ACTA MATHEMATICA VIETNAMICA
Volume 18, Number 2, 1993

PURSUIT PROBLEMS WITHOUT DISCRIMINATION OF EVASION OBJECT IN LINEAR DIFFERENTIAL GAMES

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1. Introduction

We consider a pursuit process as a linear differential game of the form

$$\dot{x} = Ax + Bu - Cv, \; x(0) = x_0 \tag{1}$$

with terminal set $M = \{0\}$ and constraints

$$\int_0^\infty ||u(s)||^2 ds \leq \rho_0, \tag{2}$$

$$||v|| \leq 1, \tag{3}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^p$ is the control vector of the pursuer, $v \in \mathbb{R}^q$ is the control vector of the evader, A, B and C are matrices with appropriate dimensions and ρ_0 is a given positive number.

Constraint (2) means that the inequality

$$\rho(t) = \rho_0 - \int_0^t ||u(s)||^2 ds \geq 0 \tag{4}$$

must be satisfied throughout the pursuit process. The pursuit process is complete when the state of system (1) becomes zero.

We are interested in completing the pursuit process by a pursuit strategy of the form

$$u = \mathsf{U}(x, \rho), \tag{5}$$

Received October 5, 1991; inrevised form June 15,1993

where ρ is defined in (4). This leads us to consider the pursuit problem as a simple nonlinear differential game (G):

$$\begin{aligned} \dot{x} &= Ax + Bu - Cv, \ x(0) = x_0 \\ \dot{\rho} &= -\|u\|^2, \ \rho(0) = \rho_0 \end{aligned} \tag{6}$$

with terminal set $\{0\} \times [0, +\infty)$, phase constraint

$$\rho(t) \geq 0$$

and geometric constraint (3) on the control vector v of the evader.

In this paper, our aim is to find a state-feedback strategy $u = \mathsf{U}(x, \rho)$ for completing the pursuit process in the game (G). This is a pursuit problem without discrimination of evasion object. The situation of nondiscriminating the evasion object in the linear differential game with geometric constraints on controls has been studied in recent works of L.S. Pontriagin and A.S. Misenco [1, 2, 3]. In these works the authors have given a state-feedback strategy which is able to complete the pursuit process. Our approach here is based on some ideas suggested in these works.

In Section 2, we shall construct some estimating functions and pursuit strategies. The ability of this strategy to complete the pursuit process will be discussed in Section 3. A simple example is presented in Section 4.

2. Estimating function and pursuit strategy

2.1 Basic assumptions

Throughout this paper we shall assume that the matrices A, B and C satisfy the following conditions :

I. *Kalman's Condition:*

$$Rank \ [B, AB, A^2B, ..., A^{n-1}B] = n,$$

II. $Rank \ [B, C] = Rank \ B.$

First, we consider the pursuit problem with discrimination of evasion object in game (1)-(3), in which at each moment the control vector v is supposed to be known.

Let s be a fixed positive number. It is well known (see [4], p. 213–214) that under the Kalman condition the optimal control problem

$$\begin{cases} \dot{x} = Ax + Bu; \ x(0) = x_0, \ x(s) = 0 \\ \text{minimize} \quad \int_0^s ||u(t)||^2 dt \end{cases} \tag{7}$$

has the unique solution

$$U(x_0, s, t) = -E^* \exp(-tA^*)K(s)x_0, \tag{8}$$

satisfying

$$\int_0^t ||U(x_0, s, t)||^2 dt = < x_0, K(s)x_0 >,$$

where $K(s)$ is defined by

$$K(s) = (\int_0^s \exp(-tA)BB^* \exp(-tA^*)dt)^{-1}. \tag{10}$$

On the other hand, because of Condition II, the matrix equation

$$BD = C, \tag{11}$$

has a solution. Let D be a solution of (11). We consider the pursuit strategy

$$u(t, v) = U(x_0, s, t) + Dv. \tag{12}$$

Let $v(t)$, $0 \leq t \leq s$, be a measurable function satisfying (3). Put $u(t) = u(t, v(t))$, $0 \leq t \leq s$. Then the trajectory $x(t)$ of equation (1) on $[0, s]$ is the optimal trajectory for problem (7) and

$$(\int_0^s ||u(t)||^2 dt)^{1/2} \leq < x_0, K(s)x_0 >^{1/2} + ||D||\sqrt{s}. \tag{13}$$

In particular, $x(s) = 0$.

From the above observations, it follows that if

$$< x_0, K(s)x_0 >^{1/2} + ||D||\sqrt{s} - \sqrt{\rho_0} \leq 0, \tag{14}$$

then the pursuit process in game (1–3) will be complete at time s with the use of the pursuit strategy (12).

REMARK 2.1. $K(s)$, $s \in (0,+\infty)$, are positive definite matrices. The function $K(s)$ is analytic on $(0,+\infty)$. Further

$$\lim_{s\to+0} K(s)^{-1} = 0. \tag{15}$$

2.2. Estimating function

Let $d > 0$. On $R^n \times (0,+\infty) \times (0,+\infty)$ we define the function F_d by

$$F_d(x,\rho,s) = < x, K(s)x >^{1/2} + d\sqrt{s} - \sqrt{\rho}. \tag{16}$$

Clearly, F_d is analytic on $(R^n \setminus \{0\}) \times (0,+\infty) \times (0,+\infty)$.

