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VARIATIONAL INEQUALITIES FOR INTEGRAND MARTINGALES AND ADDITIVE RANDOM SEQUENCES

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Abstract. New variational convergence results for lower semi-continuous integrands, reversed martingales and for lower semi-continuous additive random sequences are obtained in connection with Doob's theorem for reversed martingales as well as Birkhoff's ergodic theorem and Kingman's theorem for superadditive random sequences. Applications to the law of large numbers are given.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let (E, d) be a Suslin metrizable space with its Borel σ-algebra $\mathcal{B}(E)$.

Let $f : \Omega \times E \to]-\infty, +\infty]$ be a lower semi-continuous random integrand, that is, f is $\mathcal{F} \otimes \mathcal{B}(E)$-measurable and for any ω in Ω, $f(\omega, .)$ is lower semi-continuous on E. First we give an existence theorem of conditional expectation for a class of lower semi-continuous random integrands which has nice applications to the variational convergence versions for lower semi-continuous integrand reversed supermartingales (resp. martingales) and upper semi-continuous integrand reversed submartingales and also for lower semi-continuous additive random sequences introduced in this paper.

The key ingredient of the proofs is based on the existence and the properties of conditional expectation for random lower semi-continuous integrands mentioned above and on the parametrized lipschitzean approximations for this class of integrands. In Section 2 we give the existence and the properties of the conditional expectation for lower semi-continuous random integrands which allows to obtain the properties of lower semi-continuous integrands martingales and lower semi-continuous additive random sequences. Section 3 is devoted to lower semi-continuous integrand reversed martingales and its applications to

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the variational versions of the Cesaro convergence a.s. (or law of large numbers) for this class of integrands. In Section 4 we give a variational inequality for upper semi-continuous integrand reversed submartingales. In Section 5 we present some variational inequalities for lower semi-continuous additive random sequences and we indicate some open problems.

Finally it turns out that the results obtained are good enough to deduce the variational convergence results for the class of integrands presented in this paper.

2. Conditional expectation for lower semi-continuous random integrands

We begin with the following general result:

THEOREM 2.1. *Let S be a Suslin topological space, $\mathcal{G}$ a complete σ-subalgebra of $\mathcal{F}$, f a $\mathcal{F} \otimes \mathcal{B}(S)$-measurable integrand from $\Omega \times S$ to $[0, +\infty]$. Then there is a $\mathcal{G} \otimes \mathcal{B}(S)$-measurable integrand, denoted by $E^{\mathcal{G}} f$, such that for any $\mathcal{G}$-measurable set A and for any $(\mathcal{G}, \mathcal{B}(S))$-measurable function u from Ω to S, one has*

$$\int_A E^{\mathcal{G}} f(\omega, u(\omega)) P(d\omega) = \int_A f(\omega, u(\omega)) P(d\omega).$$

Moreover, $E^{\mathcal{G}} f$ is unique a.s. and is called the conditional expectation of f with respect to $\mathcal{G}$.

References. For a related result, see Estigneev ([17]). This theorem is an unpublished result obtained by the first author. See Derras ([14], Theorem3.2).

PROOF. The uniqueness of $E^{\mathcal{G}} f$ is a consequence of the measurable projection theorem and an argument due to Castaing-Valadier ([11]), Theorem III.23, Theorem VIII.36, p.264).

Since $f = \sup_{n \geq 1} u_n$ where $(u_n)_{n \geq 1}$ is an increasing sequence of positive elementary $\mathcal{F} \otimes \mathcal{B}(S)$- mesurable functions, by the monotone convergence theorem for conditional expectation, it is enough to prove the theorem when f is the characteristic function χ_G of a $\mathcal{F} \otimes \mathcal{B}(S)$-measurable set G. Let

$$\mathcal{D} = \{G \in \mathcal{F} \otimes \mathcal{B}(S) : E^{\mathcal{G}}(\chi_G) \text{ exists } \}$$

