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THE SOLUTION OF A CLASS OF DUAL INTEGRAL EQUATIONS INVOLVING HANKEL TRANSFORM

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1. Introduction

Let $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$ ($\mu \geq -1/2$) be the Zemanian spaces of test and generalized functions, respectively (see [5]). Denote by B_μ the Hankel integral transform defined on $\mathcal{H}'_\mu$ [5]. It is known that this operator is an automorphism on $\mathcal{H}'_\mu$ ($\mu \geq -1/2$) with $B_\mu^{-1} = B_\mu$. For a suitable ordinary function $f(x)$ (for example, $f \in L_1(R_+)$, $R_+ = (0, \infty)$) the operator B_μ is defined by

$$\tilde{f}(t) := B_\mu[f](t) := \int_0^\infty \sqrt{xt} J_\mu(xt) f(x) dx, \ t \in R_+,$$

where $J_\mu(x)$ is the Bessel function of the first kind.

Let $J_k = (a_k, b_k)$ ($k = 1, 2, ..., K$) be certain bounded intervals in R_+ such that $\overline{J}_k \cap \overline{J}_j = \emptyset$ ($k \neq j$), m a non-negative integer number. Consider the following dual integral equations

$$\begin{aligned} &r_k B_\mu[t^{2m} A(t) \tilde{u}(t)](x) = f_k(x), \ x \in J_k \ (k = \overline{1, K}), \\ &u(x) := B_\mu[\tilde{u}(t)](x) = 0, \ x \in R_+ \setminus J_0, \end{aligned} \tag{1.1}$$

where $J_0 = \bigcup_{k=1}^{K} J_k$, r_k denotes the restriction operator on $J_k, A(t)$ and $f_k(x)$ are known functions, $\tilde{u}(t)$ is an unknown regular generalized function in $\mathcal{H}'_\mu$. Concerning the function $A(t)$ we make the following assumptions:

i) $A(t) \in C(\overline{R}_+)$, Re $A(t) \geq 0$,

ii) $L(t) := A(t) - 1 = 0(t^{-p})$, $t \to \infty$, $p \gg 1$.

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Equations of the form (1.1) when $J_0 = (a, b),\ a \geq 0$ were considered by many authors (see, for example, [1,3]). Formal solutions of such equations by the method of Erdelyi-Kober fractional integrals have been given in [3] for the case $J = (0, b)$ and in [1] for the case $J_0 = (a, b), a > 0,\ A(t) \equiv 1$. The validation of the case $J_0 = (0, b),\ A(t) \equiv 1$ may be found in [4].

The aim of the present paper is to propose a method for reducing equation (1.1) to an equivalent system of Fredholm integral equations of second kind, which has a unique solution in certain subclass of square-integrable functions. The method is based on the theory of generalized integral transformations [5] and allows, in particular, to find the exact solution when $A(t) \equiv 1$.

Throughout this paper N_0 denotes the set of non-negative integers, $N = \{1, 2, 3, \ldots\}, \mu \geq -1/2$, and D_x denotes the differential operator d/dx.

2. Some auxiliary differential and integral operators

In the sequel we shall need the following differential operators

$$M_\mu^m \varphi(x) := x^{-\mu-1/2}(D_x x^{-1})^m x^{m+\mu+1/2}\varphi(x),$$
$$N_\mu^m \varphi(x) := x^{m+\mu+1/2}(x^{-1}D_x)^m x^{-\mu-1/2}\varphi(x),$$

where $m \in N_0,\ \mu \geq -1/2$.

Note that the operators M_μ^1, N_μ^1 have been introduced in [5] and denoted there by M_μ and N_μ, respectively. By induction one gets the relations

$$M_\mu^m = \prod_{j=0}^{m-1} M_{\mu+j}, \quad N_\mu^m = \prod_{j=0}^{m-1} N_{\mu+m-1-j} \tag{2.1}$$

It is obvious that

$$M_\mu^m[x^{-m-\mu+1/2}p_{m-1}(x^2)] = N_\mu^m[x^{\mu+1/2}p_{m-1}(x^2)] = 0, \tag{2.2}$$

where $p_{m-1}(x)$ is an arbitrary polynomial of degree $m - 1$.

