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A SUBCLASSIFICATION OF UNIMODAL DISTRIBUTIONS

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Abstract. We introduce and study classes $\mathcal{P}_\alpha, \alpha > 0$, of α-times unimodal distributions. Various characterizations of these distributions are obtained and relations between them and fractional calculus are established.

1. Introduction

A real variable ($r.v.$) or its distribution function F (shortly, distribution) or its density f is called *unimodal about a* if F is convex on $(-\infty, a)$ and concave on (a, ∞).

A famous theorem of Khintchine [3] (see also Shepp [9]) asserts that F is unimodal about 0 if and only if it is a distribution of a product of two independent r.v.'s one of which is uniform on the interval (0,1).

Let X and Y be independent r.v.'s with distributions G and F, respectively. We denote by $G \circ F$ the distribution of the product XY. Then F is unimodal about 0 if and only if $F = U \circ G$, U being the uniform distribution on (0,1) and G a distribution. The latter equation can be written in the following integral form

$$F(x) = \int_0^1 G(x/t)dt, \qquad x \in \mathcal{R}. \tag{1.1}$$

Let P be the class of all distributions on the real line $\mathcal{R}$ equipped with the weak convergence $\Rightarrow$. We denote by $\mathcal{P}_1$ the class of all distributions on $\mathcal{R}$ which are unimodal about 0. For $u = 2, 3 \dots$ let P_n be the subclass of $\mathcal{P}_1$ consisting of distributions $F = U \circ G$ for which $G \in \mathcal{P}_{n-1}$. Every distribution in $\mathcal{P}_n$ will be called *n-times unimodal about 0.*

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The simplest example of distributions in $\mathcal{P}_n$ is the distribution $U^n = U \circ \cdots \circ U$ (n times). More generally, every distribution in $\mathcal{P}_n$ is of the form

$$F = U^n \circ G, \qquad G \in \mathcal{P}, \tag{1.2}$$

which, by virtue of (1.1), is equivalent to the following formula

$$F(x) = \frac{1}{(n-1)} \int_0^1 G(x/u)(-\log u)^{n-1} du, \quad x \in \mathcal{R} \tag{1.3}$$

The above formula suggests the following interpolation of classes $\mathcal{P}_n$: For every positive number α let $\mathcal{P}_a$ denote the class of all distributions F on $\mathcal{R}$ such that for some distribution G

$$F(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 G(x/u)(-\log u)^{\alpha-1} du, \quad x \in \mathcal{R}. \tag{1.4}$$

It should be noted that for every distribution G and for every $\alpha > 0$ the right-hand side of (1.4) is finite and defines a distribution.

As in the integer case, the distributions in $\mathcal{P}_\alpha$ will be called *α-times unimodal about 0*. Note that the class of α-times unimodal distributions introduced above differs from the class of α-unimodal distributions introduced earlier by Olshen and Savage [5] (see also Dharmadhikari and Joag-dev [1], p.72). To see this the reader is referred to Example 5.3 and should compare Theorem 2.2 of this paper and Theorem 3.5 in Dharmadhikari and Joag-dev [1], p.74.

The main purpose of this paper is to discuss representation problems of multiply unimodal distributions. In a subsequent paper we will study infinitely divisible distributions in $\mathcal{P}_\alpha$.

The paper is organized as follows. In §2 we prove a generalization of Khintchine-Shepp representation of unimodal distributions. The general properties of the class $\mathcal{P}_\alpha$ is studied in §3. In §4 we prove a representation of α-times unimodal distributions by fractional integrals. In §5 an analogue of Olshen Savage representation is obtained which gives a clear aspect of extention of the concept to multidimensional spaces. Finally, in §6 we present a result of Yamazato about the convolution of elements of $\mathcal{P}_\alpha$ and give some examples of infinitely divisible distributions in $\mathcal{P}_\alpha$.

2. Generalized Khintchine-Shepp Representation.

The formula (1.2) is a generalization of Khintchine-Shepp's representation to n-times unimodal distributions (n=1,2 ...). We will replace n by a positive number α extending the representation to the fractional case.

