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ON THE CALCULATION OF GENERALIZED GRADIENTS FOR A MARGINAL FUNCTION[1]

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1. Introduction

Many problems encountered in practice lead to minimizing a marginal function of the type

$$m(\omega) := min\{f(\omega, x)/\ g(\omega, x) \in C\},$$

where $\omega \in R^r, x \in R^s, C = R^p_+ \times \{0\}^q$ and $f : R^r \times R^s \to R^1, g : R^r \times R^s \to R^{p+q}$ are given functions. A specific feature of this function is that generally it is not differentiable, although the functions f, g may be smooth (continuously differentiable), or even linear in x. Under appropriate assumptions, m is locally Lipschitz and nondifferentiable optimization techniques are available for the solution of the problem (see, for example, [17], [18],[19], [21], [22], [30]). One of the crucial points of the existing algorithms is that at every $\omega \in R^r$ at least one element from the generalized subdifferential $\partial m(\omega)$ of m at ω (in the sense of Clarke [7]) must be computed. The reader is referred to [2], [9],[13], [25], [28], [29], [31], [32] and references therein for many results concerning the "outer" estimation for the generalized subdifferential that play an important role in deriving necessary optimality conditions. These results enable us to construct (from the given data on f, g) a certain set $D(\omega)$ such that

$$\partial m(\omega) \subset D(\omega)$$

and, hence, the set of stationary points for m (i.e. the points ω satisfying $0 \in \partial m(\omega)$) is contained in the set of zero-points for $D(.)$ (i.e. the points ω

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satisfying $0 \in D(\omega)$). The latter are also called "quasistationary" points for m. Since stationary points for m are very difficult to find, in many cases quasistationary points are taken instead as "practical" resolution. They require less computation, due to the fact that $D(\omega)$ is always simpler than $\partial m(\omega)$. When quasistationary points do not satisfy our goals, we have a possible way to find (exact) stationary points by using the mentioned "nonsmooth" optimization techniques, and unwillingly face with the long standing problem of "inner" estimation for ∂m, or, more precisely, of finding at least one element of ∂m. Despite the fact that the results on "outer" estimation are rich enough, the ones concerning "inner" estimations are still rather rare and they have been obtained only for extremely particular situations. We are referred to Gauvin-Dubeau [10], [11] for results concerning the case when the multiplier is unique and the same for every points of the solution set $S(\omega)$, and to Outrata [23] for the case when the solution set $S(\omega)$ reduces to a singleton. A concrete case was considered in [6] where the authors succeeded in finding a calculation rule for a specific problem arised in the design of water distribution networks. In [25] Penot gave several results on "outer" and "inner" estimations for a more general problem under some other assumptions that do not cover the results mentioned. In the present paper, we provide some other methods for finding a generalized gradient of the marginal function. One of them is a further development of a result of Janin given in [16], [25]. The organization of the paper is as follows. The next section describes general results. The remainder presents some special results for the case of practical importance, where f, g are linear in x.

2. Preliminary Results

Throughout the paper f, g are supposed to be smooth in both variables and the problem of defining m is convex (with respect to x). This means that, for any ω, the function $f(\omega, .)$ and the first p components of $g(\omega, .)$ are convex, and the rest q components of $g(\omega, .)$ are affine. So that the local and the global minimizers of the problem in question always coincide, and the value of m is well-defined. For every $\omega \in R^r$, we set

$$M(\omega) := \{x \in R^s \mid g(\omega, x) \in C\},$$

$$S(\omega) := \{x \in M(\omega) \mid f(\omega, x) = m(\omega)\}.$$

Under the convexity assumption (with respect to x), the set-valued mapping S is closed [14]. We make the blanket assumption that the set $M(\omega)$ is nonempty and uniformly bounded for ω in some neighborhood of a given point $\bar{\omega}$. Then, S is upper semi-continuous at $\bar{\omega}$ and, hence, is locally compact at $\bar{\omega}$. For further development we assume that the following *regularity condition* [27] holds, for any $\bar{x} \in S(\bar{\omega})$,

(**R**) $0 \in int\{g(\bar{\omega}, \bar{x}) + \nabla_x g(\bar{\omega}, \bar{x})R^s - C\}$.

Obviously, (**R**) is equivalent to

$$g(\bar{\omega}, \bar{x}) \in int\{\nabla_x g(\bar{\omega}, \bar{x})R^s + C\}. \tag{2.1}$$