Using (2.1) and Lemma 5.3.3 in [5] one can prove that M_μ^m (respectively, N_μ^m) defines a continuous mapping (an isomorphism) from $\mathcal{H}_{\mu+m}$ into $\mathcal{H}_\mu$ (from

$\mathcal{H}_\mu$ onto $\mathcal{H}_{\mu+m}$). These operators may be extended to generalized functions by the equations

$$< M_\mu^m f, \varphi > := < f, (-1)^m N_\mu^m \varphi >, \ \varphi \in \mathcal{H}_\mu, \ f \in \mathcal{H}'_{\mu+m}, \tag{2.3}$$

$$< N_\mu^m f, \varphi > := < f, (-1)^m M_\mu^m \varphi >, \ \varphi \in \mathcal{H}_{\mu+m}, \ f \in \mathcal{H}'_\mu, \tag{2.4}$$

where $< f, \varphi >$ denotes a value of a generalized function f or a test function φ.

Let $\mathcal{D}'(J)$ denote the space of distributions on the interval $J = (a,b) \subset R_+$. For $f \in \mathcal{D}'(J)$ the operators $M_\mu^m f$ and $N_\mu^m f$ are defined by (2.3) and (2.4), respectively, where φ belongs to the set $C_0^\infty(J)$ of infinitely differentiable functions with support contained in J.

It is not difficult to verify the following formula

$$B_\mu[t^m f](x) = M_\mu^m B_{\mu+m}[f](x), \ f \in \mathcal{H}'_{\mu+m}. \tag{2.5}$$

Let us consider the following fractional integrals

$$\begin{aligned} M_{\mu,a}^{-m}[f](x) &:= \frac{x^{-m-\mu+1/2}}{2^{m-1}\Gamma(m)} \int_a^x f(t) t^{\mu+1/2} (x^2 - t^2)^{m-1} dt, \\ M_{\mu,a}^{-0}[f] &:= f, \quad M_\mu^{-m}[f] := M_{\mu,0}^{-m}[f], \end{aligned} \tag{2.6}$$

$$\begin{aligned} N_{\mu,b}^{-m}[f](x) &:= \frac{(-1)^m x^{\mu+1/2}}{2^{m-1}\Gamma(m)} \int_x^b f(t) t^{-m-\mu+1/2} (t^2 - x^2)^{m-1} dt, \\ N_{\mu,b}^{-0}[f] &:= f, \quad N_\mu^{-m}[f] := N_{\mu,\infty}^{-m}[f], \end{aligned} \tag{2.7}$$

where $m \in N$, $\Gamma(m)$ is the gamma-function.

If it is necessary to indicate the belonging of x to a certain bounded interval $J = (a,b) \subset R_+$ we shall use the notations $M_{\mu,J}^{-m}[f]$ and $N_{\mu,J}^{-m}[f]$ for $M_{\mu,a}^{-m}[f]$ and $N_{\mu,b}^{-m}[f]$ $(a < x < b)$, respectively. In this case we make the assumptions $t^{\mu+1/2} f(t) \in L_1(J)$ for $M_{\mu,J}^{-m}[f]$ and $t^{-m-\mu+1/2} f(t) \in L_1(J)$ for $N_{\mu,J}^{-m}[f]$.

Let $I_{p,q}$ and $K_{p,q}$ be the Erdelyi-Kober fractional integrals given in [1,3]. From (2.6) and (2.7) we have

$$\begin{aligned} M_\mu^{-m}[f](x) &= 2^{-m} x^m I_{\mu/2-1/4,m}[f](x), \ x > 0, \\ N_\mu^{-m}[g](x) &= (-2)^{-m} K_{\mu/2+1/4,m}[t^m g](x), \ x > 0. \end{aligned} \tag{2.8}$$

Note that if $f(t),\ t^m g(t) \in L_1(R_+)$ then $M_\mu^{-m}[f]$ and $N_\mu^{-m}[g]$ belong to $L_1(R_+)$ (cf. [4]).

