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SOME RESULTS ON RINGS WHOSE CYCLIC COFAITHFUL MODULES ARE GENERATORS

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1. Introduction

Throughout this paper all rings are associative with identity and all modules are unitary. For a module M over a ring R we write M_R $({}_RM)$ to indicate that M is a right (left) R-module. We denote the category of all right R-modules by Mod-R and for any $M \in$ Mod-R, $\sigma[M]$ stands for the full subcategory of Mod-R whose objects are submodules of M-generated modules (see Wisbauer [13]).

For any nonempty subset X of M_R (resp. ${}_RM$), the *right (resp. left) annihilator* of X is denoted by $r(X)$ (resp. $l(X)$). If $\mathrm{r}(M) = 0$ then M_R is called *faithful.*

The trace ideal of M in R is denoted by trace(M). Concepts which are unnotified by "right" or "left" mean that both sides are satisfied (e.g. X is an ideal means X is two-sided ideal, R is FSG means R is right and left FSG ...). A submodule N of M is called essential in M if $N \cap H \neq 0$ for any nonzero submodule H of M. In this case one says also that M is an essential extension of N. A submodule C of a module M is called a closed submodule of M if C is the only essential extension of C in M. If every non-zero submodule of a non-zero module M is essential in M, then M is called a uniform module. A module M_R is called a CS-module if every submodule of M is essential in a direct summand of M. A ring R is called right CS if R_R is a right CS-module.

A ring R is right bounded if every essential right ideal contains a nonzero ideal, it is strongly right bounded if every nonzero right ideal contains a nonzero ideal.

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A ring is called *quasi-Frobenius* (briefly, QF) if it is right (or left) self-injective and right (or left) Artinian and a ring R for which every faithful right R-module is a generator is called *right pseudo-Frobenius* (briefly, *right PF*). It is well-known that every QF ring is PF.

A module M_R is called *cofaithful* if there exists a finite subset $\{m_1, m_2, \cdots, m_n\}$ of elements of M such that

$$r(\{m_1, m_2, \ldots, m_n\}) = 0.$$

By [12, Lemma 2.1] a module M_R is cofaithful if and only if $\sigma[M] = \text{Mod}-R$. A module M_R with this property is called a subgenerator of Mod-R (see Wisbauer [13, p.118]). Therefore, instead of cofaithful right R-modules we sometime use the terminology "right R-subgenerators". Clearly any cofaithful module is faithful.

A ring R is called *right finitely pseudo-Frobenius* (briefly, *right FPF*) if every finitely generated faithful right R-module is a generator. For a detailed study of FPF rings we refer to the book of Faith and Page [9]. A ring R is said to be *generated by faithful right cyclics* (briefly, *right GFC*) if every cyclic faithful right R-module is a generator. GFC rings were first considered by Birkenmeier and many results about these rings were obtained in [3], [4], [5].

A generalization of right self-injective and right FPF rings has been introduced and investigated in [12] (see also [6]): A ring R is called *right FSG* if every finitely generated cofaithful right R-module (= right R-subgenerator) is a generator. Now we define a class of rings which is a generalization of GFC and FSG rings: A ring R is called *right CSG* if every cyclic cofaithful right R-module (= right R-subgenerator) is a generator. The purpose of this work is to present a study of the class of these rings.

The texts by Anderson and Fuller [1], Faith [8], Faith and Page [9], Stenström [11] and Wisbauer [13] are general references for ring-theoretic notions not defined in the paper.

2. Preliminaries

The following lemma provides a charaterization of cofaithful modules:

LEMMA 2.1. *Let $M_R \in$ Mod-R. Then the following conditions are equivalent:*

(i) M_R is cofaithful.

(ii) There exists a finite suset $\{m_1, m_2, \dots, m_n\}$ of elements of M such that $r(\{m_1, m_2, \dots, m_n\}) = 0$.

(iii) There exists a postive integer n such that R can be embedded as an R-submodule of M^n.

(iv) M generates every injective right R-module.

(v) $\sigma[M] =$ Mod-R.

(vi) Cyclic submodules of $M^{(\mathbb{N})}$ form a set of generators in mod-R.

PROOF. See [12, Lemma 2.1]. □

Note that the class of right CSG rings is a nontrivial generalization of the class of right GFC rings and of FSG rings as the following examples show.

EXAMPLE 2.2. Let D be a division ring (e.g. $D = \mathbb{R}$) and $S = End_D(V)$, where V is an infinite dimensional vector space over D (e.g. $V = \mathbb{R}^{(\mathbb{N})}$). Then S is right CSG because of the right self-injectivity of S. Let $f_1 \in S$ be a map with one-dimensional rank as follows:

$$f_1 : D^{(\mathbb{N})} \longrightarrow D^{(\mathbb{N})}$$

$$(x_1, x_2, \dots) \longmapsto (x_1, 0, 0, \dots).$$

It is easy to see that $f_1 S$ is cyclic faithful but it can not generate S because S is the direct product of infinitely many copies of $f_1 S$.

