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ON THE ACTION OF THE STEENROD ALGEBRA ON THE MODULAR INVARIANTS OF SPECIAL LINEAR GROUP

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1. Introduction

For an odd prime p, let SL_n denote the special linear subgroup of $GL(n, \mathbb{Z}/p)$, which acts naturally on the cohomology algebra $H^*(B(\mathbb{Z}/p)^n)$. Here and in what follows, the cohomology is always taken with coefficients in the prime field $\mathbb{Z}/p$.

According to [3], $H^*(B(\mathbb{Z}/p)^n) = E(x_1, \dots, x_n) \otimes P(y_1, \dots, y_n)$ with $\dim x_i = 1$, $y_i = \beta x_i$, where β is the Bockstein homomorphism, $E(.,\dots,.)$ and $P(.,\dots,.)$ are the exterior and polynomial algebras over $\mathbb{Z}/p$ generated by the variables indicated. Let $(e_{k+1}, \dots, e_n)$, $k \geq 0$, be a sequence of non-negative integers. Following Mùi [2], we define

$$[k; e_{k+1}, \dots, e_n] = [k; e_{k+1}, \dots, e_n](x_1, \dots, x_n, y_1, \dots, y_n)$$

by

$$[k; e_{k+1}, \dots, e_n] = \frac{1}{k!} \begin{vmatrix} x_1 & \cdots & x_n \\ \vdots & \cdots & \vdots \\ x_1 & \cdots & x_n \\ y_1^{p^{e_{k+1}}} & \cdots & y_n^{p^{e_{k+1}}} \\ \vdots & \cdots & \vdots \\ y_1^{p^{e_n}} & \cdots & y_n^{p^{e_n}} \end{vmatrix}.$$

The precise meaning of the right hand side is given in [2]. For $k = 0$, we write

$$[0; e_1, \dots, e_n] = [e_1, \dots, e_n] = \det(y_i^{p^{e_j}}).$$

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We set

$$L_{n,s} = [0, \dots, \hat{s}, \dots, n],\ 0 \le s \le n,$$
$$L_n = L_{n,n} = [0, \dots, n-1].$$

Each $[k; e_{k+1}, \dots, e_n]$ is an invariant of SL_n and $[e_1, \dots, e_n]$ is divisible by L_n. Then Dickson invariants $Q_{n,s}, 0 \le s \le n$, and Mùi invariants $M_{n,s_1,\dots,s_k}, 0 \le s_1 < \dots < s_k < n$, are defined by

$$Q_{n,s} = L_{n,s}/L_n,$$
$$M_{n,s_1,\dots,s_k} = [k; 0, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1].$$

Note that $Q_{n,n} = 1, Q_{n,0} = L_n^{p-1}$, $M_{n,0,\dots,n-1} = [n; \emptyset] = x_1 \dots x_n$.

Mùi proved in [2] that $H^*(B(\mathbb{Z}/p)^n)^{SL_n}$ is the free module over the Dickson algebra $P(L_n, Q_{n,1}, \dots, Q_{n,n-1})$ generated by 1 and $M_{n,s_1,\dots,s_k}$ with $0 \le s_1 < \dots < s_k < n$.

The Steenrod algebra $A(p)$ acts on $H^*(B(\mathbb{Z}/p)^n)$ by well-known rules. Since this action commutes with the action of SL_n, it induces an action of $A(p)$ on $H^*(B(\mathbb{Z}/p)^n)^{SL_n}$.

Let τ_s, and ξ_i be the Milnor elements of dimensions $2p^s - 1$ and $2p^i - 2$, respectively, in the dual algebra $A(p)^*$ of $A(p)$. Milnor showed in [5] that

$$A(p)^* = E(\tau_0, \tau_1, \dots) \otimes P(\xi_1, \xi_2, \dots).$$

So $A(p)^*$ has a basis consisting of all monomials $\tau_S \xi^R = \tau_{s_1} \dots \tau_{s_t} \xi_1^{r_1} \dots \xi_m^{r_m}$, with $S = (s_1, \dots, s_t),\ 0 \le s_1 < \dots < s_t, R = (r_1, \dots, r_m)$. Let $St^{S,R} \in A(p)$ denote the dual of $\tau_S \xi^R$ with respect to this basis of $A(p)^*$. Then $A(p)$ has a new basis consisting of all operations $St^{S,R}$. In particular, for $S = \emptyset, R = (k)$, $St^{S,R}$ is nothing but the Steenrod operation P^k.