In the sequel we shall also need $M_{\mu,J}^{-m}[f]$, where $f \in \mathcal{D}'(J)$. To extend $M_{\mu,J}^{-m}$ let us use the following equality [3]

$$I_{\mu/2-1/4,m}[f](x) = S_{\mu/2-1/4+m,0}S_{\mu/2-1/4,m}[f](x),\ x > 0, \tag{2.9}$$

where $S_{p,q}$ denotes the modified Hankel transform defined by [4]

$$S_{p,q}[f](x) := 2^q x^{-q} B_{2p+q}[t^{1/2-q} f(t)](x),\ x > 0. \tag{2.10}$$

A sufficient condition for the fulfilment of (2.9) is, for example, $t^{\mu+1/2}f(t) \in L_1(0,1)$ and $t^{1/2-m}f(t) \in L_1(1,\infty)$. Remark also that if $2p+q \geq -1/2$, $x^{1/2-q}f(x) \in L_1(R_+)$ and $\varphi(x) \in \mathcal{H}_{2p+2q+1/2}$, then

$$\int_0^\infty S_{p,q}[f](t)\varphi(t)dt = \int_0^\infty f(x)\widetilde{S}_{p,q}[\varphi](x)dx, \tag{2.11}$$

where

$$\widetilde{S}_{p,q}[\varphi](x) := 2^{-1}x^2 S_{p-1/2,q+1}[\varphi](x). \tag{2.12}$$

The equality (2.11) was established by using (2.10) and the Parseval equation for the Hankel transformation. We shall make use of the following theorem in [4].

THEOREM 2.1. *If $2p+q \geq -1/2$ then the operator $S_{p,q}$ is an isomorphism from $\mathcal{H}_{2p+2q-1/2}$ on to $\mathcal{H}_{2p-1/2}$ with $S_{p,q}^{-1} = S_{p+q,-q}$.*

As a corollary of Theorem 2.1 note that if $2p+q \geq -1/2$ then $S_{p,q}$ defined by (2.10) is an isomorphism from $\mathcal{H}_{2p+2q+1/2}$ on to $\mathcal{H}_{2p+1/2}$ with $S_{p,q}^{-1} = S_{p+q,-q}$ by virtue of Lemma 5.3.2 [5].

We are now going to define $S_{p,q}$ for generalized functions. Based on (2.11) and the above remark we put

$$< S_{p,q}[f], \varphi > := < f, \widetilde{S}_{p,q}[\varphi] >,$$

where

$$f \in \mathcal{H}'_{2p+1/2},\ \varphi \in \mathcal{H}'_{2p+2q+1/2},\ 2p+q \geq -1/2.$$

According to Theorem 1.10.2 in [5] the generalized operator $S_{p,q}$ is an isomorphism from $\mathcal{H}'_{2p+1/2}$ onto $\mathcal{H}'_{2p+2q+1/2}$. Moreover, $S_{p,q}^{-1} = S_{p+q,-q}$.

Let $2p+q \geq -1/2$ and $2p+2q \geq -1/2$. The generalized Erdelyi-Kober operator $I_{p,q}$ as a mapping from $\mathcal{H}'_{2p+1/2}$ into $\mathcal{H}'_{2p+2q+1/2}$ is defined by the equation

$$I_{p,q}[f] := S_{p+q,0}S_{p,q}[f].$$

We now define the generalized operator M_μ^{-1} acting from $\mathcal{H}'_\mu$ into $\mathcal{H}'_{\mu+m}$ by the formula (2.9), where $f \in \mathcal{H}'_\mu$. One can prove the following equality

$$< M_\mu^{-m}[f], \varphi > = < f, (-1)^m N_\mu^{-m}[\varphi] >, \ \varphi \in C^\infty(R_+). \tag{2.13}$$

Let $f \in \mathcal{D}'(J)$, $J = (a,b) \subset R_+$. Denote by f_a the extension of f such that $f_a \in \mathcal{H}'_\mu$ and $f_a = 0$ on $(0,a)$ if $a > 0$. We define the generalized operator $M_{\mu,J}^{-m}[f]$ by

$$M_{\mu,J}^{-m}[f](x) := r_J M_\mu^{-m}[f_a](x), \tag{2.14}$$

where r_J denotes the restriction operator to the interval J. Using (2.13) one can prove that the generalized operator $M_{\mu,J}^{-m}[f]$ does not depend on the choice of the extension $f_a \in \mathcal{H}'_\mu$.