The reader is referred to [24], [33] for the relation between this and other regularity conditions in the literature such as Slater condition, Mangasarian-Fromovitz constraint qualification,... Under this assumption, for any $\bar{x} \in S(\bar{\omega})$, one can find $\bar{y} \in C^+$ (the polar cone of C) such that for all $y \in C^+, x \in R^s$ the following holds

$$L(\bar{\omega}, y, \bar{x}) \leq f(\bar{\omega}, \bar{x}) + < \bar{y}, g(\bar{\omega}, \bar{x}) > \leq L(\bar{\omega}, \bar{y}, x) \; ; \tag{2.2}$$

here $L(\omega, y, x) := f(\omega, x) + < y, g(\omega, x) >$. Let $K(\bar{\omega}, \bar{x})$ denote the set of multipliers $\bar{y} \in C^+$ satisfying (2.2). It was pointed out in [9] (Proposition1.1) that Condition (**R**) implies the regularity at $(\bar{\omega}, \bar{x})$ in the sense of [9] which, in turn, means the regularity at $\bar{x}$ in the sense of [8] uniformly for ω on a neighborhood of $\bar{\omega}$. Since $S(\bar{\omega})$ is compact and S is upper semi-continuous at $\bar{\omega}$, one can easily see that the regularity condition in the sense of [8] holds at any $\bar{x} \in S(\omega)$ uniformly for ω in some neighborhood of $\bar{\omega}$. The latter implies that the set of multipliers $K(\omega, x)$ is uniformly bounded for ω in some neighborhood U of $\bar{\omega}$ and for $x \in S(\omega)$, ([8], Remark2.1). So that one can find $\alpha > 0$ such that

$$K(\omega, x) \subset C^+_\alpha := \{y \in C^+ \mid \|y\| \leq \alpha\}, \; \forall \omega \in U, x \in S(\omega).$$

From (2.2) we have

$$m(\omega) = max\{L(\omega, y, \bar{x}) \mid y \in C_\alpha^+\}, \tag{2.3}$$

where $\bar{x} \in S(\omega)$.

The following assumption will be used in the sequel.

(**A**) *The set-valued mapping $\omega \to S(\omega)$ admits a local selection $\bar{x}(\omega) \in S(\omega)$ which is smooth at $\bar{\omega}$.*

In what follows $\nabla_\omega f$ (resp. $\nabla_x f$) will denote the partial derivative of a function f with respect to ω (resp. x). An easy way to compute a generalized gradient of m is given by the following

PROPOSITION 2.1. *Let* (**R**) *and* (**A**) *hold. Then m is Lipschitz regular at $\bar{\omega}$ and, for any $\bar{y} \in K(\bar{\omega}, \bar{x})$ with $\bar{x} = \bar{x}(\bar{\omega})$, we have*

$$\nabla_\omega f(\bar{\omega}, \bar{x}) + [\nabla_\omega g(\bar{\omega}, \bar{x})]^* \bar{y} \in \partial m(\bar{\omega}). \tag{2.4}$$

PROOF. Under Condition (**A**), for any fixed y, the function $L(\omega, y, \bar{x}(\omega))$ is smooth at $\bar{\omega}$ and, hence, is Lipschitz regular at $\bar{\omega}$.
As $m(\omega) = max\{L(\omega, \bar{y}, \bar{x}(\omega)) \mid \bar{y} \in C_\alpha^+\}$ and C_α^+ is compact, we can invoke a result from [7] to deduce that m is locally Lipschitz (moreover, Lipschitz regular) at $\bar{\omega}$ and

$$\partial m(\bar{\omega}) = conv\{\nabla L(., \bar{y}, \bar{x}(.)) \mid_{\omega=\bar{\omega}} \mid \bar{y} \in K(\bar{\omega}, \bar{x})\}. \tag{2.5}$$

Note that

$$\nabla L(., \bar{y}, \bar{x}(.)) \mid_{\omega=\bar{\omega}} = \nabla_\omega L(\bar{\omega}, \bar{y}, \bar{x}(\bar{\omega})) + \nabla_x L(\bar{\omega}, \bar{y}, \bar{x}).\nabla \bar{x}(\bar{\omega}). \tag{2.6}$$

By the optimality condition (derived from the right-hand-side inequality of (2.2)) we have

$$\nabla_x L(\bar{\omega}, \bar{y}, \bar{x}) = 0. \tag{2.7}$$

The combination of (2.5),(2.6),(2.7) gives

$$\partial m(\bar{\omega}) = \{\nabla_\omega L(\bar{\omega}, \bar{y}, \bar{x}) \mid \bar{y} \in K(\bar{\omega}, \bar{x})\}$$

which implies (2.4) (the operation "conv" could be omitted in the preceding formula because the mapping $\bar{y} \to \nabla_\omega L(\bar{\omega}, \bar{y}, \bar{x})$ is affine and $K(\bar{\omega}, \bar{x})$ is a convex set). The proposition is thus proved.

REMARK 2.1. Under Condition (**A**) we, in fact, obtained more than (2.4), namely we have found not only one element but the whole generalized subdifferential $\partial m(\bar{\omega})$. To establish (2.4), the following weakened condition is sufficient.