We start with some facts about the uniform distribution U on (0,1). We associate with every r.v. X or its distribution F the following diagonal matrix W_F defined by

$$W_F(t) = \begin{pmatrix} \omega_0(t) & 0 \\ 0 & \omega_1(t) \end{pmatrix}, \quad t \in \mathcal{R}, \tag{2.1}$$

where $\omega_k(t) = E|X|^{it} sgn^k X, \quad k = 1, 2.$

The matrix W_F is called the characteristic transform of X. For the basic properties of characteristic transform see Zolotarev [11]. In particular, we have the following relation

$$W_{F \circ G} = W_F W_G \tag{2.2}$$

Moreover, $F_n \Rightarrow F$ if and only if, $W_{F_n}(t)$ uniformly converges to $W_F(t)$.

We say that a distribution F is o-infinitely divisible if for every n=1,2 ..., there exists F_n such that $F = F_n^n$.

2.1. PROPOSITION. *The uniform distribution U on (0,1) is o-infinitely divisible. Let U^α be its power under the operation o of order $\alpha > 0$. The density f_α of U^α is given by*

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)}(-\log x)^{\alpha-1}, \quad 0 < x < 1. \tag{2.3}$$

PROOF. Let F_α be the distribution with density f_α. From (2.1) and (2.3) it follows that

$$W_U(t) = \begin{pmatrix} (it+1)^{-1} & 0 \\ 0 & (it+1)^{-1} \end{pmatrix} \tag{2.4}$$

$$W_{F_\alpha}(t) = \begin{pmatrix} (it+1)^{-\alpha} & 0 \\ 0 & (it+1)^{-\alpha} \end{pmatrix} \tag{2.5}$$

($t \in \mathcal{R}$), which implies that for every $n = 1, 2 \ldots$, $U = F_{1/n}^n$, i.e. U is o-infinitely divisible (cf. Zolotarev [12]) and that

$$U^\alpha = F_\alpha, \quad \alpha > 0. \; \square \tag{2.6}$$

2.2. THEOREM. *A distribution F of $\mathcal{R}$ is α-times unimodal about zero if and only if there exists independent r.v.'s Y and Z such that U^α is the distribution of Z and F is the distribution of YZ. In other words, F belongs to P_α if and only if for some G from $\mathcal{P}$*

$$F = U^\alpha \circ G. \tag{2.7}$$

PROOF. It follows from (1.4) and Proposition 2.1. □

COROLLARY. *If $F \in \mathcal{P}_\alpha$ and $H \in \mathcal{P}_\beta$ then $F \circ H \in \mathcal{P}_{\alpha+\beta}$.*

2.4. COROLLARY. *Every α-times unimodal distribution F ($\alpha > 0$) is absolutely continuous on $\mathcal{R} \setminus \{0\}$ and its density F is given by*

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_x^\infty (-\log x/y)^{\alpha-1} G(dy)/y, & x > 0 \\ \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (-\log x/y)^{\alpha-1} G(dy)/y, & x < 0 \end{cases} \tag{2.8}$$

where $G \in \mathcal{P}$.

2.5. COROLLARY. *Suppose f is an α-times unimodal (about 0) density. If $\alpha \geq 1$, then f is non-increasing on $(0, \infty)$ and non decreasing on $(-\infty, 0)$. If $\alpha > 1$, then 0 is a discontinuity point of f anf*

$$\begin{cases} f(0+) = \infty & \text{if } 1 - G(0+) > 0 \\ f(-0) = \infty & \text{if } G(0-) > 0 \end{cases} \tag{2.9}$$

2.6. COROLLARY. *If f is a continuous or bounded density on $\mathcal{R}$, then it can not be α-times unimodal about zero for every $\alpha > 1$.*

PROOF OF COROLLARY 2.3. It follows from (2.7). □

PROOF OF COROLLARY 2.4. It is a consequence of formulas (1.4) and (2.7). □

PROOF OF COROLLARY 2.5. The first statement follows from (2.8). Suppose that $x > 0$ and $\alpha > 1$. Then (2.8) implies that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty 1_{(0,y)}(x)(-\log x/y)^{\alpha-1} G(dy)/y. \tag{2.10}$$

Note that for every y the integrand of (2.10) is a non-increasing function of x. Therefore it tends to ∞ as x tends to 0 and $x > 0$, which shows that $f(0+) = \infty$ whenever $1 > G(0+)$. Similarly, we have $f(0-) = \infty$ whenever $G(0-) > 0$. □

PROOF OF COROLLARY 2.6. It follows directly from Corollary 2.5. □

2.7. EXAMPLE. The standard normal distribution $N(0,1)$ is unimodal but not α-times unimodal about 0 for $\alpha > 1$ because its density is continuous on $\mathcal{R}$.