It is easy to check that $\mathcal{D}$ is a Dynkin system in $\Omega \times S$. Indeed, it is obvious that $\Omega \times S$ belongs to $\mathcal{D}$. If E and F belong to $\mathcal{D}$ with $E \subset F$, then one puts

$$E^{\mathcal{G}}(\chi_{F\setminus E}) = E^{\mathcal{G}}(\chi_F) - E^{\mathcal{G}}(\chi_E).$$

Let $(D_n)_{n\geq 1}$ be a disjoint sequence in $\mathcal{D}$. Then for every $p \geq 1$ one sets,

$$E^{\mathcal{G}}(\chi_{D_1 \cup D_2 \cup \cdot \cup D_2}) = E^{\mathcal{G}}(\chi_{D_1} + \cdots + \chi_{D_P}) = \sum_{i=1}^{p} E^{\mathcal{G}}(\chi_{D_i}).$$

Thus $D_1 \cup D_2 \cup \cdots D_p$ belong to $\mathcal{D}$.

Let

$$E^{\mathcal{G}}(\chi_{\bigcup_{n=1}^{\infty} D_n}) := \sup_{p\geq 1} E^{\mathcal{G}}(\chi_{D_1 \cup \cdots \cup D_p}).$$

Then $\mathcal{D}$ is a Dynkin system in $\Omega \times S$. Morever $\mathcal{F} \times \mathcal{B}(S) \subset \mathcal{D}$. It follows that

$$\mathcal{D} \supset \mathcal{D}(\mathcal{F} \times \mathcal{B}(S)) = \sigma(\mathcal{F} \times \mathcal{B}(S)) = \mathcal{F} \otimes \mathcal{B}(S)$$

The proof is complete.

COROLLARY 2.2. *Let $\mathcal{G}$ be a complete σ-subalgebra of $\mathcal{F}$. Let (E, d) be a Suslin metrizable space with $d \leq 1$. Let g be a $\mathcal{F} \otimes \mathcal{B}(E)$-measurable integrand from $\Omega \times E$ to* R *satisfying the following condition:*

(i) There is $\lambda > 0$ such that

$$| g(\omega, x) - g(\omega, y) | \leq \lambda d(x, y)$$

for (ω, x, y) in $\Omega \times E \times E$.

(ii) For any $(\mathcal{G}, \mathcal{B}(E))$-measurable mapping u from Ω to E, $g(., u(.))$ is integrable.

Then there is an a.s. unique mapping $E^{\mathcal{G}} g$ from $\Omega \times E$ to R *such that*

(1) For any x in E, $E^{\mathcal{G}} g(., x)$ is $\mathcal{G}$-measurable.

(2)$| E^{\mathcal{G}} g(\omega, x) - E^{\mathcal{G}} g(\omega, y) | \leq \lambda d(x, y)$ for (ω, x, y) in $\Omega \times E \times E$.

(3) For any $\mathcal{G}$-measurable set A in Ω and any $(\mathcal{G}, \mathcal{B}(E))$-measurable mapping u from Ω to E, one has

$$\int_A E^{\mathcal{G}} g(\omega, u(\omega)) P(d\omega) = \int_A g(\omega, u(\omega)) P(d\omega)$$

PROOF. Let $\overline{u}$ be a $(\mathcal{G},\mathcal{B}(E))$-measurable mapping from Ω to E. Then for any x in E, we have

$$g(\omega,x) \geq m(\omega) := g(\omega,\overline{u}(\omega)) - \lambda.$$

Since m is integrable, by replacing g by $g-m$, we can suppose $g(\omega,x) \geq 0$ for all (ω,x) in $\Omega \times E$. According to Theorem 2.1., there is an a.s unique $\mathcal{G}\otimes\mathcal{B}(E)$-*measurable mapping* h from $\Omega \times E$ to $[0,+\infty]$ such that for any $\mathcal{G}$-measurable set A and for any $(\mathcal{G},\mathcal{B}(E))$-measurable mapping u from Ω to E one has

$$\int_A h(\omega,u(\omega))P(d\omega) = \int_A g(\omega,u(\omega))P(d\omega). \tag{2.2.1}$$