(**A'**) *S admits a selection $\bar{x}(\omega) \in S(\omega)$ which is locally Lipschitz and differentiable at $\bar{\omega}$.*

Indeed, although it is not the case for applying the mentioned result of [7], but one can invoke Lemma 3.3 of the next section to claim that m is still locally Lipschitz (not necessarily Lipschitz regular) and, since $\bar{x}(.)$ is differentiable at $\bar{\omega}$, the following holds

$$\{\nabla L(., \bar{y}, \bar{x}(.)) \mid_{\omega=\bar{\omega}} \mid \bar{y} \in K(\bar{\omega}, \bar{x})\} \subset \partial m(\bar{\omega}).$$

From (2.6)-(2.7) it follows that

$$\nabla_\omega L(\bar{\omega}, \bar{y}, \bar{x}(\bar{\omega})) \in \partial m(\bar{\omega})$$

which implies (2.4).

REMARK 2.2. As locally Lipschitz functions are differentiable almost everywhere, the preceding remark provides a possibility to compute generalized gradients of m almost everywhere, if a locally Lipschitz selection of S does exist. Further, if one can compute generalized gradients ζ_n of m at any point of a sequence $\{\omega_n\}$ with limit $\bar{\omega}$, then any accumulation point of $\{\zeta_n\}$ will give a generalized gradient of m at $\bar{\omega}$, due to the closedness of ∂m. The reader is referred to [23] and the references therein for various conditions guaranteeing that S reduces to a single-valued map and possesses locally Lipschitzianess and/or differentiability property. The same formula as (2.4) was obtained by Outrata [23] and Gauvin-Dubeau [11], but for the case where either $S(\bar{\omega})$ is a singleton or $K(\bar{\omega}, \bar{x})$ is unique for every $\bar{x} \in S(\bar{\omega})$.

REMARK 2.3. It should be noted that Conditions (**A**), (**A'**) require the existence of a certain local selection $\bar{x}(\omega) \in S(\omega)$, but no concrete data of this selection is involved in the calculation of generalized gradients, according to

(2.4). There are simple examples showing that (2.4) does not hold in general (see [23]), so that the imposed conditions are essential.

REMARK 2.4. Condition (**A**) looks "implicit" and seems to be rather restrictive. However, as we will see later, it holds "almost everywhere" for many problems encountered in practice.

Formula (2.4) gives a very easy way to compute a generalized gradient of the marginal function at those points, where Condition (**A**) holds (such a point will be called *easy* in the sequel). For the other points the computation is more expensive. Besides the method taking accumulation points of sequences of generalized gradients at easy points, one can make another approach via directional derivatives. To do this, we introduce the following notion of [25], for any $\epsilon > 0$,

$$S_\epsilon(\omega) := \{x \in M(\omega) \mid f(\omega, x) < m(\omega) + \epsilon\}.$$

Let us recall the following definition of [25]:

DEFINITION 2.1. The problem is well-set at $\bar{\omega}$ in the direction $w \in R^r$ if for any sequences $\epsilon_n \longrightarrow 0+, t_n \longrightarrow 0+, w_n \longrightarrow w$ there exist subsequences $\{\epsilon_{n_k}\}, \{t_{n_k}\}, \{w_{n_k}\}$ and $x_{n_k} \in S_{\epsilon_{n_k}}(\bar{\omega}+t_{n_k}w_{n_k})$ such that $\lim x_{n_k} = x_0 \in S(\bar{\omega})$.

For $g = (g_1, g_2, ..., g_p, g_{p+1}, ..., g_{p+q})$ define

$$I(\bar{\omega}, \bar{x}) = \{i \in \{1, ..., p\} \mid g_i(\bar{\omega}, \bar{x}) = 0\}.$$

We are ready to state a result of Janin [16] (see also [25]).

PROPOSITION 2.2 ([25], COROLLARY 5.17). *Suppose that the following conditions hold*

(**E**) *The problem is well-set at* $\bar{\omega}$ *in the direction* w

(**J**) *For each subset* $I \subseteq I(\bar{\omega}, \bar{x})$ *there exists a neighborhood* $\mathcal{O}$ *of* $(\bar{\omega}, \bar{x})$ *such that the rank of the family* $\{\nabla_x g_i(\omega, x) \mid i \in I\}$ *is constant for all* $(\omega, x) \in \mathcal{O}$.

Then m *is differentiable at* $\bar{\omega}$ *in the direction* w *and*

$$\begin{aligned} m'(\bar{\omega}; w) &:= \lim_{(t,w')\to(0,w)} [m(\bar{\omega} + tw') - m(\bar{\omega})]/t \\ &= inf\{f'(\bar{\omega}, \bar{x})(w, v) \mid \bar{x} \in S(\bar{\omega}), g'(\bar{\omega}, \bar{x})(w, v) \in T_cC\}, \end{aligned} \tag{2.8}$$

where $c = g(\bar{\omega}, \bar{x})$ and $T_c C$ denotes the tangent cone to C at c.

From this we obtain the following result.