Let's consider equation

$$F_d(x,\rho,s) = 0, \tag{17}$$

for every (x,ρ) fixed, $(x,\rho) \in \Omega = (R^n \setminus \{0\}) \times (0,+\infty)$. By virtue of Remark 2.1, one can see that equation (17) can only have a finite number of positive solutions and all of these solutions must lie in the open interval $(0, \frac{\rho}{d^2})$.

Now, on Ω we define the estimating function T_d by

$$T_d(x,\rho) = \min\{s > 0 \mid F_d(x,\rho,s) = 0\}, \tag{18}$$

where $\min\{\emptyset\} = +\infty$. Denote

$$\mathrm{dom}T_d = \{(x,\rho) \in \Omega : T_d(x,\rho) < +\infty\}.$$

PROPOSITION 2.1. *Let $\{(x_i,\rho_i) \mid i = 0,1,2,..,\}$ be a sequence with $(x_i,\rho_i) \in$ dom T_d. If $\{\rho_i : i = 0,1,2,...\}$ is bounded and $\lim_{i\to+\infty} T_d(x_i,\rho_i) = 0$, then $\lim_{i\to\infty} x_i = 0$.*

PROOF. Put $m(s) = \min\{< x, K(s)x >^{1/2} \mid ||x|| = 1\}$ for every $s > 0$. As a consequence of (15),

$$\lim_{s\to+0} m(s) = +\infty. \tag{19}$$

Denoting $s_i = T_d(x_i,\rho_i)$, by assumptions we have

$$0 = < x_i, K(s_i)x_i >^{1/2} + d\sqrt{s_i} - \sqrt{\rho_i} \geq ||x_i||m(s_i) + d\sqrt{s_i} - \sqrt{\rho_i} >$$

$$> ||x_i||m(s_i) - \sqrt{\rho_i},\ i = 0,1,2,\dots .$$

Consequently,

$$||x_i||m(s_i) < \sup_i \sqrt{\rho_i} < +\infty.$$

Hence, it follows from (19) that $\lim_{i\to\infty} x_i = 0$.

REMARK 2.2. In general, the estimating function T_d is discontinuous on dom T_d. In the case when $A = 0$ and rank $B = n$ the function T_d is continuous on dom T_d.

Put

$$E_d = \{(x,\rho) \in \text{dom } T_d : \frac{\partial F_d}{\partial s}(x,\rho,T_d(x,\rho)) = 0\}. \tag{20}$$

One can verify that dom $T_d \setminus E_d$ is open and dense in dom T_d. By application of Implicit Function Theorem, we have

PROPOSITION 2.2. *T_d is analytic on dom $T_d \setminus E_d$. Let $(x_0,\rho_0) \in$ dom T_d and $s_0 = T_d(x_0,\rho_0)$. For every $\epsilon > 0$ and $\delta > 0$ we denote*

$$\begin{aligned} V(\epsilon,\delta) = \{(x,\rho,s) \mid ||((x,\rho) - (x_0,\rho_0)|| \le \epsilon \text{ and} \\ ||s - s_0|| \le \delta\} \cap \{((x,\rho,s) : F_d(x,\rho,s) = 0\}. \end{aligned} \tag{21}$$

We denote by P the projection $(x,\rho,s) \longmapsto (x,\rho)$. The following proposition shows an important property of T_d. The proof is simple and will be omitted.

PROPOSITION 2.3. *For every $\delta > 0$, there exists $\epsilon > 0$ such that*

$$(x,\rho) \in P(V(\epsilon,\delta)) \Rightarrow (x,\rho,T_d(x,\rho)) \in V(\epsilon,\delta). \tag{22}$$

Denote

$$G_d(x,\rho,s) = \frac{\partial F_d}{\partial s}(x,\rho,s). \tag{23}$$

It is clear that if $(x,\rho) \in$ dom T_d, $s = T_d(x,\rho)$, then

$$G_d(x,\rho,s) \le 0. \tag{24}$$

2.3. *Pursuit strategy*

Let d be a fixed positive, T_d be the estimating function defined above. We consider the differential equation

$$\begin{cases} \dot{x} &= Ax + Bu - Cv \\ \dot{\rho} &= -||u||^2 \end{cases} \tag{25}$$

on dom T_d.

We shall construct a pursuit strategy U_d on dom T_d which may assure the fastest decrease in value of the function T_d along the solution trajectories of equation (25).