EXAMPLE 2.3. Let F be a field with a proper subfield K, set $F_n = F$ and $K_n = K$ for each $n \in \mathbb{N}$, set $Q = \prod_{n=1}^{\infty} F_n$ and define the ring R by

$$R = \{(x_n) \in Q : x_n \in K_n \text{ for all but a finite number of } n\}.$$

Then R is a strongly regular continuous ring which is not self-injective (see Goodearl [10, Example 13.8]). By [3, Proposition 1.2], R is right GFC and so it is right CSG. However by [12, Proposition 2.8], R is not right FSG.

3. Results

First we prove a lemma which provides an internal characterization of a right CSG ring.

Lemma 3.1. *For a ring R the following conditions are equivalent:*

(i) R is a right CSG ring.

(ii) If $M = mR$ is a cofaithful module, then $l(r(m))R = R$.

(iii) For any right ideal X of R such that R/X is cofaithful, $l(X)R = R$.

Proof. (i) $\Longrightarrow$ (ii). If $M = mR$ is a cofaithful module, then M is a generator. By [1, p. 111] and [1, Exercise 8], $R = \text{trace}M = \text{trace}(R/r(m)) = l(r(m))R$.

(ii) $\Longrightarrow$ (i) is obvious.

(i) $\Longrightarrow$ (iii) can be obtained as in the case (i) $\Longrightarrow$ (ii).

(iii) $\Longrightarrow$ (i). Let X be any right ideal of R such that R/X is cofaithful. Since trace$(R/X) = l(X)R = R$, R/X is a generator. proving the fact R is right CSG. □

Recall that an idempotent e of R is said to be semicentral if $eR = eRe$ (equivalently if $r \in R$ then $er = ere$). Basic properties of right semicentral idempotent can be found in [2].

Proposition 3.2. *Let R be a right CSG ring, and e be a right semicentral idempotent. Then eRe is a right CSG ring.*

Proof. Write $R = eR \oplus (1-e)R$. Let X be a right ideal of eRe such that eR/X is a cofaithful right eRe-module. By [2, Lemma 1], X is then a right ideal of R. Since eR/X is a cofaithful right eRe-module, there exist $er_1, \ldots, er_n$ such that:

$$r_{eR}(\{er_1 + X, er_2 + X, \ldots, er_n + X\}) = 0.$$

It follows that

$$r_R(\{er_1 e + X, \ldots, er_n e + X\}) = 0.$$

Consequently, R/X is a cofaithful right R-module. By Lemma 3.1, $l(X)R = R$. It follows that $(el(X)e)eRe = eRe$. Hence $eR = eRe$ is a right CSG ring. □

By the same argument as in the proof of Proposition 2.7 in [12] we obtain the following result that states precisely when CSG rings are self-injective.

Proposition 3.3. *A ring is right self-injective if and only if it is right CSG and its right injective hull is cyclic.*

We note that every right GFC ring is right bounded. But it is unknown whether every right CSG ring is right bounded, too. The answer to this question seems to be negative and we are still trying to find a right self-injective ring which is not right bounded. Of course, if the answer of this question were positive, then we would have an interesting consequence: Any right self-injective ring is right bounded. However, exploiting technique in [9, 1.3B], we get:

PROPOSITION 3.4. *Let R be a right CSG ring. If I is any essential right ideal, then R/I is not cofaithful.*

By the same argument as in the proofs of Propositions 2.4 and 2.5 in [12] we obtain some characterizations of QF-rings in terms of CSG rings.

THEOREM 3.5. *(i) A ring R is QF if and only if R is a right CSG ring satisfying the descending chain condition (briefly, DCC) on right annihilators and R contains a faithful injective right ideal.*

(ii) A ring R is QF if and only if R is a right CSG, ring satisfying DCC on right annihilators and its injective hull is cyclic.

THEOREM 3.6. *The following conditions are equivalent for a ring R:*

(1) R is a QF-ring.

(2) R is right CSG and there is a quasi-injective cyclic cofaithful right module with the ascending chain condition (briefly, ACC) or DCC on essential submodules.

(3) R is a left CS right CSG ring and there is a continuous cyclic cofaithful right R-module with ACC on essential submodules.

(4) R is a right CSG, ring satisfying ACC on essential left or essential right ideals and its injective hull is cyclic.

Now we have a property of rings each of whose cyclic faithful right ideals is cofaithful. Recall that a ring R is right Baer if the right annihilator of every nonempty subset of R is generated by an idempotent.

PROPOSITION 3.7. *Let R be a right Baer ring. Then the following conditions are equivalent:*

(i) Every cyclic faithful right ideal of R is cofaithful.

(ii) Every ideal of R which is cyclic faithful as a right ideal is cofaithful.

(iii) Every cyclic faithful, torsionless right R-module is cofaithful.

PROOF. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are obvious.