PROPOSITION 2.3. *Under the blanket assumption, if Condition (J) is fulfilled, then (2.8) holds for every $w \in R^r$.*

PROOF. It remains to show that the problem is well-set at $\bar{\omega}$ in any direction $w \in R^r$. Fix $w \in R^r$ and take sequences $\{\epsilon_n\}, \{t_n\}, \{w_n\}$ in accordance to the definition above. As $S(\omega) \subset S_\epsilon(\omega)$ for all $\epsilon > 0$, we take $x_n \in S(\bar{\omega} + t_n w_n)$ and have $x_n \in S_{\epsilon_n}(\bar{\omega} + t_n w_n)$, for all n. Since $\bar{\omega} + t_n w_n \longrightarrow \bar{\omega}$ and S is locally compact at $\bar{\omega}$, the sequence $\{x_n\}$ is contained in a certain compact set and, therefore, possesses a subsequence $\{x_{n_k}\}$ converging to a certain point x_0. Since S is closed at $\bar{\omega}$, we have $x_0 \in S(\bar{\omega})$ and the proposition is proved.

REMARK 2.5. The reader is referred to [10], [11], [23] for various conditions guaranteeing that the marginal function is Lipshitz regular (or its negative is Lipschitz regular in some other cases). Under Lipschitz regularity, $m^o(\bar{\omega}, w) = m'(\bar{\omega}, w)$ and formula (2.8) provides a possibility to compute $\partial m(\bar{\omega})$ (via standard codes) if the problem defining the value $m'(\bar{\omega}, w)$ is solvable. However, this problem is not linear (in $\bar{x}$) and may not be solved (numerically) in general. Furthermore, a serious difficulty may also come from the fact that the cone $T_{g(\bar{\omega},\bar{x})}C$ varies together with $\bar{x}$. Fortunately, for the problems which are linear in x one can overcome these difficulties and find $m'(\bar{\omega}, w)$ by solving a linear program. This will be done in the next section.

3. The Linear Case

In this part we consider the case when f, g are linear with respect to x. Problems of this kind have many important practical applications (such as in the design of water distribution networks [5]-[6], in maximum strength truss topology design,...) and, therefore, need a special treatment. Thus, in this part we suppose

$$f(\omega, x) = < c(\omega), x >,$$

$$g_i(\omega, x) = < a_i(\omega), x > + \alpha_i(\omega), i = 1, 2, ..., p + q,$$

where $c(.), a_i(.), i = 1,2,...p+q$, are vector functions from R^r to R^s, and $\alpha_i(.), i = 1,2,...,p+q$, are real functions on R^r. It is assumed throughout that these functions are smooth. Clearly,

$$f'(\bar\omega,\bar x)(w,v) = < c'(\bar\omega)w, \bar x > + < c(\bar\omega), v >,$$
$$g_i'(\bar\omega,\bar x)(w,v) = < a_i'(\bar\omega)w, \bar x > + < a_i(\bar\omega), v > + < \alpha_i'(\bar\omega), w > .$$

Fix $\bar\omega$ and set

$$I(\bar x) = I(\bar\omega,\bar x) = \{i \le p \mid g_i(\bar\omega,\bar x) = 0\}.$$

It is easy to see that $g'(\bar\omega,\bar x)(w,v) \in T_{g(\bar\omega,\bar x)}C$ if and only if

$$g_i'(\bar\omega,\bar x)(w,v) \le 0, i \in I(\bar x),$$
$$g_j'(\bar\omega,\bar x)(w,v) = 0, j = p+1,...,p+q.$$

Assuming that Condition (J) holds, from (2.8) we conclude that $m'(\bar\omega;w)$ is the value of the following problem

$$(P_L^*)\begin{cases} \text{minimize } < c'(\bar\omega)w, \bar x > + < c(\bar\omega), v > \\ \text{subject to } \bar x \in S(\bar\omega), v \in R^s, \text{ satisfying} \\ < a_i'(\bar\omega)w, \bar x > + < a_i(\bar\omega), v > + < \alpha_i'(\bar\omega), w > \le 0, i \in I(\bar x), \\ < a_j'(\bar\omega)w, \bar x > + < a_j(\bar\omega), v > + < \alpha_j'(\bar\omega), w > = 0, j = p+1,...,p+q \end{cases}$$

Observe that $S(\bar\omega)$ is a polyhedron and, for fixed $\bar\omega$ and w, the problem (P_L^*) is "almost" linear (in $\bar x, v$). The "nonlinearity" here is due to the condition $i \in I(\bar x)$, where $I(\bar x)$ varies together with $\bar x$. Fortunately, under some unrestrictive conditions we can show that (P_L^*) may be replaced by a linear program. This is obvious when $S(\bar\omega)$ reduces to a singleton. For the other case, $riS(\bar\omega)$ is nonempty and one can easily see that there exists a constant set $I \subset \{1,2,...,p\}$ such that

$$I = I(\bar x), \ \forall \bar x \in riS(\bar\omega),$$

and

$$I \subset I(\bar x), \ \forall \bar x \in S(\bar\omega) \setminus riS(\bar\omega).$$

For any $\bar{x} \in S(\bar{\omega})$, set

$$\psi_w(\bar{\omega}) = min\{f'(\bar{\omega},\bar{x})(w,v) \mid g_i'(\bar{\omega},\bar{x})(w,v) \leq 0, i \in I(\bar{x}), g_j'(\bar{\omega},\bar{x})(w,v) = 0,$$
$$j = p+1,...,p+q\}$$

and define $\varphi_w(\bar{x})$ in the same way with the displacement of $I(\bar{x})$ by the constant set I.