Let D be a countable dense subset of E. By (i), (ii) and (2.2.1) there is a set N in $\mathcal{G}$ with $P(N)=0$ such that

$$| h(\omega,x) - h(\omega,y) | \leq \lambda d(x,y)$$

for $(\omega,x,y) \in (\Omega\setminus N)\times D \times D$. For $(\omega,x) \in \Omega \times E$, define

$$k(\omega,x) = \begin{cases} h(\omega,x) & \text{for } (\omega,x) \in (\Omega\setminus N)\times D \\ \lim\limits_{n\to\infty} h(\omega,x_n) & \text{for } \omega \in \Omega\setminus N, \quad x = \lim\limits_{n\to\infty} x_n, (x_n)_{n\geq 1} \subset D \\ 0 & \text{for } (\omega,x) \in N \times E \end{cases}$$

Then for any in $E, k(.,x)$ is $\mathcal{G}$-measurable and

$$| k(\omega,x) - k(\omega,y) | \leq \lambda d(x,y)$$

for $(\omega,x,y) \in \Omega \times E \times E$. To finish the proof, put $E^{\mathcal{G}}g := k$. It is enough to check that for any $\mathcal{G}$-measurable set A and for any $(\mathcal{G},\mathcal{B}(E))$-measurable mapping u from Ω to E, we have

$$\int_A E^{\mathcal{G}}g(\omega,u(\omega))P(d\omega) = \int_A g(\omega,u(\omega))P(d\omega).$$

By Castaing-Valadier ([11], Theorem III-6) there is a sequence of $\mathcal{G}$-measurable mapping $(u_j)_{j\geq 1}$ from Ω to D such that $\lim\limits_{j\to\infty} u_j(\omega) = u(\omega)$ for $\omega \in \Omega$. Then by

(2.2.1) and by definition of k, we have

$$\int_A k(\omega, u_j(\omega))P(d\omega) = \int_A h(\omega, u_j(\omega))P(d\omega)$$
$$= \int_A g(\omega, u_j(\omega))P(d\omega).$$

Note that for $j \geq 1$ and ω in Ω,

$$0 \leq g(\omega, u_j(\omega)) \leq g(\omega, \bar{u}(\omega)) + \lambda d(u_j(\omega), \bar{u}(\omega)) \leq g(\omega, \bar{u}(\omega)) + \lambda.$$

Hence, by the dominated convergence theorem, we get

$$\lim_{j \to \infty} \int_A g(\omega, u_j(\omega))P(d\omega) = \int_A g(\omega, u(\omega))P(d\omega).$$

Let x be a fixed element in D. Then for any $j \geq 1$, any $\omega \notin N$, we have

$$0 \leq k(\omega, u_j(\omega)) \leq k(\omega, x) + \lambda d(x, u_j(\omega))$$
$$\leq h(\omega, x) + \lambda$$

By the dominated convergence theorem the first member of (2.2.2) converges to $\int_A k(\omega, u(\omega))P(d\omega)$ when j goes to ∞. Finally we have

$$\int_A k(\omega, u(\omega))P(d\omega) = \int_A g(\omega, u(\omega))P(d\omega)$$

for any $\mathcal{G}$-measurable set A and any $\mathcal{G}$-measurable mapping u from Ω to E. The uniqueness of $E_g^{\mathcal{G}}$ is evident which completes the proof.

Now we can prove a general existence theorem of conditional expectation for lower semi-continuous random integrands.

