$$

which coincides with the first (and dominating) term of the exponent in (4.6). Thus, by considering points $x \in S$, defined above, together with the arguments following (4.5) we obtain the conclusion (4.12) (for $x \to \infty$, such that $x \in S$) and (4.7) follows as desired.

Similar arguments can be made for the other cases. We omit the details.

ACKNOWLEDGEMENT. I wish to thank Dr. Charles Goldie for drawing my attention to Bojanic and Seneta (1971) and for mentioning that the conclusions can be obtained via representation formulas.

REFERENCES

[1] N.H. Bingham, C.M. Goldie and J.L. Teugels, "Regular variation," Cambridge University Press, 1987.

[2] R. Bojanic and E. Seneta, *Slowly varying functions and asymptotic relations*, J. Math. Anal. Appl. **34** (1971), 302–315.

[3] A. Gut, *Stopped random walks. Limit theorems and applications*, "Springer, New York, (1988)."

[4] C.C. Heyde, *On almost sure convergence for sums of independent random variables*, Sankya Ser. A **30** (1968), 353–358.

[5] A. Rosalsky, *Strong stability of normed weighted sums of pairwise i.i.d. random variables*, Bull. Inst. Math. Academia Sinica **15** (1987), 203–219.

UPPSALA UNIVERSITY
DEPARTMENT OF MATHEMATICS
S-75106 UPPSALA
SWEDEN