REMARK 3.1. It is obvious that

$$m'(\bar{\omega};w) = min\{\psi_w(\bar{x}) \mid \bar{x} \in S(\bar{\omega})\} \tag{3.1}$$

and

$$\begin{cases} \psi_w(\bar{x}) = \varphi_w(\bar{x}), & \text{if } \bar{x} \in riS(\bar{\omega}) \\ \psi_w(\bar{x}) \geq \varphi_w(\bar{x}), & \text{if } \bar{x} \in S(\bar{\omega}) \setminus riS(\bar{\omega}). \end{cases}$$

LEMMA 3.1. *If $\varphi_w(.)$ is continuous on $S(\bar{\omega})$, then*

$$min\{\psi_w(\bar{x}) \mid \bar{x} \in S(\bar{\omega})\} = min\{\varphi_w(\bar{x}) \mid \bar{x} \in S(\bar{\omega})\}$$

and, hence, $m'(\bar{\omega};w)$ is the value of the linear problem obtained from (P_L^*) by replacing $I(\bar{x})$ with I.

PROOF. By contradiction, suppose that

$$\psi_o := min\{\psi_w(\bar{x}) \mid \bar{x} \in S(\bar{\omega})\} \neq \varphi_o := min\{\varphi_w(\bar{x}) \mid \bar{x} \in S(\bar{\omega})\}.$$

Then $\psi_o > \varphi_o$ (due to Remark 3.1) and there exists $\epsilon > 0$ such that $\psi_o > \varphi_o + \epsilon$. Let $\bar{x}_o \in argmin\, \varphi_w(.)$, i.e. $\varphi_w(\bar{x}_o) = \varphi_o$. Since $\varphi_w(.)$ is continuous, for some $\bar{x}' \in riS(\bar{\omega})$ close enough to $\bar{x}_o$ we have $\varphi_w(\bar{x}') \leq \varphi_o + \epsilon$. On the other hand, $\varphi_w(\bar{x}') = \psi_w(\bar{x}')$ for $\bar{x}' \in riS(\bar{\omega})$, so that

$$\psi_w(\bar{x}') \leq \varphi_o + \epsilon < \psi_o.$$

This contradicts the minimality of ψ_o and the lemma is proved.

REMARK 3.2. The continuity condition on φ_w is unrestrictive and, as it is the value of a linear program, we can refer to many works for conditions guaranteeing such a property. On the other hand, taking into account the specific features of the problem in question we are able to give one more simple criterion which can easily be verified in practice. To this end, let $\Phi_{\bar\omega}(z)$ denote the value of the following (linear) problem

$$Q_{\bar\omega}(z)\begin{cases} \text{minnimize } < c(\bar\omega), v > \\ \text{subject to} \\ < a_i(\bar\omega), v > \leq z_i, & i \in I, \\ < a_j(\bar\omega), v >= z_j, & p+1 \leq j \leq p+q \end{cases}$$

where $z = (z_1, ..., z_{p+q}) \in R^{p+q}$. Defining a vector function $z(.) : R^r \times R^s \longrightarrow R^{p+q}$ such that $z(.) = (z_1(.), ..., z_{p+q}(.))$, where

$$z_i(w, x) = - < a_i'(\bar\omega)w, x > - < \alpha_i'(\bar\omega), w >,\ i = 1, ..., p+q, \tag{3.2}$$

we have

$$\varphi_w(\bar x) = \Phi_{\bar\omega}[z(w, \bar x)] + < c'(\bar\omega)w, \bar x > . \tag{3.3}$$

Obviously, $z(w, .)$ is continuous and, therefore, $\varphi_w(.)$ is continuous if $\Phi_{\bar\omega}(.)$ is continuous.

To verify the continuity of $\Phi_{\bar\omega}(.)$ we have the following criterion

LEMMA 3.2. *$\Phi_{\bar\omega}$ is continuous if and only if it is finite at a finite number of points $z^i, i = 1, 2, ..., N$, with $0 \in intconv\{z^i \mid i = 1, ..., N\}$. In particular, $\Phi_{\bar\omega}$ is continuous if and only if it is finite at the following $(p+q+1)$ points $\bar z^1 = (1, 0, 0, ..., 0), \bar z^2 = (0, 1, 0, ..., 0), ..., \bar z^{p+q} = (0, 0, 0, ..., 1), \bar z^{p+q+1} = (-1, -1, -1, ..., -1)$.*

PROOF. There is no difficulty to show that $\Phi_{\bar\omega}(.)$ is sublinear and, hence, it is continuous if and only if it is finite on a neighborhood of the origin. The later is equivalent to saying that it is finite at a finite number of points whose convex hull contains the origin in its interior.