THEOREM 2.3. *Let $\mathcal{G}$ be a complete σ-subalgebra of $\mathcal{F}$. Let (E, d) be a Suslin metrizable space. Let $f : \Omega \times E \to]-\infty, +\infty]$ be a $\mathcal{F} \otimes \mathcal{B}(E)$-measurable integrand such that for any $\omega \in \Omega, f(\omega, .)$ is lower semi-continuous on E. Let m be a positive integrable random variable. Assume that there is a $(\mathcal{G}, \mathcal{B}(E))$-measurable mapping $\bar{u}$ from Ω to E such that $f(., \bar{u}(.))$ is integrable and $f(\omega, x)$ $+ m(\omega) \geq 0$ for (ω, x) in $\Omega \times E$.*

Then there is an a.s. unique $\mathcal{G} \otimes \mathcal{B}(E)$-measurable integrand $E^{\mathcal{G}} f$ with following properties:

(1) $E^{\mathcal{G}}f(\omega,.)$ is lower semi-continuous on E for any ω in Ω.

(2) for any $\mathcal{G}$-mesurable mapping u from Ω to E and any $\mathcal{G}$-measurable set A, one has

$$\int_A E^{\mathcal{G}}f(\omega,u(\omega))P(d\omega) = \int_A f(\omega,u(\omega))P(d\omega)$$

PROOF. We can suppose that $d(x,y) \leq 1$ for (x,y) in $E \times E$. For each integer $k \geq 1$ and (ω,x) in $\Omega \times E$, put

$$g_k(\omega,x) = \inf_{y\in E}[kd(x,y) + f(\omega,y) + m(\omega)]$$

Then by the measurable projection theorem, see ([11], Theorem III.23), $g_k(.,x)$ is $\mathcal{F}$-measurable for any x in E and we have

$$| g_k(\omega,x) - g_k(\omega,y) | \leq kd(x,y)$$

for (ω,x,y) in $\Omega\times E\times E$ and for ω in Ω, x in E, $\uparrow \lim_{k\to\infty} g_k(\omega,x) = f(\omega,x)+m(\omega)$. Then we can apply Corollary 2.2 to each g_k since for any $(\mathcal{G},\mathcal{B}(E))$-measurable mapping u from Ω to E, we have

$$\begin{aligned} 0 \leq g_k(\omega,u(\omega)) &\leq g_k(\omega,\bar{u}(\omega)) + kd(u(\omega),\bar{u}(\omega)) \\ &\leq f(\omega,\bar{u}(\omega)) + m(\omega) + k \end{aligned}$$

for any ω in Ω. So $g_k(.,u(.))$ is integrable. By Corollary 2.2, there is an a.s. unique mapping $E^{\mathcal{G}}g_k$ from $\Omega \times E$ to $\mathbb{R}^+$ such that

$$\text{For any } x \text{ in } E, E^{\mathcal{G}}g_k(.,x) \text{ is } \mathcal{G} - \text{ measurable.} \tag{2.3.1}$$

$$| E^{\mathcal{G}}g_k(\omega,x) - E^{\mathcal{G}}g_k(\omega,y) | \leq kd(x,y) \tag{2.3.2}$$

for (ω,x,y) in $\Omega \times E \times E$.

For any $\mathcal{G}$-measurable set A in $\mathcal{G}$ and for any $(\mathcal{G},\mathcal{B}(E))$-measurable mapping u from Ω to E, one has

$$\int_A E^{\mathcal{G}}g_k(\omega,u(\omega))P(d\omega) = \int_A g_k(\omega,u(\omega))P(d\omega). \tag{2.3.3}$$

Since $(g_k)_{k\geq 1}$ is increasing, using again measurable projection theorem ([11], Theorem III-23) and an argument given by Castaing-Valadier ([11], Theorem

VIII 3.6, p.264) we find a set N in $\mathcal{G}$ with $P(N) = 0$ such that

$$E^{\mathcal{G}} g_k(\omega, x) \leq E^{\mathcal{G}} g_{k+1}(\omega, x)$$

for (ω, x, k) in $(\Omega \setminus N) \times E \times \mathbf{N}^*$. Put

$$h(\omega, x) = \begin{cases} \sup\limits_{k \geq 1} E^{\mathcal{G}} g_k(\omega, x) & \text{for } (\omega, x) \in (\Omega \setminus N) \times E, \\ 0 & \text{for } (\omega, x) \in N \times E. \end{cases}$$