The action of P^k on Dickson and Mùi invariants was explicitly computed by Hung and Minh [4]. The action of $S^{S,R}$ on the invariant $[n; \emptyset] = x_1 \dots x_n$ was computed by Mùi [3] .

In this paper, we compute the action of $St^{S,R}$ on $[k; e_{k+1}, \dots, e_n]$ and prove a nice relation between the invariants $[k; e_{k+1}, \dots, e_n + s], 0 \le s \le n$, and the

Dickson invariants. Using these results, we explicitly compute the action of P^k on Mùi invariants $M_{n,s_1,\dots,s_k}$, which was first computed in Hung and Minh [4] by another method.

To state the main results, we introduce some notations. Let $J = (J_0, \dots, J_m)$ with $J_s \subset \{k+1, \dots, n\}$, $0 \le s \le m$, and $\coprod_{s=0}^{m} J_s = \{k+1, \dots, n\}$ (disjoint union). We define the sequence $R_J = (r_{J_1}, \dots, r_{J_m})$, r_{J_0} and the function $\Phi_J : \{k+1, \dots, n\} \to \{0, \dots, m\}$ by setting

$$r_{J_s} = \sum_{j \in J_s} p^{e_j}, \ 0 \le s \le m,$$
$$\Phi_J(i) = s \quad \text{if} \quad i \in J_s, \ k+1 \le i \le n.$$

The main result of this paper is

1.1. THEOREM. *Suppose that $e_i \neq e_j$ for $i \neq j$, $S = (s_1, \dots, s_t)$, $0 \le s_1 < \dots < s_t < m$. Under the above notations we have*

$$St^{S,R}[k; e_{k+1}, \dots, e_n] = \begin{cases} (-1)^{t(k-t)}[k-t; s_1, \dots, s_t, e_{k+1} + \Phi_J(k+1), \dots, e_n + \Phi_J(n)], & \\ \qquad\qquad R = R_J, \ \textit{for some } J, & \\ 0, & \textit{otherwise.} \end{cases}$$

We have also the following relation from which we can compute $St^{S,R}[k; e_{k+1}, \dots, e_n]$ in terms of Dickson and Mùi invariants.

1.2. PROPOSITION. *For $0 \le k < n$,*

$$[k; e_{k+1}, \dots, e_{n-1}, e_n + n] = \sum_{s=0}^{n-1} (-1)^{n+s-1} [k; e_{k+1}, \dots, e_{n-1} e_n + s] Q_{n,s}^{p^{e_n}}.$$

Using Theorem 1.1 and Proposition 1.2 we explicitly compute the action of $St^{S,R}$ on Mùi invariant $M_{n,s_1,\dots,s_k}$ when S, R are special. Particularly, we prove

1.3. THEOREM (Hung and Minh [4]). *For* $s_0 = -1 < s_1 < \ldots < s_k < s_{k+1} = n$,

$$P^t M_{n,s_1,\ldots,s_k} = \begin{cases} M_{n,t_1,\ldots,t_k}, & t = \sum_{i=1}^{k} \frac{p^{s_i}-p^{t_i}}{p-1}, \text{ with } s_{i-1} < t_i \le s_i, \\ \sum_{i=1}^{k+1} (-1)^{k+1-i} M_{n,t_1,\ldots,\hat{t}_i,\ldots,t_{k+1}} Q_{n,t_i}, & t = \sum_{i=1}^{k+1} \frac{p^{s_i}-p^{t_i}}{p-1}, \text{ with } s_{i-1} < t_i \le s_i, \\ & 1 \le i \le k+1,\ t_{k+1} < s_{k+1} = n, \\ 0, & \text{otherwise.} \end{cases}$$

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2. Proof of Theorem 1.1

First we recall Mùi's results on the homomorphism $d_m^* P_m$ and the operations $St^{S,R}$.