2.8. EXAMPLE. By Sato and Yamazato's results [11] and by Corollary 2.6 it follows that all nondegenerate distributions in L are absolutely continuous on $\mathcal{R}$ and unimodal. Moreover, if the density f is continuous on $\mathcal{R}$ (for example, if its Gaussian component is not trivial), then for every $\alpha > 1$ f is not α-times unimodal.

2.9. REMARK. Independently, Example (2.8) was proved by Yamazato for the case $\alpha > 1$ (private communication) who applied the results to prove Example 2.7. Earlier, Zolotarev (private communication) showed that $N(0,1)$ is not 2-times unimodal.

The following theorem is a generalization of Theorem 8 in Zolotarev [12].

2.9. THEOREM. *A distribution F is α-times unimodal about 0 if and only if the matrix $(1+it)^{\alpha}W_F(t)$ is a characteristic transform.*

PROOF. It follows from (2.4), (2.6) and (2.7). □

3. The monotoncity, convexity and continuity of Classes $\mathcal{P}_\alpha$

We begin with proving the following theorem:

3.1. THEOREM.

(i) $\mathcal{P}_\alpha (\alpha > 0)$ is closed under weak convergence and convex combinations.

(ii) If $\alpha > \beta$,

$$\mathcal{P}_\alpha \not\subseteq \mathcal{P}_\beta. \tag{3.1}$$

(iii) δ_0 is the only distribution belonging to all classes $\mathcal{P}_a$. Here we write δ_α for the degenerate distribution at a.

(iv) For any $\beta > 0$,

$$\overline{\bigcup_{\alpha>\beta} \mathcal{P}_\alpha} = \mathcal{P}_\beta$$

where the bar denotes the closure in weak topology and for $\beta = 0$ we write $\mathcal{P}_\beta = \mathcal{P}_0$.

PROOF. (i) The convexity of $\mathcal{P}_\alpha$ is clear. Further, we have $\delta_0 \in \mathcal{P}_\alpha$. Let $\{F_n\}$ be a sequence in $\mathcal{P}_\alpha$ converging to a distribution F other than δ_0. By Theorem

2.2 there exists a sequence $\{G_n\} \subset \mathcal{P}$ such that for every n

$$F_n = U^\alpha \circ G_n, \tag{3.2}$$

Taking the characteristic transform of both sides of (3.2) we get

$$W_{F_n} = W_{U^\alpha}(t) W_{G_n}(t).$$

Therefore, the sequence G_n is convergent to a limit G. Thus we have $F = U^\alpha \circ G$ which shows that F is an element of $\mathcal{P}_\alpha$ and consequently, the set P_α is closed in weak topology.

(ii) Suppose $\alpha > \beta$ and $F = U^\alpha \circ G \in P_\alpha$. Then $F = U^\beta \circ (U^{\alpha-\beta} \circ G)$ which implies that $\mathcal{P}_\alpha$ is a subset of $\mathcal{P}_\beta$. It remains to prove that the inclusion cannot be replaced by an equality.

In fact, we shall prove that $U^\beta \in \mathcal{P}^\beta \setminus \mathcal{P}^\alpha$. Suppose that for some G from $\mathcal{P}$ we have

$$U^\beta = U^\alpha \circ G. \tag{3.3}$$

Taking the characteristic transform of both sides of (3.3) we get the formula

$$I = W_{U^{\alpha-\beta}}(t) W_G(t), \tag{3.4}$$

I being the unit matrix. Consequently,

$$U^{\alpha-\beta} \circ G = \delta_1 \tag{3.5}$$

which contradicts the fact that the left-hand side of (3.5) is absolutely continuous on $\mathcal{R} \setminus \{0\}$ (cf. Corollary 2.4).