Let $t_0 \geq 0$ and $u(t)$ and $v(t)$ be measurable functions defined on a neighbourhood of t_0. Suppose $(x(t), \rho(t))$ is a solution of (25) satisfying the initial condition $(x(t_0), \rho(t_0)) = (\bar{x}, \bar{\rho}) \in \text{dom}T_d$. Put $s(t) = T_d(x(t), \rho(t))$. We have

$$F_d(x(t), \rho(t), s(t)) \equiv 0. \tag{26}$$

Assume that $(\bar{x}, \bar{\rho}) \in \text{dom}T_d \setminus E_d$, i.e. $G_d(\bar{x}, \bar{\rho}, s(t_0)) \neq 0$. In view of Proposition 2.2 the function $s(t)$ is absolutely continuous in a neighbourhood of t_0 and

$$\frac{d}{dt} F_d(x(t), \rho(t), s(t)) = 0.$$

Hence,

$$\frac{ds}{dt} = -(\frac{\partial F_d}{\partial x}(Ax + Bu - Cv) + \frac{\partial F_d}{\partial \rho}(-||u||^2))(G_d(x, \rho, s))^{-1}. \tag{27}$$

Thus, in order to minimize $\frac{ds(t_0)}{dt}$ the control vector $u(t_0)$ must be chosen among the solutions of the optimization problem

$$\min_u J(x, \rho, s(t_0), u),$$

where

$$J(x, \rho, s, u) = \frac{\partial F_d}{\partial x}(Ax + Bu) - \frac{\partial F_d}{\partial \rho}||u||^2. \tag{28}$$

By direct computing we obtain

$$J(x, \rho, s, u) = < x, K(s)x >^{1/2} < K(s)x, Ax + Bu > \ -\sqrt{\rho}||u||^2/2. \tag{29}$$

Therefore, the problem $\min_u J(x, \rho, s, u)$ has the solution

$$U(x, \rho, s) = -\sqrt{\rho} < x, K(s)x >^{1/2} B^* K(s)x. \tag{30}$$

Now, for the estimating function T_d we define the pursuit strategy U_d on dom T_d by

$$U_d(x, \rho) = U(x, \rho, T_d(x, \rho)). \tag{31}$$

Like T_d, the function U_d is discontinuous on Ω. But on dom $T_d \backslash E_d$ it is analytic.

Denote by $\overline{D}$ the solution of problem min $\{||D|| : BD = C\}$ and $\bar{d} = ||\overline{D}||$. Put

$$E_d(x, \rho, s, v) = \frac{\partial F_d}{\partial x}(Ax + BU_d - Cv) - \frac{\partial F_d}{\partial \rho}||U_d||^2.$$

LEMMA 2.1. *For any $v \in \mathbf{R}$, $(x, \rho) \in domT_d$ and $s = T_d(x, \rho)$, we have*

$$E_d(x, \rho, s, v) - G_d(x, \rho, s) \leq -d(1 - \bar{d}^2||v||^2)\frac{1}{2\sqrt{s}}. \tag{32}$$

PROOF. First, one can verify that $K(s)$ satisfies the following Riccati equation

$$\dot{K}(s = AK(s) + K(s)A - K(s).BB^*K(s) \quad \text{on} \quad (0, +\infty).$$

By using this equation we get

$$G_d(x, \rho, s) = < x, K(s)x >^{-1/2} (< K(s)x, Ax > -\frac{1}{2}||B^*K(s)x||^2) + \frac{d}{2\sqrt{s}}, \tag{33}$$

and

$$\begin{aligned} E_d(x, \rho, s) = & < x, K(s)x >^{-1/2} (< K(s)x, Ax > -\frac{1}{2}\sqrt{\rho}||B^*K(s)x||^2) \\ & - < K(s)x, Cv > . \end{aligned} \tag{34}$$

Since $s = T_d(x, \rho)$, we have

$$1 - \sqrt{\rho} < x, K(s)x >^{-1/2} = -d\sqrt{s} < x, K(s)x >^{-1/2} . \tag{35}$$

By the definition of $\overline{D}$ and (33)–(35), we have

$$\begin{aligned} & E_d(x, \rho, s, v) - G_d(x, \rho, s) = \\ & = \langle x, K(s)x\rangle^{-1/2}.\frac{1}{2}[(1 - \sqrt{\rho}\langle x, K(s)x\rangle^{-1/2})||B^*K(s)x||^2 \\ & - 2\langle K(s)x, Cv\rangle] - \frac{d}{2\sqrt{s}} \\ & = \langle x, K(s)x\rangle^{-1/2}.\frac{1}{2}(-d\sqrt{s}\langle x, K(s)x\rangle^{-1/2})||B^*K(s)x||^2 \\ & - 2\langle B^*K(s)x, Dv\rangle) - \frac{d}{2\sqrt{s}} \end{aligned}$$

$$\leq \langle x, K(s)x\rangle^{-1/2}.\frac{1}{2}(-d\sqrt{s})x, K(s)x\rangle^{-1/2})||B^*K(s)x||^2$$
$$+ 2\bar{d}||v||\langle B^*K(s)x, Dv\rangle) - \frac{d}{2\sqrt{s}}$$
$$\leq \langle x, K(s)x\rangle^{-1/2}.\frac{1}{2}(\sup_{a>0}(-d\sqrt{s}\langle x, K(s)x\rangle^{-1/2})a^2$$
$$+ 2\bar{d}||v||.a) - \frac{d}{2\sqrt{s}}$$
$$\leq \ -d(1 - \bar{d}^2||v||^2.\frac{1}{d^2})\frac{1}{2\sqrt{s}}.$$

COROLLARY 2.1. *If $d \geq \bar{d}$, then*

$$E_d(x, \rho, T_d(x, \rho), v) \leq G_d(x, \rho, T_d(x, \rho)), \tag{36}$$

for all of vectors v, $||v|| \leq 1$.