Let A_{p^m} be the alternating group on p^m letters. Suppose that X is a topological space, WA_{p^m} is a contractible A_{p^m}-free space. Then we have the Steenrod power map

$$P_m : H^q(X) \longrightarrow H^{p^m q}(WA_{p^m} \underset{A_{p^m}}{\times} X^{p^m}),$$

which sends u to $1 \otimes u^{p^m}$ at the cochain level (see [6; Chap.VII]).

The inclusion $(\mathbf{Z}/p)^m \subset A_{p^m}$ together with the diagonal map $X \to X^{p^m}$ and the Künneth formula induces the homomorphism

$$d_m^* : H^*(WA_{p^m} \underset{A_{p^m}}{\times} X^{p^m}) \longrightarrow H^*(B(\mathbf{Z}/p)^m) \otimes H^*(X).$$

Set $\tilde{M}_{m,s} = M_{m,s} L_m^{h-1}, 0 \le s < m, \tilde{L}_m = L_m^h, h = (p-1)/2$. We have

2.1. THEOREM (Mùi [4; 1.3]). *Let* $u \in H^q(X), \mu(q) = (h!)^q (-1)^{hq(q-1)/2}$ *Then*

$$d_m^* P_m u = \mu(q)^m \sum_{S,R} (-1)^{r(S,R)} \tilde{M}_{m,s_1} \ldots \tilde{M}_{m,s_t} \tilde{L}_m^{r_0} Q_{m,1}^{r_1} \ldots Q_{m,m-1}^{r_{m-1}} \otimes St^{S,R} u\ .$$

Here the summation runs over all (S,R) *with* $S=(s_1,\dots,s_t)$, $0\le s_1<\dots<s_t<m$, $R=(r_1,\dots,r_m)$, $r_0=q-t-2(r_1+\dots+r_m)\ge 0$, $r(S,R)=t+s_1+\dots+s_t+r_1+2r_2+\dots+mr_m$.

2.2. PROPOSITION (Mùi [2], [3]).

(i) $d_m^*P_m$ *is a natural monomorphism preserving cup product up to a sign. Precisely,*

$$d_m^*P_m(uv) = (-1)^{mhqr}d_m^*P_mud_m^*P_mv\ ,$$

with $q=\dim u, r=\dim v$.

(ii) $d_m^*P_my_i=\sum_{s=0}^{m}(-1)^{m+s}Q_{m,s}\otimes y_i^p$.

(iii) $d_m^*P_m(x_1\dots x_n)=$

$$\mu(n)^m\sum_{0\le s_1<\dots<s_t<m}(-1)^{t(n-t)+r(S,0)}\tilde M_{m,s_1}\dots\tilde M_{m,s_t}\tilde L_m^{n-t}\otimes[n-t;s_1,\dots,s_t].$$

Here x_i *and* y_i *are defined as in the introduction.*

2.3. LEMMA. *If* $e_i\ne e_j$ *for* $i\ne j$, *then*

$$d_m^*P_m[e_1,\dots,e_n] = \sum_{J=(J_0,\dots,J_m)}(-1)^{mn+r(\emptyset,R_J)}\tilde L_m^{2r_{J_0}}Q_{m,1}^{r_{J_1}}\dots Q_{m,m-1}^{r_{J_{m-1}}} \otimes[e_1+\Phi_J(1),\dots,e_n+\Phi_J(n)],$$

Where R_J *and* Φ_J *are defined as in Theorem 1.1.*

PROOF. Let Σ_n be the symmetric group on n letters. Then

$$[e_1,\dots,e_n]=\sum_{\sigma\in\Sigma_n}\operatorname{sign}\sigma\prod_{i=1}^{n}y_i^{p^{e_{\sigma(i)}}}.$$

From Proposition 2.2, we have

$$d_m^*P_m\Big(\prod_{i=1}^n y_i^{p^{e_{\sigma(i)}}}\Big)=\prod_{i=1}^n\Big(d_m^*P_my_i\Big)^{p^{e_{\sigma(i)}}} = \prod_{i=1}^n\Big(\sum_{s=0}^m(-1)^{m+s}Q_{m,s}^{p^{e_{\sigma(i)}}}\otimes y_i^{p^{e_{\sigma(i)}+s}}\Big).$$