(iii) It is clear that δ_0 belongs to all classes $\mathcal{P}_\alpha$. Suppose that there exists another distribution F with the same property. Then for every n=1,2 ... there is G_n such that

$$F = U^n \circ G_n. \tag{3.6}$$

Let X, Y and Z_n be independent r.v.'s with distributions F, U and G_n respectively. Passing to the absolute value and changing the probability space if necessary, one may assume, without loss of generality, that X is positive with probability one. Then Z_n's share the same property. Let f, g, h_n denote the characteristic functions of $\log X, \log Y$, and $\log Z_n$, respectively. From (3.6) we get the relation

$$f(t) = g^n(t)h_n(t), \quad t \in P \tag{3.7}$$

Let $N_a(f)$ be the Khintchine functional defined on characteristic functions f by

$$N_a(f) = -Re \int_0^a \log f(t)dt.$$

Choosing $a > 0$ such that $N_a(f) < \infty$ we get the equation

$$N_a(f) = nN_a(g) + N_a(h_n)$$

which implies that

$$N_a(g) \le \frac{1}{n} N_a(f), \quad n = 1, 2 \ldots \tag{3.8}$$

Consequently, $N_a(g) = 0$ which contradicts the fact that

$$N_a(g) = \frac{1}{2} \int_0^a \log(1 + t^2)dt > 0.$$

(iv) follows from Theorem 2.2 and from the relation

$$\lim_{a \to 0} U^\beta \circ U^\alpha \circ G = U^\beta \circ G$$

where $\beta \ge 0$ and $G \in \mathcal{P}$. □

4. Representation by Fractional Integrals

Given a function H on $\mathcal{R} \setminus \{0\}$ we define two auxiliary functions H_+ and H_- by

$$\begin{cases} H_+(t) &= e^t H(e^{-t}) \\ H_-(t) &= e^t H(-e^{-t}), \quad t \in \mathcal{R} \end{cases} \tag{4.1}$$

It is evident that H_+ and H_- determine H uniquely.

Let F be a distribution from $\mathcal{P}_\alpha$. Then for some distribution G the formula (1.4) holds. By a simple change of variables in (1.4) we get:

$$F(x) = \begin{cases} G(0) & x = 0 \\ \frac{x}{\Gamma(\alpha)} \int_x^\infty (\log u/x)^{\alpha-1} G(u) \frac{du}{u^2} & x > 0 \\ \frac{x}{\Gamma(\alpha)} \int_{-\infty}^{-x} (\log u/x)^{\alpha-1} G(u) \frac{du}{u^2} & x < 0 \end{cases} \tag{4.2}$$

Putting $x = sgn\ xe^{-t}$ and $u = e^{-v}$ and taking into account the formulas (4.1) and (4.2) we have

$$\begin{cases} F_+(t) &= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-v)^{\alpha-1} G_+(v) dv \\ F_-(t) &= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-v)^{\alpha-1} G_-(v) dv, t \in \mathcal{R} \end{cases} \tag{4.3}$$

which are integrals of fractional order α of G_+ and G_-. That means

$$F_\pm(t) = D^\alpha F_\pm(t), \quad t \in \mathcal{R} \tag{4.5}$$

The reader is referred to Post (1930) and Letnikov (1968) for an account of fractional calculus.

We summarize the above results in the following theorem

4.1. THEOREM. *For every α-times distribution F on $\mathcal{R}$ there exists an unique distribution G such that F_+ anf F_- are integrals of order α of G_+ and G_-., respectively. Conversely, for every distribution G on $\mathcal{R}$ the formula (4.4) defines an α-times distribution.*

4.2. REMARK. From the above theorem it follows that if F is α-times unimodal then F_+ and F_- are α-times differentiable and (4.5) holds.

5. An analogue of Olshen-Savage's Representation

As mentioned in §1, Olshen and Savage [5] introduced and studied an important concept of α-unimodality. Their results extend, enrich and illuminate Khintchine's original results on unimodality. Following Olshen and Savage [5] we say that a r.v. X is *α-unimodal about 0* if for every bounded, nonnegative, Borel measurable function g on $\mathcal{R}$, the quantity $t^\alpha E[g(tX)]$ is non-decreasing in $t \in (0, \infty)$.

The following theorem was proved in Olshen and Savage [5], (see also Dhatmadhikari and Joag-dev [1], Theorem 3.5,p.74).