Then h is $\mathcal{G} \otimes \mathcal{B}(E)$-measurable and $h(\omega, .)$ is lower semi-continuous on E for any ω in Ω. For any $\mathcal{G}$- measurable set A and for any $(\mathcal{G}, \mathcal{B}(E)$-measurable mapping u from Ω to E, we have, by (2.3.3) and by the monotone convergence theorem,

$$\begin{aligned} \int_A h(\omega, u(\omega)) P(d\omega) &= \uparrow \lim_{k \to \infty} \int_A E^{\mathcal{G}} g_k(\omega, u(\omega)) P(d\omega) \\ &= \uparrow \lim_{k \to \infty} \int_A g_k(\omega, u(\omega)) P(d\omega) \\ &= \int_A [f(\omega, u(\omega)) + m(\omega)] P(d\omega) \\ &= \int_A f(\omega, u(\omega)) P(d\omega) + \int_A m(\omega) P(d\omega) \\ &= \int_A f(\omega, u(\omega)) P(d\omega) + \int_A E^{\mathcal{G}} m(\omega) P(d\omega), \end{aligned}$$

where $E^{\mathcal{G}} m$ is the conditional expectation of the integrable random variable m. Put $E^{\mathcal{G}} f = h - E^{\mathcal{G}} m$. Then $E^{\mathcal{G}} f$ is $\mathcal{G} \otimes \mathcal{B}(E)$-measurable, $E^{\mathcal{G}} f(\omega, .)$ is lower semi-continuous on E for any ω in Ω and

$$\int_A E^{\mathcal{G}} f(\omega, u(\omega))(d\omega) = \int_A f(\omega, u(\omega)) P(d\omega)$$

for any A in $\mathcal{G}$ and any $(\mathcal{G}, \mathcal{B}(E))$-measurable mapping from Ω to E.

References. Castaing ([9]), Castaing-Valadier [11], Castaing-Clauzure ([8]), Dynkin-Estigneev ([16]), Estigneev [17], Thibault [33], [34])), Truffert ([35]), Valadier ([36],[37]), particularly, there is a rich bibliography on the subject in Truffert ([35]).

Theorem 2.3 allows us to introduce the notion of lower semi-continuous (lsc) integrand martigales as follows.

DEFINITION 2.4. Let $(\mathcal{B}_n)_{n\geq 1}$ be a decreasing sequence of complete σ-sub algebras of $\mathcal{F}$ and $\mathcal{B}_\infty = \bigcap_{n=1}^{\infty} \mathcal{B}_n$. A sequence $(f_n)_{n\geq 1}$ of integrands from $\Omega \times E$ to $]-\infty, +\infty]$ is a *lsc reversed supermartingale* (resp. submartingale)(resp. martingale) if $(f_n)_{n\geq 1}$ has the following properties:

(i) For each $n \geq 1, f_n$ is $\mathcal{B}_n \otimes \mathcal{B}(E)$-measurable and $f_n(\omega, .)$ is lower semi-continuous on E for any ω in Ω.

(ii) There is a $(\mathcal{B}_\infty, \mathcal{B}(E))$-measurable mapping $\bar{u}$ from Ω to E such that for all $n \geq 1$, $f_n(., \bar{u}(.))$ is integrable.

(iii) For each $n \geq 1$, there is a positive integrable random variable α_n such that $f_n(\omega, x) + \alpha_n(\omega) \geq 0$ for all (ω, x) in $\Omega \times E$.