Expanding this product and using the definitions of Φ_J, R_J and the assumption of the lemma, we get

$$d_m^* P_m\Big(\prod_{i=1}^n y_i^{p^{e_{\sigma(i)}}}\Big) = \sum_J (-1)^{mn+r(\emptyset,R_J)} Q_{m,0}^{r_{J_0}} \dots Q_{m,m-1}^{r_{J_{m-1}}} \otimes \prod_{i=1}^n y_i^{p^{e_{\sigma(i)}+\Phi_J(\sigma(i))}}$$

Hence, from the above equalities we obtain

$$\begin{aligned} &d_m^* P_m[e_1, \dots, e_n] \\ &\quad = \sum_J (-1)^{mn+r(\emptyset,R_J)} Q_{m,0}^{r_{J_0}} \dots Q_{m,m-1}^{r_{J_{m-1}}} \otimes \sum_{\sigma\in\Sigma_n} \operatorname{sign}\sigma \prod_{i=1}^n y_i^{p^{e_{\sigma(i)}+\Phi_J(\sigma(i))}} \\ &\quad = \sum_J (-1)^{mn+r(\emptyset,R_J)} Q_{m,0}^{r_{J_0}} \dots Q_{m,m-1}^{r_{J_{m-1}}} \otimes [e_1 + \Phi_J(1), \dots, e_n + \Phi_J(n)]. \end{aligned}$$

Since $Q_{m,0} = \tilde{L}_m^2$, the lemma is proved.

2.4. Proof of Theorem 1.1. Let I be a subset of $\{1, \dots, n\}$ and I' its complement in $\{1, \dots, n\}$. Writing $I = \{i_1, \dots, i_k\}$ and $I' = \{i_{k+1}, \dots, i_n\}$ with $i_1 < \dots < i_k$ and $i_{k+1} < \dots < i_n$. We set $x_I = x_{i_1} \dots x_{i_k}, [e_{k+1}, \dots, e_n]_I = [e_{k+1}, \dots, e_n](y_{i_{k+1}}, \dots, y_{i_n})$ and $\sigma_I = \begin{pmatrix} 1 & \cdots & n \\ i_1 & \cdots & i_n \end{pmatrix} \in \Sigma_n$. In [2; I.4.2], Mùi showed that

$$[k; e_{k+1}, \dots, e_n] = \sum_I \operatorname{sign}\sigma_I x_I [e_{k+1}, \dots, e_n]_I.$$

From Proposition 2.2 and Lemma 2.3 we have

$$\begin{aligned} d_m^* P_m(x_I) = \mu(k)^m \sum_{0\le s_1<\dots<s_t<m} (-1)^{t(k-t)+r(S,0)} \tilde{M}_{m,s_1} \dots \tilde{M}_{m,s_t} \tilde{L}_m^{k-t} \otimes \\ [k-t; s_1, \dots, s_t]_I. \end{aligned}$$

where $[k-t; s_1, \dots, s_t]_I = [k-t; s_1, \dots, s_t](x_{i_1}, \dots, x_{i_k}, y_{i_1}, \dots, y_{i_k})$,

$$\begin{aligned} &d_m^* P_m[e_{k+1}, \dots, e_n]_I \\ &\quad = \sum_{J=(J_0,\dots,J_m)} (-1)^{m(n-k)+r(\emptyset,R_J)} \tilde{L}_m^{2r_{J_0}} Q_{m,1}^{r_{J_1}} \dots Q_{m,m-1}^{r_{J_{m-1}}} \otimes \\ &\qquad [e_{k+1} + \Phi_J(k+1), \dots, e_n + \Phi_J(n)]_I. \end{aligned}$$