(ii) $\Rightarrow$ (iii). Let M be a cyclic faithful, torsionless right R-module. Then $M \simeq R/I$ for some right ideal I of R. Hence

$$A = \text{trace}(R/I) = l(I)R.$$

Since R is a Baer ring, A is cyclic. If $r \in R$, $r \neq 0$, then there exists $m \in M$ such that $rm \neq 0$ because of the faithful property of M. Now since M is torsionless, there exists $f \in \text{Hom}_R(M, R)$ with $rf(m) = f(rm) \neq 0$, which shows that A is faithful. By (ii), A is cofaithful and so there exist $a_i \in A$, $1 \leq i \leq n$, such that $r(\{a_1, \cdots, a_n\}) = 0$. Since $a_i \in A$,

$$a_i = \sum_j f_{ij}(m_{ij}),$$

for $m_{ij} \in M$ and $f_{ij} \in \text{Hom}_R(M, R)$, and then $rm_{ij} = 0$ for all i, j (finitely) implies $ra_i = 0$ for all i, so $r(\{m_{ij}\}_{ij}) = 0$, proving that M is cofaithful.

COROLLARY 3.8. *Let R be a right Baer ring and $r\big(l(I)\big) = I$ for any right ideal of R and assume that every cyclic faithful right ideal of R is cofaithful. Then R is right CSG if and only if R is right GFC.*

PROOF. By [11, Exercise 3] and Proposition 3.7. □

The following result generalizes [4, Proposition 6]. Recall that R is called *ERT* if every essential right ideal of R is an ideal.

PROPOSITION 3.9. *Let R be a ring. Then R is ERT if and only if R is right bounded and for every ideal I which is right essential in R, R/I is strongly right bounded.*

PROOF. Let R be a ERT ring and I an ideal of R which is right essential in R. If A/I is a right ideal of R/I, then A is essential in R. Since R is ERT, A is a two-sided ideal of R. Hence A/I is an ideal of R/I, proving that R/I is strongly right bounded.

Conversely, let X be an essential right ideal of R, but X is not a left ideal of R. Let I be the sum of all ideals of R contained in X. Since X is not a left ideal of R, $I \subsetneq X$. Moreover, $I \neq 0$, because X is essential in R and R is right bounded. By assumption, R/I is strongly right bounded. It follows that these exists an ideal K of R containing I such that $K/I \subseteq X/I$ and K/I is a non-zero two-sided ideal of R/I. Hence $I \subsetneq K \subsetneq X$. However K is an ideal contained in X, hence $K \subseteq I$, i.e. $K = I$. This contradiction proves that X is a two-sided ideal of R. □

For a factor ring, we have a criterion:

THEOREM 3.10. *Let I be an ideal of R. Then the following conditions are equivalent:*

(i) R/I is right CSG.

(ii) If Y is a right ideal of R and $I \subseteq Y$ such that R/Y is a cofaithful R/I-module, then $R = (I : Y)R$, where $(I : Y) = \{x \in R$ such that $xY \subseteq I\}$.

PROOF. Let R/I be right CSG and Y a right ideal of R containing I in which R/Y is a cofaithful right R/I-module. By Lemma 3.1, $l(R/I)R/I = R/I$. But $l(Y/I) = (I : Y)/I$, hence

$$1 + I = \sum_{i=1}^{n} x_i r_i + I,$$

where $x_i \in (I : Y)$ and $r_i \in R$. Therefore $R = (I : Y)R$.

Conversely, assume Y/I is a right ideal of R/I such that R/Y is cofaithful, then by assumption

$$R = (I : Y)R.$$

So $1 = \sum_{i=1}^{n} x_i r_i$ where $x_i \in (I : Y)$ and $r_i \in R$. It follows that

$$1 + I = \sum_{i=1}^{n} (x_i + I)(r_i + I) \in l(Y/I).R/I.$$

Consequently, $R/I = l(Y/I).R/I$. By Lemma 3.1, R/I is a right CSG ring. □

Finally we give an example of non-CSG rings.

THEOREM 3.11. *If R is any ring, then $T_n(R)$ is neither right nor left CSG for $n > 1$, where $T_n(R)$ is the $n \times n$ lower (upper) triangular matrix ring over R.*

Consequently, $T_n(R)$ is neither right nor left self-injective.

PROOF. Let $T_n(R)$ be the $n \times n$ lower triangular matrix ring over R and

$$I = \begin{pmatrix} 0 & 0 & 0 & . & . & . & 0 \\ R & R & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ R & R & R & . & . & . & R \end{pmatrix}.$$

Then I is a right ideal of $T_n(R)$. Let $a,\ b \in I$, with

$$a = \begin{pmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ 1 & 0 & . & . & . & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 0 & 0 & . & . & . & 0 \\ 1 & 1 & 0 & . & . & . & 0 \\ 1 & 1 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 1 & 1 & 1 & . & . & . & 1 \end{pmatrix}.$$

It follows that I is a cofaithful right $T_n(R)$-module with $r(\{a,b\}) = 0$. Note that $I \simeq T_n(R)/J$, where

$$J = \begin{pmatrix} R & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 \end{pmatrix}.$$

It is easy to prove that $l(J) \cdot T_n(R) \neq T_n(R)$. Then by Lemma 3.1, $T_n(R)$ is not right CSG. □

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