PROPOSITION 2.1. *The following equalities are valid*

$$M_\mu^m M_{\mu,J}^{-m}[f](x) = f(x), \ x \in J, \tag{2.15}$$

$$M_{\mu,J}^{-m} M_\mu^m[f](x) = f(x) + x^{-m-\mu+1/2} p_{m-1}(x^2), \ x \in J, \tag{2.16}$$

where $f \in \mathcal{D}'(J)$ and $p_{m-1}(x)$ is a certain polynomial of degree $m-1$.

PROOF. First we prove (2.15). For any $\varphi \in C_0^\infty(J)$ we have (see (2.3))

$$\begin{aligned} &< M_\mu^m M_{\mu,J}^{-m}[f], \varphi > = < M_{\mu,J}^{-m}[f], (-1)^m N_\mu^m[\varphi] > = \\ &< r_J M_\mu^{-m}[f_a], (-1)^m N_\mu^m[\varphi] = < M_\mu^{-m}[f_a], (-1)^m N_\mu^m[\varphi] > . \end{aligned} \tag{2.17}$$

For every $\varphi \in C_0^\infty(R_+)$ there exists the equality

$$N_\mu^{-m} N_\mu^m[\varphi] = \varphi. \tag{2.18}$$

By virtue of (2.13) and (2.18) we get from (2.17) the following relation

$$< M_\mu^m M_{\mu,J}^{-m}[f], \varphi > = < f_a, \varphi > = < f, \varphi >, \ \forall \varphi \in C_0^\infty(J).$$

This implies (2.15).

We now prove (2.16). According to (2.15) we have

$$M_\mu^m M_{\mu,J}^{-m} M_\mu^m[f] = M_\mu^m[f]. \tag{2.19}$$

Therefore, taking into account (2.2), from (2.19) we obtain (2.16). Q.E.D.

3. Solution of the equations (1.1)

Without loss of generality we may assume that $0 \le a_1 < a_2 \cdots < a_K$. First consider the case $a_1 > 0$. We introduce some classes of functions. Denote by $\overset{o}{C}{}_\mu^{m-1}(J_0)$ the class of functions $u(x)$ satisfying the conditions

$$N_\mu^n[u](x) \in C(R_+), \ N_\mu^n[u](x) = 0, \ x \in \overline{R}_+ \setminus J_0 \ (n = 0, 1, ..., m-1), \tag{3.1}$$

$$N_\mu^n[u](x) \in L_2(J). \tag{3.2}$$

Denote by $0_\mu^m(J_k)$ the class of functions $v \in L_2(R_+)$ with supp $v \subset J_K$ such that

$$\int_{a_k}^{b_k} x^{-m-\mu+1/2} v(x) x^{2n} dx = 0, \ n = 0, 1, ..., m-1. \tag{3.3}$$

Let $\theta(x)$ be the Heaviside function. Consider the function

$$u(x) = \frac{x^{\mu+1/2}}{2^{m-1}\Gamma(m)} \sum_{k=1}^{K} \int_{a_k}^{b_k} \theta(t-x) t^{-m-\mu+1/2} v_k(t)(t^2 - x^2)^{m-1} dt, \tag{3.4}$$

where $v_k(t) \in 0_\mu^m(J_k)$.

THEOREM 3.1. *In order that $u(x)$ belongs to the class $\overset{o}{C}{}_\mu^{m-1}(J_0)$ it is necessary and sufficient that it is represented by the form (3.4).*

PROOF. *Sufficiency.* If $x \in J_K = (a_k, b_k)$, by virtue of (3.3) and (3.4) we have

$$N_\mu^n[u](x) = \frac{(-1)^n x^{n+\mu+1/2}}{2^{m-n-1}\Gamma(m-n)} \int_x^{b_k} t^{-m-\mu+1/2} v_k(t)(t^2-x^2)^{m-n-1} dt, \tag{3.5}$$

$(n = 0, 1, ..., m-1)$,

$$u(x) = (-1)^m N_{\mu,J_k}^{-m}[v_k](x), \tag{3.6}$$

where $N_{\mu,J_k}^{-m}[v_k]$ is defined by the formula (2.7). Now, let $b_{k-1} \le x \le a_k$. From (3.4) and (3.3) it follows

$$N_\mu^n[u](x) = \frac{(-1)^n x^{n+\mu+1/2}}{2^{m-n-1}\Gamma(m-n)} \sum_{j=k}^{K} \int_{a_j}^{b_j} t^{-m-\mu+1/2} v_j(t)(t^2-x^2)^{m-n-1} dt = 0 \tag{3.7}$$

$(n = 0, 1, ..., m-1)$.