Set $q = \dim\,[k; e_{k+1}, \ldots, e_n] = k + 2(p^{e_{k+1}} + \ldots + p^{e_n})$. An easy computation shows that $\mu(q) = (-1)^{n-k}\mu(k)$, and $r(S,0) + r(\emptyset, R) = r(S,R)$. Hence from Proposition 2.2 and the above equalities we get

$$\begin{aligned} &d_m^* P_m[k; e_{k+1}, \ldots, e_n] \\ &= \mu(q)^m \sum_{S,J} (-1)^{t(k-t)+r(S,R_J)} \tilde{M}_{m,s_1} \ldots \tilde{M}_{m,s_t} \tilde{L}_m^{k-t+2r_{J_0}} Q_{m,1}^{r_{J_1}} \ldots Q_{m,m-1}^{r_{J_{m-1}}} \otimes \\ &\qquad \sum_I \text{sign}\ \sigma_I [k-t; s_1, \ldots, s_t]_I [e_{k+1} + \Phi_J(k+1), \ldots, e_n + \Phi_J(n)]_I. \end{aligned}$$

Then, using the Laplace development we obtain

$$\begin{aligned} &d_m^* P_m[k; e_{k+1}, \ldots, e_n] \\ &= \mu(q)^m \sum_{S,J} (-1)^{t(k-t)+r(S,R_J)} \tilde{M}_{m,s_1} \ldots \tilde{M}_{m,s_t} \tilde{L}_m^{k-t+2r_{J_0}} Q_{m,1}^{r_{J_1}} \ldots Q_{m,m-1}^{r_{J_{m-1}}} \otimes \\ &\qquad [k-t; s_1, \ldots, s_t, e_{k+1} + \Phi_J(k+1), \ldots, e_n + \Phi_J(n)]. \end{aligned}$$

Theorem 1.1 now follows from this equality and Theorem 2.1.

3. Proof of Proposition 1.2

First we prove the stated relation for $k = 0$,

$$[e_1, \ldots, e_{n-1}, e_n + n] = \sum_{s=0}^{n-1} (-1)^{n+s-1} [e_1, \ldots, e_{n-1}, e_n + s] Q_{n,s}^{p^{e_n}}. \tag{3.1}$$

We will prove (3.1) and the following relation together by induction on n,

$$\begin{aligned} [e_1, \ldots, e_{n-1}, e_n + n - 1] = &\sum_{s=0}^{n-2} (-1)^{n+s} [e_1, \ldots, e_{n-1}, e_n + s] Q_{n-1,s}^{p^{e_n}} \\ &+ [e_1, \ldots, e_{n-1}] V_n^{p^{e_n}}. \end{aligned} \tag{3.2}$$

Here, $V_n = L_n / L_{n-1}$.

We denote (3.1) and (3.2) when $n = m$ by 3.1(m) and 3.2(m), respectively.

When $n = 2$, the proof is straightforward. Suppose that $n > 2$ and that 3.1($n-1$) and 3.2($n-1$) are true.

By the Laplace development and 3.1($n-1$) we have

$$\begin{aligned}
[e_1,\ldots,e_{n-1},e_n+n-1]
&= \sum_{t=1}^{n-1}(-1)^{n+t}[e_1,\ldots,\hat{e}_t,\ldots,e_{n-1},e_n+n-1]y_n^{p^{e_t}} \\
&\qquad + [e_1,\ldots,e_{n-1}]y_n^{p^{e_n+n-1}} \\
&= \sum_{t=1}^{n-1}(-1)^{n+t}\Big(\sum_{s=0}^{n-2}(-1)^{n+s}[e_1,\ldots,\hat{e}_t,\ldots,e_{n-1},e_n+s]Q_{n-1,s}^{p^{e_n}}\Big)y_n^{p^{e_t}} \\
&\qquad + [e_1,\ldots,e_{n-1}]y_n^{p^{e_n+n-1}} \\
&= \sum_{s=0}^{n-2}(-1)^{n+s}\Big(\sum_{t=1}^{n-1}(-1)^{n+t}[e_1,\ldots,\hat{e}_t,\ldots,e_{n-1},e_n+s]y_n^{p^{e_t}}\Big)Q_{n-1,s}^{p^{e_n}} \\
&\qquad + [e_1,\ldots,e_{n-1}]y_n^{p^{e_n+n-1}} \\
&= \sum_{s=0}^{n-2}(-1)^{n+s}[e_1,\ldots,\ldots,e_{n-1},e_n+s]Q_{n-1,s}^{p^{e_n}} \\
&\qquad + [e_1,\ldots,e_{n-1}]\sum_{s=0}^{n-1}(-1)^{n+s-1}Q_{n-1,s}^{p^{e_n}}y_n^{p^{e_n+s}}
\end{aligned}$$

Since $V_n = \sum_{s=0}^{n-1}(-1)^{n+s-1}Q_{n-1,s}y_n^{p^s}$ (see [1], [2]), 3.2(n) is proved.