Let $(x, \rho) \in domT_d \setminus E_d$. Denote by grad $T_d(x, \rho)$ the gradient vector of function T_d at (x, ρ),

$$\text{grad } T_d(x, \rho) = (\frac{\partial T_d}{\partial x}, \frac{\partial T_d}{\partial \rho})(x, \rho).$$

Lemma 2.1 together with (27) implies the following

COROLLARY 2.2. *If $d \geq \bar{d}$ and $(x, \rho) \in domT_d \setminus E_d$, then*

$$< \text{grad } T_d(x, \rho), (Ax + BU_d - Cv, -||U_d||^2) > \leq 1, \tag{37}$$

for all vectors v, $||v|| \leq 1$.

3. Existence of solution for the pursuit problem

Let d be a given number, $d \geq \bar{d}$; U_d and T_d are the functions constructed in Section 2. We consider the following differential equation

$$\begin{cases} \dot{x} &= Ax + BU_d - Cv \\ \dot{\rho} &= -||U_d||^2 \end{cases} \tag{38}$$

on dom T_d, where $v(\cdot)$ is some measurable function with $||v|| \leq 1$.

Let $(x_0, \rho_0) \in$ domT_d. In view of the pursuit, we shall consider the existence of solutions $(x(\cdot), \rho(\cdot))$ for the differential equation (38) with the initial condition

$(x(o), \rho(o)) = (x_0, \rho_0)$ defined on an interval $[0, T)$ with $\lim_{t \to T} x(t) = 0$. Such solutions are called solutions of the pursuit problem.

In general, the right hand-side of equation (38) is discontinuous. Thus, to study this differential equation, one may make the use of the differential inclusion derived from it (see [5]). However, such an approach seems to be unconvenient for our purpose. We shall therefore use a direct approach to examine equation (38) and to get some affirmative answers for the existence problem.

3.1. Main theorem

Consider the differential equation (38) with function $v(\cdot)$ defined on $[0, +\infty)$ and satisfying the following condition

($\star$) There exist a partition $0 = t_0 < t_1 < t_2 < \cdots$ of the interval $[0, +\infty)$, a positive number ϵ_k and C^∞-differential functions v_k defined on (t_k, t_{k+1}), $k = 0, 1, 2, \ldots$ such that

$$v(t) = v_k(t) \qquad \forall t \in (t_k, t_{k+1}).$$

THEOREM 3.1. *Let $d > \bar{d}$. If $v(\cdot)$ is a function satisfying condition ($\star$) with $||v|| \leq 1$, then for each $(x_0, \rho_0) \in domT_d$ there exists a solution $(x(\cdot), \rho(\cdot))$ of the differential equation (38) on an interval $[0, T)$ satisfying the following*

(i) $x(o), \rho(o)) = (x_0, \rho_0)$ and $\lim_{t \to T} x(t) = 0$;

(ii) The function $T_d(x(\cdot), \rho(\cdot))$ is strictly decreasing on $[0, T)$ and

$$\frac{d}{dt} T_d(x(t), \rho(t)) < -1$$

wherever the derivative exists;

(iii) $T < T_d(x_0, \rho_0)$.

Before giving the proof of Theorem 3.1, we shall study the local existence of solutions of equation (38). Let $v(\cdot)$ be a function satisfying condition ($\star$), $||v|| \leq 1$. Let $(\bar{x}, \bar{\rho}) \in \text{dom}T_d$, $\bar{t} \geq 0$, and $\bar{s} = T_d(\bar{x}, \bar{\rho})$. Consider equation (38) with the initial condition

$$(x(\bar{t}), \rho(\bar{t})) = (\bar{x}, \bar{\rho}). \tag{39}$$

We shall only be concerned with the existence of solutions for (38)–(39) on some interval $[\bar{t}, \bar{t}+\epsilon)$. Hence, without loss of generality, we may assume that $v(\cdot)$ is C^∞-differentiable in a neighbourhood of $\bar{t}$.

In the case when $(\bar{x}, \bar{\rho}) \in \mathrm{dom}T_d \setminus E_d$, the right hand-side of (38) is differentiable in a neighbourhood of $(\bar{x}, \bar{\rho}, \bar{t})$. Hence, a unique local solution of (38)–(39) exists in some neighbourhood of $\bar{t}$, even in the case when $v(\cdot)$ is measurable. In view of Corollary 2.2, this local solution satisfies Property (ii) pointed out in the statement of Theorem 3.1.

Now, consider the singular case when $(\bar{x}, \bar{\rho}) \in E_d$, i.e. $G_d(\bar{x}, \bar{\rho}, \bar{s}) = 0$.

Applying the Weierstrass Separation Theorem and Proposition 2.3, we have

LEMMA 3.1. *There exists a neighbourhood V of $(\bar{x}, \bar{\rho}, \bar{s})$ with the following properties*

(i) On V, F_d can be represented in the form

$$F_d(x, \rho) = [(-1)^k (s-\bar{s})^k + \sum_{j=0}^{k-1} (s-\bar{s})^j < a_j(x,\rho), (\bar{x}, \bar{\rho}) >] F(x, \rho, s), \quad (40)$$

where F is some analytic function with no zero in V and $a_j(\cdot)$ are some differentiable functions;

(ii) $(x, \rho) \in P(U) \Rightarrow (x, \rho, T_d(x, \rho)) \in U$, (41)

where $U = V \cap \{(x, \rho, s) : F_d(x, \rho, s) = 0\}$.