It is clear that if $x > b_K$ then $N_\mu^n[u](x) = 0$ $(n = 0, 1, 2, ..., m)$. Therefore, from (3.5) and (3.7) we get (3.1). Taking into account (2.15), from (3.6) we obtain (3.2).

Necessity. Let $u(x) \in \overset{o}{C}{}_\mu^{m-1}(J_0)$. We put

$$v_k(x) := \begin{cases} (-1)^m N_\mu^m[u](x), & x \in J_k, \\ 0, & x \in R_+ \setminus \end{cases} \quad (k = 1, 2, ..., K). \tag{3.8}$$

By virtue of (3.2) we have $v_k \in L_2(J_k)$. We need only to prove that the functions v_k satisfy (3.3). Indeed, taking integration by parts, we get conditions (3.3) in view of (3.1). It is not difficult to see that by virtue of (3.1) and (3.2) the form (3.4) follows from (3.8). Q.E.D.

Taking the Hankel transform B_μ of $u(x)$ defined by (3.4) we get

$$\tilde{u}(t) := B_\mu[u](t) = \frac{1}{t^m} \sum_{k=1}^{K} B_{\mu+m}[v_k](t). \tag{3.9}$$

One can easily prove the following result:

PROPOSITION 3.2. *A function $u(x)$ belongs to the class $\overset{o}{C}{}^{m-1}_{\mu}(J_0)$ if and only if its Hankel transform B_μ has the form (3.9), where $v_k \in 0^m_\mu(J_k)$, $k = 1, 2, ..., K$.*

We now turn to the equation (1.1). We shall find a the function $u(x) := B_\mu[\tilde{u}](x)$ in the class $\overset{o}{C}{}^{m-1}_{\mu}(J_0)$ for $m \in N$. Using (2.5) we rewrite this equation in the form

$$r_k M^m_\mu B_{\mu+m}[t^m A(t)\tilde{u}(t)](x) = f_k(x), \ x \in J_k \quad (k = 1, 2, ..., K), \tag{3.10}$$

where $u(x) := B_\mu[\tilde{u}](x) \in \overset{o}{C}{}^{m-1}_{\mu}(J_0)$ if $m \in N$, $u(x) \in L_2(J_0)$ if $m = 0$, and $f_k(x) \in \mathcal{D}'(J_k)$ for which there exist generalized operators $M^{-m}_{\mu,J_k}[f_k]$.

Taking into account that $r_k M^m_\mu[g] = M^m_\mu[r_k g]$ one can apply M^{-m}_{μ,J_k} to (3.10). According to Proposition 2.1 we have

$$\begin{aligned} &r_k B_{\mu+m}[t^m A(t)\tilde{u}(t)](x) = F_k(x) + G_k(x) \\ &(x \in J_k, \ k = 1, 2, ..., K), \end{aligned} \tag{3.11}$$

where

$$F_k(x) := M^{-m}_{\mu,J_k}[f_k](x), G_k(x) := x^{-m-\mu+1/2} p_{k,m-1}(x^2), \tag{3.12}$$

$p_{k,m-1}(x)$ is a polynomial of degree $m-1$ and identically equals zero if $m = 0$.