Now we prove 3.1(n). From 3.2(n) and the relation $Q_{n,s} = Q_{n-1,s-1}^p + Q_{n-1,s}V_n^{p-1}$ (see [1], [2]) we obtain

$$\begin{aligned}
[e_1,\ldots,e_{n-1},e_n+n]
&= \sum_{s=1}^{n-1}(-1)^{n+s-1}[e_1,\ldots,e_{n-1},e_n+s]Q_{n-1,s-1}^{p^{e_n+1}} \\
&\qquad + [e_1,\ldots,e_{n-1}]V_n^{p^{e_n+1}} \\
&= \sum_{s=1}^{n-1}(-1)^{n+s-1}[e_1,\ldots,e_{n-1},e_n+s]Q_{n,s}^{p^{e_n}} \\
&\qquad - [e_1,\ldots,e_{n-1},e_n+n-1]V_n^{(p-1)p^{e_n}} \\
&\quad + \Big(\sum_{s=1}^{n-2}(-1)^{n+s}[e_1,\ldots,e_{n-1},e_n+s]Q_{n-1,s}^{p^{e_n}} \\
&\qquad + [e_1,\ldots,e_{n-1}]V_n^{p^{e_n}}\Big)V_n^{(p-1)p^{e_n}}.
\end{aligned}$$

Combining this equality and 3.2(n) we get

$$[e_1, \dots, e_{n-1}, e_n + n] = \sum_{s=1}^{n-1} (-1)^{n+s-1} [e_1, \dots, e_{n-1}, e_n + s] Q_{n,s}^{p^{e_n}} - (-1)^n [e_1, \dots, e_n] Q_{n-1,0}^{p^{e_n}} V_n^{(p-1)p^{e_n}}.$$

Since $Q_{n,0} = Q_{n-1,0} V_n^{p-1}$, the proof of 3.1($n$) is completed.

For $0 < k < n$, Proposition 1.2 follows from (3.1) and [2; I.4.7] which asserts that

$$[k; e_{k+1}, \dots, e_n] = (-1)^{k(k-1)/2} \sum_{s_1 < \dots < s_k < n} (-1)^{s_1 + \dots + s_k} M_{n,s_1,\dots,s_k} [s_1, \dots, s_k, e_{k+1}, \dots, e_n]/L_n.$$

The proposition is completely proved.

4. Some applications

In this section, using Theorem 1.1 and Proposition 1.2, we prove Theorem 1.3 and explicitly compute the action of $St^{S,R}$ on the Mùi invariant $M_{n,s_1,\dots,s_k}$ when S, R are special. First we prove Theorem 1.3.

4.1. PROOF OF THEOREM 1.3. Recall that $P^t = St^{\emptyset,(t)}$. From Theorem 1.1 we have

$$P^t M_{n,s_1,\dots,s_k} = \begin{cases} [k; 0, \dots, \hat{t}_1, \dots, \hat{t}_{k+1}, \dots, n], & t = \sum_{i=1}^{k+1} \frac{p^{s_i} - p^{t_i}}{p-1}, \text{ with } \\ & s_{i-1} < t_i \le s_i, 1 \le i \le k+1, \\ 0, & \text{otherwise.} \end{cases}$$

If $t_{k+1} = s_{k+1} = n$, then $[k; 0, \dots, \hat{t}_1, \dots, \hat{t}_{k+1}, \dots, n] = M_{n,t_1,\dots,t_k}$. Suppose that $t_{k+1} < n$. By Proposition 1.2 we have

$$[k; 0, \dots, \hat{t}_1, \dots, \hat{t}_{k+1}, \dots, n] = \sum_{s=0}^{n-1} (-1)^{n+s-1} [k; 0, \dots, \hat{t}_1, \dots, \hat{t}_{k+1}, \dots, n-1, s] Q_{n,s}$$

$$= \sum_{i=1}^{k+1} (-1)^{k+1-i} M_{n,t_1,\dots,\hat{t}_i,\dots,t_{k+1}} Q_{n,t_i}.$$

Hence Theorem 1.3 follows.