Note that in (40) if $(\bar{x}, \bar{\rho}) \in E_d$, then $k > 1$.

LEMMA 3.2. *If $G_d(\bar{x}, \bar{\rho}, \bar{s}) = 0$ and $D_d(\bar{x}, \bar{\rho}, \bar{s}, v(\bar{t})) < 0$, then there exists a solution $(x(\cdot), \rho(\cdot))$ such that $T_d(x(\cdot), \rho(\cdot))$ is differentiable on $[\bar{t}, t^*)$ and*

$$(x(t), \rho(t)) \in \mathrm{dom}T_d \setminus E_d, \ \forall t \in (\bar{t}, t^*). \quad (42)$$

PROOF. The proof can be carried out by an analogous way as in [2] (see part II, 2 in [2]). Only a sketch of the proof is provided here.

First, by assumptions we can choose the neighbourhood V of $(\bar{x}, \bar{\rho}, \bar{s})$ specified in Lemma 3.1 and $\epsilon > 0$ such that

$$H(\bar{x}, \bar{\rho}, \bar{\epsilon}, \bar{t}) = E_d(x, \rho, s, v(t)) < 0$$

on $\widetilde{V} = V \times (\bar{t} - \epsilon, \bar{t} + \epsilon)$. On $\widetilde{V}$ consider, the differential equation

$$\begin{aligned}\frac{d}{ds}x &= (Ax + BU(x,\rho,s) - Cv(t)).\frac{-G(x,\rho,s)}{H(x,\rho,s,t)}\\ \frac{d}{ds}\rho &= -\|U(x,\rho,s)\|^2.\frac{-G(x,\rho,s)}{H(x,\rho,s,t)}\\ \frac{d}{ds}t &= \frac{-G(x,\rho,s)}{H(x,\rho,s,t)},\end{aligned} \tag{43}$$

with the initial condition

$$(x(\bar{s}, \rho(\bar{s}), t(\bar{s})) = (\bar{x}, \bar{\rho}, \bar{t}), \tag{44}$$

where $G(x,\rho,s) = G_d(x,\rho,s)$.

By using the representation of F_d given in Lemma 3.1, we can rewrite equation (43) in the following form

$$\begin{aligned}\frac{d}{ds}x =& x_1 \frac{(-1)^{k-1}k}{H(\bar{x},\bar{\rho},\bar{s},\bar{\lambda})}(s-\bar{s})^{k-1} + \alpha_3(s,t)(s-\bar{s})^k +\\ &+ \alpha_2(s,t)(s-\bar{s})^{k-1}(t-\bar{t}) + \alpha_1(x,\rho,s,t)((x,\rho)-(\bar{x},\bar{\rho}))\end{aligned}$$

$$\begin{aligned}\frac{d}{ds}\rho =& \rho_1 \frac{(-1)^{k-1}k}{H(\bar{x},\bar{\rho},\bar{s},\bar{\lambda})}(s-\bar{s})^{k-1} + \beta_3(s,t)(s-\bar{s})^k +\\ &+ \beta_2(s,t)(s-\bar{s})^{k-1}(t-\bar{t}) + \beta_1(x,\rho,s,t)((x,\rho)-(\bar{x},\bar{\rho}))\end{aligned}$$

$$\begin{aligned}\frac{d}{ds}t =& x_1 \frac{(-1)^{k-1}k}{H(\bar{x},\bar{\rho},\bar{s},\bar{t})}(s-\bar{s})^{k-1} + \gamma_3(s,t)(s-\bar{s})^k +\\ &+ \gamma_2(s,t)(s-\bar{s})^{k-1}(t-\bar{t}) + \gamma_1(x,\rho,s,t)((x,\rho)-(\bar{x},\bar{\rho})).\end{aligned} \tag{45}$$

This implies that the unique local solution of (43)-(44) has the form

$$(x(s), \rho(s), t(s)) = (\bar{x}, \bar{\rho}, \bar{t}) + (s-\bar{s})^k(\tilde{x}(s), \tilde{\rho}, \tilde{t}(s)), \tag{46}$$

where $\tilde{x}, \tilde{\rho}, \tilde{t}$ are C^∞-differentiable functions.

$$\tilde{f}(\bar{s}) = \frac{(-1)^{k+1}k}{H(\bar{x},\bar{\rho},\bar{s},\bar{t})} \neq 0.$$

Hence, on an interval $(\delta, \bar{s})$ the function $\tilde{t}(s)$ has an inverse function $s(t)$ that is defined on an interval $[\bar{t}, t^*)$ and has the form

$$s(t) = \bar{s} + (t - \bar{t})^{1/k}\tilde{s}(t) \tag{47}$$

where $\tilde{s}(t)$ is differentiable on $(\bar{t}, t^*)$ and $\tilde{s}(\bar{t}) < 0$.

Putting (47) into (46), we obtain the function

$$(x(t), \rho(t)) = (\bar{x}, \bar{\rho}) + (t - \bar{t})(\tilde{x}(t), \tilde{\rho}(t)) \tag{48}$$

defined on $[\bar{t}, t^*)$. This function satisfies equation $F_d(x, \rho, T_d(x, \rho)) = 0$ on $[\bar{t}, t^*)$.