Now in (3.11) we substitute $A(t)$ and $\tilde{u}(t)$ by using condition ii) and (3.9), respectively. Note that $B_{\mu+m}B_{\mu+m}[v] = v$. Then after some transformation we obtain the following system of integral equations

$$\begin{aligned} &v_k(x) + \sum_{j=1}^{K} \int_{a_j}^{b_j} \ell_{\mu+m}(x,t) v_j(t) dt = F_k(x) + G_k(x) \\ &(x \in J_k, \ k = 1, 2, ..., K), \end{aligned} \tag{3.13}$$

where

$$\ell_{\mu+m}(x,t) := \int_0^\infty L(\lambda)\sqrt{x\lambda}J_{\mu+m}(x\lambda)\sqrt{t\lambda}J_{\mu+m}(t\lambda)d\lambda. \tag{3.14}$$

Obviously, under the assumption ii) the integral (3.14) is absolutely convergent. Hence $\ell_{\mu+m}(x,t) \in L_2(J_k \times J_j)$.

Using transformations inverse to those used above from (3.13) we come to the dual equation (1.1) where $\tilde{u}(t)$ is defined by (3.9). Due to Theorem 3.1

and Propositionn 3.2 the function $u(x) := B_\mu[\tilde{u}(t)](x)$ belongs to the class $\overset{o}{C}{}^{m-1}_\mu(J_0)$. Therefore, we obtain the following theorem:

THEOREM 3.3. *The dual equations (1.1) considered in* $\overset{o}{C}{}^{m-1}_\mu(J_0)$ *with respect to* $u = B_\mu[\tilde{u}]$ *are equivalent to the system of integral equations (3.13), where* $v_k(x) \in 0^m_\mu(J_k)$. *Moreover, they are connected with the function* $u(x)$ *by the formular (3.4).*

Note that $a_1 = \min_k\{a_k\} > 0$. Then as it is clear from (3.12), $G_k(x) \in C^\infty(\overline{J}_k) \subset L_2(J_k)$ $(k = 1, 2, ..., K)$. Suppose that the functions $f_k(x)$ are given so that the functions $F_k(x)$ defined by (3.12) belong to $L_2(J_k)$, $k = 1, 2, .., K$. Using the assumptions i), ii), (3.3) and the Parseval equality, one can veryfy that system (3.13) has at most one solution in $\prod_{k=1}^K O^m_\mu(J_) \subset \prod_{k=1}^K L_2(J_k)$. Since the kernel functions are symmetric and square-integrable, the dual system has at most one solution either.

Thus by the Fredholm alternative, we have:

THEOREM 3.4. *The system of integral equations (3.13) has a unique solution in* $\prod_{k=1}^{K} 0^m_\mu(J_k)$.

EXAMPLE 3.1. Let $J = (a, b)$, $0 < a < c < b$. Denote by $\delta_J(x - c)$ the restriction to J of the Dirac $\delta(x - c)$ function. Let us consider the equations

$$\begin{aligned} &B_0[t^2\tilde{u}(t)](x)\sqrt{c}\delta_J(x-c), \ a < x < b, \\ &u(x) := B_0[\tilde{u}](x) = 0, \ 0 \le x \le a, \ x \ge b. \end{aligned} \tag{3.15}$$

One can show that (3.15) can be reduced to the form

$$M_0N_0u(x) = -\sqrt{c}\delta_J(x-c), \ a < x < b, \tag{3.16}$$

where

$$M_0N_0u(x) := x^{-1/2}\frac{d}{dx}x\frac{d}{dx}x^{-1/2}u(x).$$

Using (2.10), Problem 5.5.1 in [5] and formula 6.575 (1) in [2] one gets

$$M^{-1}_{0,J}[\sqrt{c}\delta_j(x-c)](x) = \begin{cases} 0, \ a < x < c, \\ \frac{c}{\sqrt{x}}, \ c < x < n. \end{cases}$$

In this case $u(x)$ and $v_1(x) = v(x)$ have the forms

$$u(x) = \begin{cases} x^{1/2} \int_x^b t^{-1/2} v(t)dt, \ a < x < b, \\ 0, \ x \in R_+ \setminus (a,b), \end{cases} \tag{3.17}$$

$$v(x) = \begin{cases} -\frac{c}{\sqrt{x}} \frac{\ell n b/c}{\ell n b/a}, \ a < x < c, \\ \frac{c}{\sqrt{x}} - \frac{c}{\sqrt{x}} \frac{\ell n b/c}{\ell n b/a}, \ c < x < b, \\ 0, \ x \in R_+ \setminus (a,b). \end{cases}$$

Calculating the integral (3.17) we obtain

$$u(x) = -c\sqrt{x} \frac{\ell n b/c}{\ell n b/a} \ell n b/x + c\sqrt{x} \ell n b/x \theta(x-c) + \\ + c\sqrt{x} \ell n b/c \theta(c-x), \ a < x < b.$$

It is clear that $u(a) = u(b) = 0$. Therefore $u(x)$ satisfies the differential equation (3.16).