5.1. THEOREM. *A r.v. X is α-unimodal if and only if X is distributed as $W^{1/\alpha} Z$ where W is uniform on (0,1) and Z is independent of W.*

Given a r.v. X and a bounded, Borel measurable function g we put

$$H(X, g, x) = e^x Eg(e^x X), \quad x \in \mathcal{R}. \tag{5.1}$$

Further, if F is a function bounded on $(-\infty, a], a \in \mathcal{R}$, its right-handed difference of fractional order $\alpha > 0$ is defined by

$$\Delta_t^\alpha f(x) = \sum_0^\infty (-1)^k \binom{\alpha}{k} f(x - kt),$$

$(t > 0, x \in \mathcal{R})$, where

$$\binom{\alpha}{k} = \begin{cases} 1 & k = 0 \\ \alpha(\alpha-1)\ldots(\alpha-k-1)/k! & k = 1, 2\ldots \end{cases}$$

A function f of $\mathcal{R}$ is called *α-non-decreasing* if for every $x \in \mathcal{R}$ and for every $t > 0$,

$$\Delta_t^\alpha f(x) \geq 0. \tag{5.2}$$

Note that for $\alpha = 1$ the above concept is reduced to that of nondecreasing functions. The following theorem stands for an analogue of Olshen-Savage's representation (cf. Theorem 5.1 and the definition of α-unimodality).

5.2. THEOREM. *A r.v. X is α-times unimodal about 0 if and only if the function $H(X, g, x)$ defined by (5.1) is α-nondecreasing in x for every bounded, nonnegative, Borel measurable function g*

PROOF. We borrow some ideas from the proof of Theorem 1 and 2 in Olshen and Savage [5].

Suppose first that X is α-times unimodal about 0. By Theorem 2.2, the re exists independent r.v.'s Y and Z such that Y is α-uniform on (0,1) and $X \stackrel{d}{=} YZ$. Let g be as in the theorem. Then we have

$$\begin{aligned} H(YZ, g, x) &= \frac{e^x}{\Gamma(\alpha)} \int_0^1 Eg(e^x yZ)(-\log y)^{\alpha-1} dy \\ &= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-y)^{\alpha-1} e^Y g(e^Y Z) dy \end{aligned}$$

which implies that

$$H(X, g, x) = I^\alpha H(Z, g, x), \quad x \in \mathcal{R}. \tag{5.3}$$

Consequently, $H(X, g, x)$ is α-nondecreasing.

Conversely, let $\mathcal{S}$ be the class of all functions f on $\mathcal{R}$ with continuous and bounded derivatives $f^{(k)}, k = 1, 2 \ldots, [\alpha] + 1$. Here $[\alpha]$ denotes the integer part of α. By Dominated Convergence Theorem

$$\frac{d^k}{dx^k} Ef(e^x Z) = E \frac{d^k}{dx^k} f(e^x Z)$$

for every $k = 1, 2, \ldots |\alpha| + 1$ and every f from S and every r.v.Z. Consequently,

$$D^\alpha Ef(e^x Z) = ED^\alpha f(e^x Z).$$

Suppose that X is a r.v. such that for every bounded, nonnegative, Borel measurable function g the function $H(X, g, x)$ is α-nondecreasing in x. Let L be a linear functional on $\mathcal{S}$ defined by

$$L(f) = D^\alpha H(X, f, x)\big|_{x=0}, \quad f \in \mathcal{S}. \tag{5.4}$$

It is clear that L is nonnegative and normalized by the condition $L(1) = 1$. By virtue of the Riesz representation theorem it follows that there exists an unique probability distribution G such that

$$L(f) = \int_{-\infty}^{\infty} f(x) G(dx), \quad f \in \mathcal{S}. \tag{5.5}$$

Applying (5.4) and (5.5) to the function $f(e^t x)$ where t is fixed number, we get the formula

$$\int_{-\infty}^{\infty} f(e^t x) G(dx) = e^{-t} D^\alpha H(X, f, t). \tag{5.6}$$

Let Y and Z be r.v.'s with the same property as in proof of the "only part". By (5.3) and (5.6) we get

$$\begin{aligned} D^\alpha H(YZ, f, t) &= H(Z, f, t) \\ &= e^t \int_{-\infty}^{\infty} G(dx) \\ &= D^\alpha H(X, f, t) \end{aligned}$$

The above equalities together with the fact that

$$H(YZ, f, -\infty) = H(X, f, -\infty) = 0$$

imply the equation

$$Ef(YZ) = Ef(X), \quad f \in \mathcal{S}.$$

Consequently, X and YZ are identically distributed which implies that X is α-times unimodal. □

The following simple example can be used to distinguish the two concepts of α unimodality and α-times unimodality.