(iv) For each $n \geq 1, A$ in $\mathcal{B}_{n+1}$ and each $(\mathcal{B}_{n+1}, \mathcal{B}(E))$-measurable mapping u from Ω to E, one has

$$\int_A f_{n+1}(\omega, u(\omega))P(d\omega) \geq \int_A f_n(\omega, u(\omega))P(d\omega)$$

$$\text{resp } \int_A f_{n+1}(\omega, u(\omega))P(d\omega) \leq \int_A f_n(\omega, u(\omega))P(d\omega)$$

$$\text{resp } f_{n+1}(\omega, u(\omega))P(d\omega) = \int_A f_n(\omega, u(\omega))P(d\omega).$$

Note that by (i), (ii), (iii), for each n in $\mathbf{N}^*$, the conditional expectation $E^{\mathcal{B}_{n+1}} f_n$ and $E^{\mathcal{B}_\infty}$ of f_n with respect to $\mathcal{B}_{n+1}$ and $\mathcal{B}_\infty$, respectively, are ensured by Theorem 2.3. For instance, if $(f_n)_{n\geq 1}$ is a lsc reversed supermartingale, then $E^{\mathcal{B}_{n+1}} f_n \leq f_{n+1}$ a.s., for all $n \geq 1$, that is, there is a negligible set N such that for all (ω, x) in $(\Omega \setminus N) \times E, E^{\mathcal{B}_{n+1}} f_n(\omega, x) \leq f_{n+1}(\omega, x)$. Similarly, it is possible to introduce the notion of *lsc* integrand martingale (submartingale) (supermartingale) with respect to an increasing sequence $(\mathcal{F}_n)_{n\geq 1}$ of σ-subalgebras.

If f_n is positive, $\mathcal{B}_n \otimes \mathcal{B}(E)$-measurable on $\Omega \times E$ for all $n \geq 1$, and if semi continuity condition is not required for the conditional expectation of $E^{\mathcal{B}_{n+1}} f_n$ and $E^{\mathcal{B}_\infty} f_n$, then the integrability condition (ii) and (iii) are superflous by Theorem 2.1. In this case, we say that $(f_n)_{n\geq 1}$ is a positive integrand supermartingale (martingale)(submartingale) if $(f_n)_{n\geq 1}$ satisfies (iv).

3. Lower semi-continuous integrand reversed supermartingales.

The main result in this section relies on the following lemma.

LEMMA 3.1. *Let $(f_n)_{n\geq 1}$ be a positive lsc integrand reversed supermartingale. Let $\lambda > 0$. Let*

$$f_n^\lambda(\omega, x) = \inf_{y\in E}[\lambda d(x,y) + f_n(\omega, y)]$$

for $n \geq 1, (\omega, x)$ in $\Omega \times E$. Then $(f_n^\lambda, \mathcal{B}_n)_{n\geq 1}$ is a positive λ-Lipschitzean reversed supermartingale and there is a negligible set N_λ such that

$$\lim_{n\to\infty} f_n^\lambda(\omega, x) = \lim_{n\to\infty} E^{\mathcal{B}_\infty}[f_n^\lambda(., x)](\omega)$$

for $(\omega, x) \in (\Omega \setminus N_\lambda) \times E$.

PROOF. Without loss of generality we can suppose $d(x,y) \leq 1$ for (x,y) in $E \times E$. By the measurable projection theorem ([11], Theorem III.23), $f_n^\lambda(., x)$ is $\mathcal{B}_n$- measurable for any x in E. Moreover $| f_n^\lambda(\omega, x) - f_n^\lambda(\omega, y) |\leq \lambda d(x,y)$, $\forall(\omega, x, y) \in \Omega \times E \times E$. Now we show that $(f_n^\lambda, \mathcal{B}_n)_{n\geq 1}$ is an integrand reversed supermartingale. For each n, denote by $\mathcal{L}_E(\mathcal{B}_n)$ the set of $(\mathcal{B}_n, \mathcal{B}(E))$-measurable mappings from Ω to E. For A in $\mathcal{B}_{n+1}$ and u in $\mathcal{L}_E(\mathcal{B}_{n+1})$ we have, by a trivial modification of a result due to Hiai-Umegaki ([21], Theorem 2.2)