We now consider the case $a_1 = 0$. This case is more complicated than that considered above since there are examples having an infinite number of solutions in the class $\overset{o}{C}{}_\mu^{m-1}(J_0)$. We make the assumption $\mu \geq 0$ and introduce the following notion.

Denote by $\widetilde{0}_\mu^m(J_1)$ $(J_1 = (0,b))$ the class of function $v(x) \in L_2(J_1)$ such that the functions defined by

$$V_n[v](x) := \int_x^{b_1} t^{-m-\mu+1/2} v(t)(t^2 - x^2)^n dt, \ n = 0, 1, ..., m-1 \tag{3.18}$$

have bounded values when $x \to +0$.

The solution $U(x)$ of (1.1) is sought to be in the same form (3.4), where $v_1(x) \in \widetilde{0}_\mu^m(J_1)$, $v_k(x) \in 0_\mu^m(J_k)$ $(k = 2, 3, ..., K)$. The set of such functions $U(x)$ is denoted by $\overset{\sim}{\overset{o}{C}}{}_\mu^{m-1}(J_0)$. As in the case $a_1 > 0$, the function $v_k(x)$ are determined similarly by the system (3.13). Let $G_k(x)$ be the functions defined by the formula (3.12).

LEMMA 3.1. *Let $\mu \geq 0$. The function $G_1(x)$ satisfies the conditions (3.18) iff $F_1(x) \equiv 0$.*

PROOF. Representing function $G_1(x)$ in the form

$$G_1(x) = x^{-m-\mu+1/2} \sum_{j=0}^{m-1} C_{1j} x^{2j},$$

we have

$$V_n[G_1](x) = \sum_{j=0}^{m-1} C_{1j} U_{n,j}(x), \; n = 0, 1, ..., m-1,$$

where

$$U_{n,j}(x) = \int_x^{b_1} t^{2j-2m-2\mu+1}(t^2 - x^2)^n dt.$$

Let us consider $U_{0,j}(x)$. Since $j \leq m-1+\mu$, $\mu \geq 0$, $U_{0,j}(x)$ $(j = 0, 1, 2, ..., m-1)$ are not bounded at $x = 0$. This implies $C_{1j} = 0$ $(j = 0, 1, 2, ..., m-1)$, i.e. $G_1(x) \equiv 0$. Q.E.D.

Obviously, the integrals $\int_{a_j}^{b_j} \ell_{\mu+m}(x,t) v_j(t) dt$ $(x \in J_1)$, where $\ell_{\mu+m}(x,t)$ is defined by (3.14) satisfy the conditions (3.18). According to Theorem 3.4 we have

THEOREM 3.5. *Let $\mu \geq 0$, $F_k(x) \in L_2(J_k)$ $(k = 1, 2, ..., K)$, $F_1(x)$ satisfy the conditions (3.18). Then the system (3.13) with respect to $v_1(x) \in \widetilde{0}_\mu^m(J_1)$, $v_k(x) \in 0_\mu^m(J_k)$ $(k = 2, 3, ..., K)$ is one-valued solvable.*

Let us consider some examples. For simplicity, let $A(t) \equiv 1$.

EXAMPLE 3.2. Let $J_0 = (0, b_1) \cup (a_2, b_2)$. Consider the equations

$$\begin{aligned} &B_0[t^2 \tilde{u}(t)](x) = -\sqrt{x} d_k, \; x \in J_k \; (k = 1, 2), \\ &u(x) := B_0[\tilde{u}](x) = 0, \; x \in R_+ \setminus J_0, \end{aligned} \tag{3.19}$$

where $d_k = \text{const}$, $J_1 = (0, b_1)$, $J_2 = (a_2, b_2)$. We have

$$F_1(x) = -\frac{d_1}{2} x^{3/2} \; (0 < x < b_1), \; F_2(x) - \frac{d_2}{2} x^{-1/2}(x^2 - a_2^2) \; (a_2 < x < b_2),$$