REMARK 3.3. For a given $\bar\omega$, Problem $Q_{\bar\omega}(z)$ is a classical parametric program which is linear in both variable and parameter. The reader is referred to [4], [12] for more results.

DEFINITION 3.1. The primary problem is said to be finite at $\bar\omega$ if $Q_{\bar\omega}(.)$ is finite at $\bar z^i, i = 1, ..., p+q+1$, defined in Lemma 3.2.

From Lemmas 3.1, 3.2 we get the following

PROPOSITION 3.1. *If Condition (J) is fulfilled and the primary problem is finite at $\bar{\omega}$, then $m'(\bar{\omega}, w)$ is the value of the following linear program (in ω, v)*

$$(L_{\bar{\omega}}^{w})\begin{cases} \text{minimize } < c'(\bar{\omega})w, \bar{x} > + < c(\bar{\omega}), v > \\ \text{subject to} \\ < a_i'(\bar{\omega})w, \bar{x} > + < a_i(\bar{\omega}), v > + < \alpha_i'(\bar{\omega}), w > \leq 0, \quad i \in I, \\ < a_j'(\bar{\omega})w, \bar{x} > + < a_j(\bar{\omega}), v > + < \alpha_j'(\bar{\omega}), w > = 0, \\ j = p+1, ..., p+q, \bar{x} \in S(\bar{\omega}), \end{cases}$$

where I denotes the set of active indexes on $riS(\bar{\omega})$.

REMARK 3.4. If m is Lipschitz regular at $\bar{\omega}$, then

$$m^o(\bar{\omega}; w) = m'(\bar{\omega}; w),$$

and Proposition 3.1 gives a possibility to calculate $\partial m(\bar{\omega})$ via standard codes, since

$$\partial m(\bar{\omega}) = \partial(m^o(\bar{\omega}; 0)) = \partial(m'(\bar{\omega}; .)) \mid_{\omega=0} .$$

However, we are dealing with the calculation of only one element of $\partial m(\bar{\omega})$, so we may require a condition weaker than Lipschitz regularity.

Following [15] we say that m is ***locally convex*** at $\bar{\omega}$ if its directional derivative $m'(\bar{\omega}; .)$ is a continuous convex function (with respect to w). Obviously, Lipschitz regularity implies local convexity, but the converse is not true in general. Furthermore, to verify the local convexity of m one needs to concern only the derived problem $(L_{\bar{\omega}}^{w})$ which is linear in the parameter w and, therefore, is much simpler than the primary problem. Clearly, if $S(\bar{\omega})$ is a singleton, then m is locally convex at $\bar{\omega}$.

Assume that m is locally convex at $\bar{\omega}$. Then

$$\partial m'(\bar{\omega}; .) \mid_{w=0} \subset \partial m^o(\bar{\omega}; .) \mid_{w=0} = \partial m(\bar{\omega}), \tag{3.4}$$

and one can find an element of $\partial m(\bar{\omega})$ by computing one element of $\partial m'(\bar{\omega}; .) \mid_{w=0}$. This can be done by standard codes for calculation of subgradients of convex functions. However, if we know the behaviour of the function $\Phi_{\bar{\omega}}(.)$, then we can much simplify the procedure, since we have the following

PROPOSITION 3.2. *Suppose that (J) holds, the primary problem is finite at $\bar{\omega}$, and m is locally convex at $\bar{\omega}$. If for some $w_o \in R^r$ and for some $\bar{x}$ solving $(L_{\bar{\omega}}^{w_o})$ the function $\Phi_{\bar{\omega}}(.)$ is differentiable at $z^o = z(w_o, \bar{x})$, then*

$$[c'(\bar{\omega})]^* \bar{x} - \sum_{i=1}^{p+q} \nabla_{z_i} \Phi_{\bar{\omega}}(z_o)([a_i'(\bar{\omega})]^* \bar{x} + \alpha_i'(\bar{\omega})) \in \partial m(\bar{\omega}). \tag{3.5}$$

PROOF. By (3.3) and Remark 3.1 we have

$$m'(\bar{\omega}, w) = min\{\Phi_{\bar{\omega}}[z(w, \bar{x})] + < c'(\bar{\omega})w, \bar{x} > \ | \ \bar{x} \in S(\bar{\omega})\},$$

where the function $z(w, \bar{x})$ is defined by (3.2). Observe that $\bar{x}$ gives "min" in the preceding formula if and only if $\bar{x}$ solves $(L_{\bar{\omega}}^{w})$. Further, since m is locally convex at $\bar{\omega}$, the function $m'(\bar{\omega}; .)$ is sublinear and

$$\partial m'(\bar{\omega}; .) \mid_{w=w'} \subset \partial m'(\bar{\omega}; .) \mid_{w=0}, \ \forall w' \in R^r.$$

From this and (3.4) we have

$$\partial m'(\bar{\omega}; .) \mid_{w=w'} \subset \partial m(\bar{\omega}), \forall w' \in R^r.$$

The proposition is now easily established by direct calculation and the following lemma.