4.2. NOTATION. Denote by $S' : s_{k+1} < \dots < s_{n-1}$ the ordered complement of the sequence $S : 1 \le s_1 < \dots < s_k < n$ in $\{1, \dots, n-1\}$. Set $\Delta_i = (0, \dots, 1, \dots, 0)$ with 1 at the i-th place $(1 \le i \le n)$, $\Delta_0 = (0, \dots, 0)$ and $R = (r_1, \dots, r_n)$. Here, the length of Δ_i is n.

The following was proved by Mùi [3; 5.3] for $R = \Delta_0$.

4.3. PROPOSITION. *Set $s_0 = 0$. Under the above notations, we have*

$$St^{S',R} M_{n,1,\dots,n-1} = \begin{cases} (-1)^{(k-1)(n-1-k)+s_t-t} M_{n,s_0,\dots,\hat{s}_t,\dots,s_k}\,, & R = \Delta_{s_t}, \\ \sum\limits_{t=0}^{k} (-1)^{k(n-k)-t} M_{n,s_0,\dots,\hat{s}_t,\dots,s_k} Q_{n,s_t}\,, & R = \Delta_n\,, \\ 0\,, & \textit{otherwise.} \end{cases}$$

PROOF. Note that $M_{n,1,\dots,n-1} = [n-1; 0]$. From Theorem 1.1 we obtain

$$St^{S',R} M_{n,1,\dots,n-1} = \begin{cases} (-1)^{k(n-1-k)} [k; 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1, i]\,. \\ \qquad\qquad R = \Delta_i, \text{ with } i = s_t,\ 0 \le t \le k \text{ or } i = n, \\ 0\,, \qquad \text{otherwise.} \end{cases}$$

It is easy to see that

$$[k; 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1, s_t] = (-1)^{n-1-k+s_t-t} M_{n,s_0,\dots,\hat{s}_t,\dots,s_k}.$$

According to Proposition 1.2 we have

$$\begin{aligned} &[k; 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n] \\ &\qquad = \sum_{s=0}^{n-1} (-1)^{n+s-1} [k; 1, \dots, \hat{s}_1, \dots, \hat{s}_k, \dots, n-1, s] Q_{n,s} \\ &\qquad = \sum_{t=0}^{k} (-1)^{k-t} M_{n,s_0,\dots,\hat{s}_t,\dots,s_k} Q_{n,s_t}. \end{aligned}$$

From this the proposition follows.

By the same argument as given in the proofs of Theorem 1.3 and Proposition 4.3 we obtain the following results.

4.4. PROPOSITION. *Let* Δ_i *be as in* 4.2 *and* $s_0 = 0$. *Then*

$$St^{\emptyset,\Delta_i} M_{n,s_1,\dots,s_k} = \begin{cases} (-1)^{s_t - t} M_{n,s_0,\dots,\hat{s}_t,\dots,s_k}, & s_1 > 0,\ i = s_t, \\ \sum_{t=0}^{k} (-1)^{n-t-1} M_{n,s_0,\dots,\hat{s}_t,\dots,s_k} Q_{n,s_t}, & s_1 > 0,\ i = n, \\ 0, & \textit{otherwise}. \end{cases}$$

The following proposition was proved by Hung and Minh [4] for $s = 0$.

4.5. PROPOSITION. *For* $0 \le s \le n$,

$$St^{(s),(0)} M_{n,s_1,\dots,s_k} = \begin{cases} (-1)^{k+s_t-t} M_{n,s_1,\dots,\hat{s}_t,\dots,s_k}, & s = s_t, \\ \sum_{t=1}^{k} (-1)^{n+k+t+1} M_{n,s_1,\dots,\hat{s}_t,\dots,s_k} Q_{n,s_t}, & s = n, \\ 0, & \textit{otherwise}. \end{cases}$$

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