To continue, we shall show that function $(x(\cdot), \rho(\cdot))$ is a solution of (38), (39) and $(x(t), \rho(t)) \in \text{dom}T_d \setminus E_d$ for all $t \in (\bar{t}, t^*)$, where t^* is chosen enough close to $\bar{t}$ if necessary. For this we have only to prove that for any t close enough to $\bar{t}$, $T_d(x(t), \rho(t)) = s(t)$ and $(x(t), \rho(t)) \notin E_d$.

Let $t > \bar{t}$. Suppose that $T_d(x(t), \rho(t)) \neq s(t)$ or $(x(t), \rho(t)) \in E_d$. By the definition of T_d, there exists $s_1(t)$ such that

$$T_d(x(t), \rho(t)) \leq s_1(t) \leq s(t), \tag{49}$$

and

$$G_d(x(t), \rho(t), s_1(t)) = 0. \tag{50}$$

By using the representation of F_d in Lemma 3.1, from (50) we get

$$0 = (-1)^k k(s_1(t) - \bar{s})^{k-1} + \\ + \sum_{j=0}^{k-1} j(s_1(t) - \bar{s})^{j-1} < a_j(x(t), \rho(t)), (x(t), \rho(t)) - (\bar{x}, \bar{\rho}) > .$$

Then

$$|s_1(t) - \bar{s}| \leq C_1 ||(x(t), \rho(t)) - (\bar{x}, \bar{\rho})||^{\frac{1}{k-1}}, \tag{51}$$

where C_1 is a positive constant.

On the other hand, (47) and (48) imply that

$$|s(t) - \bar{s}| \geq C_2 |t - \bar{t}|^{1/k} \tag{52}$$

and

$$\|(x(t),\rho(t)) - (\bar{x},\bar{\rho})\| \leq C_3|t-\bar{t}|, \tag{53}$$

for any positive constant C_2, C_3.

Combining (49)–(53) we get

$$C_2|t-\bar{t}|^{1/k} \leq |a_1(t) - \tilde{s}| \leq C_3 C_2 |t-\bar{t}|^{\frac{1}{k-1}}.$$

This can not happen for t enough close to $\bar{t}$. Thus we can choose $t^*, t^* > \bar{t}$, such that $(x(t),\rho(t))$ is a solution of (38)–(39) and $(x(t),\rho(t)) \in \mathrm{dom}T_d \setminus E_d$ for all $t \in (\bar{t}, t^*)$.

PROOF OF THEOREM 3.1.

Let $v(\cdot)$ be a function satisfying condition $(\star)$, $\|v\| \leq 1$. Since $d > \tilde{d}$, from Lemma 2.1 it follows that

$$H_d(x,\rho,s,v(t)) - G_d(x,\rho,s) < 0$$

for all $(x,\rho) \in \mathrm{dom}T_d$, $s = T_d(x,\rho)$ and $t \in [0,+\infty)$. Therefore, in view of Lemma 3.2 and Corollary 2.2, for every $(\bar{x},\bar{\rho}) \in \mathrm{dom}T_d$ the equations (38)–(39) have a local solution satisfying Property (ii) contained in the statement of Theorem 3.1. Hence, by using standard arguments for the extension of solutions it follows that there exists a maximal solution $(x(\cdot),\rho(\cdot))$ of (38) with the initial condition $(x(o),\rho(o)) = (x_0,\rho_0)$ defined on an interval $[0,T)$ and satisfying property (ii).

Let $(x(\cdot),\rho(\cdot))$ defined on $[0,T)$ be such a solution. We shall show that $\lim_{t\to T} x(t) = 0$. Then, by property (ii), it is clear that $T < T_d(x_0,\rho_0)$ and $(x(\cdot),\rho(\cdot))$ satisfies all of the properties (i) (ii) and (iii).

First, we observe that $\lim_{t\to T}\rho(t) = \rho^* \geq 0$ and $\lim_{t\to T} T_d(x(t),\rho(t)) = s^* \geq 0$. We must only prove $s^* = 0$. This fact together with Proposition 2.1 implies that $\lim_{t\to T} x(t) = 0$. Assume the contradition that $s^* > 0$. We have

$$\begin{aligned} 0 &\equiv F_d(x(t),\rho(t),s(t)) = \\ &= \lim_{t\to T}(< x(t), K(s(t))x(t) >^{1/2} + d\sqrt{s(t)} - \sqrt{\rho(t)}) \\ &= \lim_{t\to T}(< x(t), K(s(t))x(t) >^{1/2} + d\sqrt{s^*} - \sqrt{\rho^*}), \end{aligned}$$

where $s(t) = T_d(x(t), \rho(t))$. It follows that $\rho^* > 0$, the set $\Omega = \{x : x(t_i) \to x,$ for some sequence $t_i \to T\}$ is nonempty compact, and $\Omega \times \{\rho^*\} \subset \operatorname{dom} T_d$. Consider the set Ω. One can see that for t close enough to T the point $(x(t), \rho(t))$ belongs to a neighbourhood of the compact set $\Omega \times \{\rho^*\} \times \{s^*\}$. Consequently, $\frac{d}{dt}x(\cdot)$ is bounded on $[0, T)$. This follows that

$$\int_0^T ||\frac{d}{dt}x(t)||dt < +\infty.$$

This fact means that $\Omega = \{x^*\}$. Thus, we have that $\lim_{t\to T} x(t) = x^*$, $(x^*, \rho^*) \in \operatorname{dom}T_d$ and $T_d(x^*, \rho^*) \leq s^*$. Therefore, the solution $(x(\cdot), \rho(\cdot))$ may be extended for the points outside $[0, T)$. This conflicts with the maximality of this solution. Thus, we get $s^* = 0$. The proof is complete.