LEMMA 3.3. *Let $h : R^r \times R^s \to R^1$ be locally Lipschitz with respect to $w \in R^r$, uniformly for x on some compact set $S \subset R^s$, and continuous with respect to x for every $w \in R^r$. If*

$$\theta(w) = min\{h(w, x) \mid x \in S\},$$

then $\theta(.)$ is locally Lipschitz and, if $h(., \bar{x})$ is differentiable at w' for some $\bar{x}$ such that $\theta(w') = h(w', \bar{x})$, the following relation holds

$$\nabla_w h(w', \bar{x}) \in \partial\theta(w'). \tag{3.6}$$

PROOF. Let

$$\eta(w) = max\{h(w, x) \mid x \in S\}.$$

It is easy to show that $\theta(.), \eta(.)$ are locally Lipschitz. Let $h(., \bar{x})$ be differentiable at w' for some $\bar{x}$ such that $\eta(w') = h(w', \bar{x})$. Then, for every $v \in R^r$,

$$< \nabla_w h(w', \bar{x}), v > = \lim_{t \to 0} [h(w' + tv, \bar{x}) - h(w', \bar{x})]/t$$
$$\leq \limsup_{t \to 0} [\eta(w' + tv) - \eta(w')]/t \leq \eta^o(w', v).$$

From this it follows that

$$\nabla_w h(w', \bar{x}) \in \partial\eta(w').$$

Observe that $-\theta(w) = max\{-h(w, x) \mid x \in S\}$ and from the previous inclusion we have

$$-\nabla_w h(w', \bar{x}) \in \partial(-\theta(w'))$$

which implies (3.6), since $\partial(-\theta(.)) = -\partial\theta(.)$. The lemma is proved, and so is the proposition.

REMARK 3.5. Formula (3.5) gives a simple way to calculate a generalized gradient of m at $\bar{\omega}$ (without using standard codes to calculate ∂m from $m'(\bar{\omega}; .)$). The "price" we have to pay for this is to learn the behaviour of the function $\Phi_{\bar{\omega}}(.)$ and to find a point where it is differentiable. Generally, it is differentiable almost everywhere (since it is continuous and convex, and hence, is locally Lipschitz). But for finding one concrete point we have to take into account the specific features of the problem encountered. Nevertheless, the following observation is very helpful.

As $\bar{\omega}$ is fixed, let us denote, for a while, $c(\bar{\omega}), a_i(\bar{\omega}), \Phi_{\bar{\omega}}(.)$ by $c, a_i, \Phi(.)$ (resp.). Then, $\Phi(z)$ is the optimal value of the following linear program

$$(A_z) \begin{cases} \text{minimize } \langle c, x \rangle \\ \langle a_i, x \rangle \leq z_i, & i \in I, \\ \langle a_j, x \rangle = z_j, & j = p+1, ..., p+q. \end{cases}$$

For given $\bar{z}$, let $\bar{x}$ be a solution to $(A_{\bar{z}})$ and let

$$I(\bar{x}) := \{i \in I \mid < a_i, \bar{x} >= \bar{z}_i\}$$
$$J(\bar{x}) := I(\bar{x}) \cup \{p+1, ..., p+q\}.$$

We then have

PROPOSITION 3.3. *If the system $\{a_j \mid j \in J(\bar{x})\}$ is linearly independent, then $\Phi(.)$ is differentiable at $\bar{z}$ and*

$$\nabla\Phi(\bar{z}) = -\bar{y},$$

where $\bar{y}$ is a Kuhn-Tucker multiplier for $(A_{\bar{z}})$ at $\bar{x}$.

PROOF. Denote by A the matrix whose rows are vectors $a_i, i = 1, ..., p+q$. Then $\bar{x}$ is a solution to $(A_{\bar{z}})$ and $\bar{y}$ is an associated Kuhn-Tucker multiplier if and only if $\bar{y}_i \geq 0$ for $i \in I$, $\bar{y}_i = 0$ for $i \in I \setminus I(\bar{x})$ and

$$c + A\bar{y} = 0, \tag{3.7}$$

$$< a_j, \bar{x} > = \bar{z}_j , \forall j \in J(\bar{x}). \tag{3.8}$$

Since the system $\{a_j \mid j \in J(\bar{x})\}$ is linearly independent, $s \geq q' := card I(\bar{x}) + q$ and, therefore, without loss of generality one can write A in the form

$$A = (A' \; A"),$$

where A' is an invertible $(q' \times q')$-matrix. Rewrite

$$x = (x', x")^T,$$

where x' denotes the first q' components of x. We now have

$$A'\bar{x}' + A"\bar{x}" = \bar{z}',$$

where z' stands for the vector whose components are z_j with $j \in J(\bar{x})$. Clearly, (3.8) is equivalent to

$$\bar{x}' = [A']^{-1}(\bar{z}' - A"\bar{x}").$$

For $i \in I \setminus I(\bar{x})$,

$$< a_i, \bar{x} > < \bar{z}_i$$

and, hence, one can find neighborhoods U of $\bar{x}$, V of $\bar{z}$ such that, for all $x \in U, z \in V$,

$$< a_i, x > < z_i , \forall i \in I \setminus I(\bar{x}). \tag{3.9}$$

Obviously, there is a neighborhood V_o of $\bar{z}$ such that $V_o \subset V$ and, for $z \in V_o$,

$$x'(z) = [A']^{-1}(z' - A"\bar{x}") \text{ implies } x(z) := (x'(z), \bar{x}") \in U.$$