$$v_1(x) = -\frac{d_1}{2} x^{3/2} \; (0 < x < b_1)$$

$$V_0[F_1](x) = V_0[v_1](x) = -\frac{d_1}{4}(b_1^2 - x^2),$$
$$v_2(x) = -\frac{d_2}{2}.\frac{x^2 - a_2^2}{\sqrt{x}} + \frac{d_2}{2\ell n b_2/a_2}(\frac{b_2^2 - a_2^2}{2} - a_2^2 \ell n b_2/a_2)\frac{1}{\sqrt{x}}$$
$$(a_2 < x < b_2),$$
$$u(x) = x^{1/2}\int_0^{b_1} \theta(t-x)t^{-1/2}v_1(t)dt + x^{1/2}\int_{a_2}^{b_2} \theta(t-x)t^{-1/2}v_2(t)dt.$$

Calculating theses integrals we get

$$u(x) = \begin{cases} -\frac{d_1}{4}x^{1/2}(b_1^2 - x^2), \ 0 \le x \le b_1, \\ 0, \ b_1 \le x \le a_2, \\ -\frac{d_2}{4}x^{1/2}(b_2^2 - x^2) + \frac{d_2(b_2^2 - a_2^2)}{4\ell n b_2/a_2}x^{1/2}\ell n b_2/x, \ a_2 \le x \le b_2, \\ 0, \ x \ge b_2. \end{cases}$$

Note that (3.19) can be reduced to the form

$$x^{-1/2}\frac{d}{dx}x\frac{d}{dx}x^{-1/2}u(x) = d_k\sqrt{x}, \ x \in J_k.$$

It is not difficult to verify that the obtained function $u(x)$ satisfies the last differential equation.

The following example shows that if conditions (3.18) are not fulfilled then (1.1) may have many solutions in $\overset{o}{C}{}_{\mu}^{m-1}(J_0)$.

EXAMPLE 3.3. Consider the following homogeneous equations:

$$\begin{aligned} &B_0[t^4\tilde{u}(t)](x) = 0, \ 0 < x < b, \\ &u(x) := B_0[\tilde{u}(t)](x) = 0, \ x \ge b. \end{aligned} \tag{3.20}$$

We shall show that if $u(x)$ is in $\overset{o}{C}{}_0^1(0,b)$ but not in $\overset{\bar{o}}{C}{}_0^1(0,b)$, then (3.20) will have non-trivial solutions. By (3.4) and (3.13) we have

$$u(x) = \frac{x^{1/2}}{2}\int_0^b t^{-3/2}v(t)(t^2 - x^2)\theta(t-x)dt, \ x > 0, \tag{3.21}$$
$$v(x) = x^{-3/2}(C_0 + C_1x^2), \ 0 < x < b,$$

where C_0 and C_1 are arbitrary constants. Note that $v \in L_2(0, b)$ if and only if $C_0 = 0$. It is clear that

$$V_0[v](x) = C_1 \ell n b/x.$$

Hence, the conditions (3.18) are not fulfilled if $C_1 \neq 0$. Calculating the integral (3.21) we get

$$u(x) = \begin{cases} \frac{C_1 x^{1/2}}{2}(\frac{b^2 - x^2}{2} - x^2 \ell n b/x), & 0 < x < b, \\ 0, & x \geq b. \end{cases}$$

From this it follows that

$$\begin{aligned} N_0(u)(x) &= C_1 x^{3/2} \ell n x/b, \ 0 < x < b, \\ &= 0, \ x \geq b, \\ N_0^2(u)(x) &= C_1 x^{1/2}, \ 0 < x < b, \\ &= 0, \ x \geq b. \end{aligned}$$

Obviously, $u(x) \in \overset{o}{C}{}_0^1(0, b)$. One can show that

$$B_0[t^4 \tilde{u}(t)](x) = M_0^2 N_0^2 u(x) = -C_1[\frac{5}{2} b^{-1/2} \delta(x - b) + b^{1/2} \delta'(x - b)].$$

Therefore, (3.20) is fulfilled for any constant C_1.

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