5.3. EXAMPLE. Let Y be a r.v. uniformly distributed on (0,1) and $X = Y^{\frac{1}{\beta}}, \beta > 0$. By Lemma 1 in Dharmadhikari and Joag-dev [1] p.73, X is β-unimodal. It is trivial to see that if $\beta > 1$, then density of X is increasing on (0,1) which, by Corollary 2.5, implies that X is not β-times unimodal.

Now let $0 < \alpha < 1$ and Z be a r.v. distributed as U^α. By Theorem 3.1, Z is not unimodal and therefore it is not α-unimodal. Note that Z is α-times unimodal.

6. Multiple Unimodality of Infinitely Divisible Distributions

We start with the following proposition which was essentially given by Yamazato (private communication).

6.1. PROPOSITION. *Suppose $\alpha > 1$. There exists symmetric α-times unimodal distributions μ and ν such that the convolution $\mu * \nu$ is not α-times unimodal (about 0).*

PROOF. Suppose that for every pair of symmetric α-times unimodal (about 0) distributions μ and ν, $\mu * \nu$ is symmetric and α-times unimodal.

Let ν be a symmetric α-times unimodal distribution and $c > 0$. Since the mixture of symmetric α-times unimodal distributions is symmetric α-times unimodal the distribution μ defined by

$$\mu = e^{-c} \sum\nolimits_0^\infty c^n \mu^{*n}/n!$$

is α-times unimodal.

Put

$$G_n(a, r) = \begin{cases} Kndx & 0 < |x| \le 1/n \\ 0 & |x| > 1/n \end{cases}$$

where $K = \frac{3}{4}(\int_0^1 (-\log y)^{\alpha-1} y^2 dy)^{-1}$ and

$$\nu_n(dx) = \int_0^1 (-\log y)^{\alpha-1} G_n(dx/y)dy.$$

Then $c_n^{-1}\nu_n$ is a symmetric α-times unimodal distribution, where $c_n - \nu_n(\mathcal{R})$. Let μ_n be an infinitely divisible distribution with the characteristic function

$$\hat{\mu}_n(t) = exp \int_{-\infty}^{\infty} (e^{itx} - 1 - \frac{itx}{1+x^2})\nu_n(dx).$$

then μ_n is symmetric α-times unimodal. Our further aim is to show that μ_n converges to $N(0,1)$.

Accordingly, let us consider the positive part of $\nu_n(dx)$. We have

$$\nu_n([\frac{1}{n},\infty]) = \int_0^1 (-\log y)^{\alpha-1} G_n([\frac{1}{n},\infty))dy = 0. \tag{6.1}$$

Given $\epsilon > 0$ choose n such that $n\epsilon > 1$. Then we have

$$\int_0^\epsilon = \int_0^1 (-\log y)^{\alpha-1} \int_0^\epsilon G_n(dx/y)dy = \frac{1}{4}.$$

By (6.1) and (6.2) we infer that μ_n converges to the standard normal distribution which implies that $N(0,1)$ is α-times unimodal. This contradicts Example 2.7. □

In a private communication Yamazato asked whether or not there exists an infinitely divisible α-times unimodal distribution. In what follows we will answer Yamazato's question by giving some examples which show that the class of infinitely divisible α-times unimodal distribution is rich enough.

6.2. EXAMPLES. Suppose a r.v. X has the following distribution:

(i) F is symmetric with log-convex characteristic function on the positive half-line (cf. Keilson and Steutel [2]).

(ii) F is the distribution of $|Y|^{-r}$, Y being symmetric stable with exponent $p < 2$ and $r > \frac{2p}{2-p}$ (cf. Shandhag,Pestana and Sreehari [8]).

(iii) F is the distribution of Y^r, Y being a Gamma r.v. with index $q > 0$ and $r > \max(1,p)$ (cf. Shandhag and Sreeheri [7]).

Then, if X and Z are independent, the product XZ is infinitely divisible. Moreover, if Z is α-times unimodal, then XZ is α-times unimodal and infinitely divisible.

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