3.2. Measurable case

Consider the case when $v(\cdot)$ is a measurable function. In this case, for an initial state $(\bar{x}, \bar{\rho}) \in \operatorname{dom}T_d \setminus E_d$ there exists a unique local solution of (38)–(39).

Denote

$$E_d^* = \{(x, \rho) \in E_d \mid (\frac{\partial G_d}{\partial s})(x, \rho, T_d(x, \rho)) = 0\}.$$

Clearly, $E_d^* \subset E_d \subset \operatorname{dom}T_d$. Note that E_d and E_d^* are contained in the images of some semi-analytic sets of dimension n and $n-1$ via the projection map $(x, \rho, s) \to (x, \rho)$, respectively.

THEOREM 3.2. *Let $d \geq \bar{d}$ and $\bar{t} \geq 0$. If $v(\cdot)$ is a measurable function on $[0, +\infty)$ with $||v|| \leq 1$, then for every $(\bar{x}, \bar{\rho}) \in \operatorname{dom}T_d \setminus E_d^*$ there exists a solution $(x(\cdot), \rho(\cdot))$ of (38)–(39) on an interval $[\bar{t}, t^*)$ which satisfies the following property*

(*) The function $T_d(x(\cdot), \rho(\cdot))$ is continuous and strictly decreasing on $[\bar{t}, t^*)$.

Furthermore,

$$\frac{d}{dt}T_d(x(t), \rho(t)) \leq -1$$

provided the derivative exists.

PROOF. Let $(\bar{x}, \bar{\rho}) \in \operatorname{dom}T_d \setminus E_d^*$. We consider only the case when $(\bar{x}, \bar{\rho}) \in E_d$. Put $\bar{s} = T_d(\bar{x}, \bar{\rho})$. By the definition of E_d^*, $G_d(\bar{x}, \bar{\rho}, \bar{s}) = 0$ and $(\frac{\partial G_d}{\partial s})(\bar{x}, \bar{\rho}, \bar{s}) \neq 0$. Therefore, the local representation of F_d at $(\bar{x}, \bar{\rho}, \bar{s})$ given in Lemma 3.1 becomes

$$F_d(x, \rho, s) = ((s - \bar{s})^2 + a_1(x, \rho)(s - \bar{s}) + a_0(x, \rho))F.$$

Denote $\Delta(x,\rho) = a_1(x,\rho)^2 - 4a_0(x,\rho)$. By virtue of Lemma 3.1, there exists $\epsilon > 0$ such that

$$W = \{(x,\rho) \mid ||(x,\rho) - (\bar{x},\bar{\rho})|| \leq \epsilon \quad \text{and} \quad \Delta(x,\rho) \geq 0\} \subset \text{dom}T_d$$

and

$$T_d(x,\rho) = \frac{1}{2}(-a_1(x,\rho) - \sqrt{\Delta(x,\rho)}) + \bar{s}$$

for all of $(x,\rho) \in W$. It is clear that W is a nonempty compact set and T_d is continuous on W.

Now, suppose that $v(\cdot)$ is a given measurable function on $[0,+\infty)$ with $||v|| \leq 1$. Take $\delta > 0$ and some C^∞-functions v_i defined on $[\bar{t}-\delta, \bar{t}+\delta]$ with

$$||v_i(t)|| < 1, \quad i = 1,2,... \tag{54}$$

such that $v_i(\cdot)$ almost everywhere converges to $v(\cdot)$ on $[\bar{t}, \bar{t}+\delta]$.

For any i, $i = 1,2,...$, we consider the differential equation on W

$$\begin{cases} \dot{x}_i = & Ax_i + BU_d - Cv_i \\ \dot{\rho}_i = & -||U_d||^2 \end{cases} \tag{55}$$

In view of Lemma 2.1, (54) yields

$$H_d(x,\rho,s,v_i(t)) < 0$$

for all of $(x,\rho) \in W$, $s = T_d(x,\rho)$ and $t \in [\bar{t}-\delta, \bar{t}+\delta]$. Applying Lemma 3, 2, we obtain that for every $(x,\rho) \in W$ and $t_0 \in [\bar{t}-\delta, \bar{t}+\delta]$ the equation (55) with initial condition $(x(t_0), \rho(t_0)) = (x,\rho)$ has a local solution satisfying condition (*). By standard arguments of the extending solution it follows that there exists a maximal solution $(x_i(\cdot), \rho_i(\cdot))$ of (55) with the condition $(x_i(\bar{t}, \rho_i(\bar{t})) = (\bar{x},\bar{\rho})$ which is defined on an interval $[t_0, t_1)$ and satisfies condition (*).