It is easy to see that, for $z \in V_o$, $x(z) = (x'(z), \bar{x}")$ is a solution to (A_z) with $\bar{y}$ being an associated Kuhn-Tucker multiplier (since (3.7)-(3.8) are satisfied and the inactive constraints remain unchanged due to (3.9)). Clearly, $x(z)$ depends smoothly on z and, hence, Condition (**A**) holds. Since $\Phi(z) =< c, x(z) >$, it is smooth in a neighborhood of $\bar{z}$ and Proposition 2.1 gives

$$\nabla\Phi(\bar{z}) = -\bar{y}.$$

The proposition is thus proved.

From Propositions 3.3-3.4 we get the following

COROLLARY 3.1. *Under the assumptions of Proposition 3.4, let $\bar{x}$ be a solution to $(L_{\bar{\omega}}^{w})$ for some w, $\bar{z} = z(w, \bar{x})$ (defined by (3.2)), and $\bar{v}$ be a solution to $Q_{\bar{\omega}}(\bar{z})$. If the system*

$$\{a_j(\bar{\omega}) / \ j \in J(\bar{v})\}$$

is linearly independent, then

$$[c'(\bar{\omega})]^*\bar{x} - \sum_{j=1}^{p+q} \bar{y}_j([a_j'(\bar{\omega})]^*\bar{x} + \alpha_j'(\bar{\omega})) \ \in \ \partial m(\bar{\omega}),$$

where $\bar{y}_j, j \in J(\bar{v})$, are components of Kuhn-Tucker multiplier for $Q_{\bar{\omega}}(\bar{z})$ at $\bar{v}$ and $\bar{y}_i = 0$ if $i \notin J(\bar{v})$.

REMARK 3.6. It follows from a result of [28] (Corollary 6J) that $-\partial\Phi_{\bar{\omega}}(0)$ is exactly the set of Kuhn-Tucker multipliers for Problem $(Q_{\bar{\omega}}(0))$ which coincides with the one for the primary problem at $\bar{\omega}$ (if we set $\bar{y}_i = 0$ for $i \in \{1, ..., p\} \setminus I$). So, if $\Phi_{\bar{\omega}}(.)$ is differentiable at some point z, then the derivative $\nabla\Phi_{\bar{\omega}}(z)$ must belong to $\partial\Phi_{\bar{\omega}}(0)$ (for $\Phi_{\bar{\omega}}(.)$ is sublinear) and, hence, $-\nabla\Phi_{\bar{\omega}}(z)$ is a multipliers for the primary problem at $\bar{\omega}$. We have thus found that formula (3.5) is in full agreement with (2.4). The difference between these formulas is also clear: for ***easy points*** (2.4) is valid with arbitrary multipliers $\bar{y}$, and for ***"uneasy" points*** it is valid only for some "special" multipliers which are the negative of derivatives of the function $\Phi_{\bar{\omega}}(.)$. Besides, unlike in (2.4), in (3.5) $\bar{x}$ must be a solution not only for the primary problem (at $\bar{\omega}$) but also for $(L^w_{\bar{\omega}})$ and, hence, belongs to a proper subset of $S(\bar{\omega})$.

REMARK 3.7. If the set of multipliers for the primary problem (at $\bar{\omega}$) reduces to a singleton, then $\partial\Phi_{\bar{\omega}}(0)$ contains one single element and $\Phi_{\bar{\omega}}(.)$ is differentiable at 0 (therefore, is linear), and from (3.5) we obtain a result previously established by Gauvin-Dubeau [11].

REMARK 3.8. Propositions 3.1-3.2 provide a method to compute a generalized gradient of the marginal function (via linear programs) at "uneasy" points. Obviuosly, the computation is much more complicated than the calculation by (2.4). In favour of this formula we shall show that "easy" points can be found "almost everywhere" and, therefore, (2.4) recommends a rather strong "heuristic" method.

The following result shows particular interest in itself and will be presented in another work [20].

PROPOSITION 3.4. *For every open set $\Omega_o \subset R^r$ there exists an open subset $\Omega \subset \Omega_o$ such that S admits a smooth selection $\bar{x}(\omega) \in S(\omega)$ on Ω. Consequently, the set of "uneasy" points are nowhere dense.*

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