Consider a family $\{x_i(\cdot), \rho_i(\cdot) : i = 1,2,...\}$ of such maximal solutions. Note that the right hand-side of (55) is bounded by a constant independent on the indices i.

Hence, there exists a number t^* such that $\bar{t} < t^* < t_i$ for all of the indices i. Then, the family $\{(x_i(\cdot), \rho_i(\cdot), T_d(x_i(\cdot), \rho_i(\cdot))) : i = 1,2,...\}$ is bounded and continuous on $[\bar{t}, t^*)$. Consequently, in view of the Arzela-Ascoli Theorem, we can assume that $(x_i(\cdot), \rho_i(\cdot), T_d(x_i(\cdot), \rho_i(\cdot)))$ converges uniformly to a continuous function $(x(\cdot), \rho(\cdot), s(\cdot))$ on $[\bar{t}, t^*)$. Since $v_i(\cdot)$ on $[\bar{t}, \bar{t}+\delta)$ converges to $v(\cdot), (x(\cdot), \rho(\cdot))$ is a solution of (38)-(39). It is clear that $(x(\cdot), \rho(\cdot))$ satisfies condition $(*)$. The proof is complete.

Note that the proof of Theorem 3.2 is based on the continuity of T_d on the compact set W. In the case when T_d is continuous on dom T_d, we have the following theorem.

THEOREM 3.3. *Suppose that* $d \geq \bar{d}$ *and* $v(\cdot)$ *is a measurable function on* $[0, +\infty)$ *with* $||v|| \leq 1$. *If* T_d *is continuous on dom* T_d, *then for every* $(x_0, \rho_0) \in domT_d$ *there exists a solution* $(x(\cdot), \rho(\cdot))$ *of the differential equation (38) on an interval* $[0, T)$ *satisfying the following*

(i) $(x(o), \rho(o)) = (x_0, \rho_0)$ and $\lim_{t \to T} x(t) = 0$,

(ii) Function $T_d(x(\cdot), \rho(\cdot))$ *decreases strictly on* $[0, T)$ *and*

$$\frac{d}{dt} T_d(x(\cdot), \rho(\cdot)) \leq -1$$

provided the derivative exists;

(iii) $T \leq T_d(x_0, \rho_0)$.

The proof of Theorem 3.3 can be carried out by an argument analogous as in the proof of Theorem 3.1 and Theorem 3.2.

4. Example

Consider games of the simple form

$$\begin{cases} \dot{x} = & Bu - Cv \\ x \in & R^n;\ u \in \mathbf{R}^p;\ v \in \mathbf{R}^q \end{cases} \tag{56}$$

Suppose that the basic assumptions in subsection 2.1 are satisfied, i.e. $p \geq n$ and rank $B = n$. In this case

$$K(s) = \left(\int_0^s BB^* dt\right)^{-1} = s^{-1}(BB^*)^{-1}.$$

Since $(BB^*)^{-1}$ is positive definite, $(BB^*)^{-1}$ can be rewritten in the form $(BB^*)^{-1} = Q^*Q$, where Q is a symmetric invertible $n \times n$-matrix. Then

$$K(s) = s^{-1}Q^*Q,$$

and

$$F_d(x, \rho, s) = \sqrt{s^{-1}}||Qx|| + d\sqrt{s} - \sqrt{\rho}.$$

For (x, ρ) fixed the equation $F_d(x, \rho, s) = 0$ has some solutions iff $\rho - 4d||Qx|| \geq 0$. Consequently,

$$\mathrm{dom}T_d = \{(x, \rho) : x \neq 0,\ \rho > 0 \quad \text{and} \quad \rho - 4d||Q(x)|| \geq 0\}.$$

By direct computation, we obtain that on dom T_d

$$T_d(x, \rho) = \frac{1}{2}\rho - \frac{1}{2}(\rho(\rho - 4d||Q(x)||))^{1/2} - 4d||Qx||$$

and

$$U_d(x, \rho) = -2\sqrt{\rho}(\sqrt{\rho} - (\rho - 4d||Q(x)||)^{1/2})^{-1}B^*Q^*(\frac{Qx}{||Qx||}).$$

It is clear that T_d is continuous on dom T_d. In view of Theorem 3.3, one can complete the pursuit process in the game (56) by using pursuit strategy U_d with $d \geq \bar{d}$. Under action of the strategy U_d, the equation (38) becomes

$$\begin{aligned}\dot{x} &= -\,2\sqrt{\rho}(\sqrt{\rho} - (\rho - 4d||Qx||^{1/2})^{-1}(\frac{x}{||Qx||}) - Cv,\\ \dot{\rho} &= -\,4\rho(\sqrt{\rho} - (\rho - 4d||Qx||)^{1/2})^{-2}.\end{aligned}$$

In this example, $E_d = \{(x, \rho) \mid x \neq 0 \quad \text{and} \quad \rho = 4d||Qx||\}$ and $E_d^* = \{\emptyset\}$.

5. Conclusion

Pursuit strategies without discrimination of evasion object for differential linear games with terminal set $\{0\}$, geometric constraint on evasion controls and integral constraint on pursuit controls have been constructed. Our method may be generalized to some other cases when the terminal sets are A-invariant subspaces or neighbourhoods of zero.

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