$$\begin{aligned}\int_A f_{n+1}^\lambda(\omega, u(\omega))P(d\omega) &= \inf_{v\in\mathcal{L}_E(\mathcal{B}_{n+1})} \int_A \lambda d(u(\omega), v(\omega)) \\ +f_{n+1}(\omega, v(\omega))P(d\omega) &\geq \inf_{v\in\mathcal{L}_E(\mathcal{B}_n)} \int_A \lambda d(u(\omega), v(\omega)) + f_n(\omega, v(\omega))P(d\omega) \\ &= \int_A f_n^\lambda(\omega, u(\omega))P(d\omega)\end{aligned}$$

Hence $(f_n^\lambda, \mathcal{B}_n)_{n\geq 1}$ is a positive integrand reversed supermartingale, since by hypothesis, there is $\bar{u}$ in $\mathcal{L}_E(\mathcal{B}_\infty)$ such that $f_n(., \bar{u}(.))$ is integrable, so by the definition of f_n^λ, we get, for any x in E and ω in Ω

$$0 \leq f_n^\lambda(\omega, x) \leq f_n^\lambda(\omega, \bar{u}(\omega)) + \lambda d(x, \bar{u}(\omega)) \leq f_n(\omega, \bar{u}(\omega)) + \lambda d(x, \bar{u}(\omega)).$$

Hence $f_n^\lambda(., u(.))$ is integrable for all u in $\mathcal{L}_E(\mathcal{B}_\infty)$. Now let D be a countable dense subset of E. Let $E^{\mathcal{B}_\infty} f_n^\lambda$ be the conditional expectation of f_n^λ given by

Corollary 2.2 which shows that $E^{\mathcal{B}_\infty} f_n^\lambda$ is λ-lipschitzean, namely

$$| E^{\mathcal{B}_\infty} f_n^\lambda(\omega, x) - E^{\mathcal{B}_\infty} f_n^\lambda(\omega, y) | \leq \lambda d(x, y)$$

for (ω, x, y) in $\Omega \times E \times E$. By Neveu ([30], Proposition V.3.11), for each x in E, there is a negligible N_x^λ such that

$$\lim_{n\to\infty} f_n^\lambda(\omega, x) = \uparrow \lim_{n\to\infty} E^{\mathcal{B}_\infty}[f_n^\lambda(., x)](\omega) \tag{3.1.1}$$

for ω in $\Omega \setminus N_x^\lambda$. Let $N^\lambda = \bigcup_{x\in D} N_x^\lambda$. Then $P(N^\lambda) = 0$ and

$$\lim_{n\to\infty} f_n^\lambda(\omega, x) = \uparrow \lim_{n\to\infty} E^{\mathcal{B}_\infty}[f_n^\lambda(., x)](\omega) \tag{3.1.2}$$

for (ω, x) in $(\Omega \setminus N^\lambda) \times D$. Since by Corollary 2.2, $E^{\mathcal{B}_\infty} f_n^\lambda$ is λ-lipschitzean, by (3.1.2) it follows that

$$\lim_{n\to\infty} f_n^\lambda(\omega, x) = \uparrow \lim_{n\to\infty} E^{\mathcal{B}_\infty}[f_n^\lambda(., x)](\omega) \tag{3.1.3}$$

for (ω, x) in $(\Omega \setminus N^\lambda) \times E$

COROLLARY 3.2. *Let $(f_n)_{n\geq 1}$ be a positive lsc reversed supermartingale. For each integer $k \geq 1$, let*

$$f_n^k(\omega, x) = \inf_{y\in E}[kd(x, y) + f_n(\omega, y)]$$

for $n \geq 1, (\omega, x)$ in $\Omega \times E$. Then for each $k \geq 1, (f_n^k, \mathcal{B}_n)_{n\geq 1}$ is a positive k-lipschitzean integrand reversed supermartingale and there is a negligible set N such that

$$\sup_{k\geq 1} \lim_{n\to\infty} f_n^k(\omega, x) = \lim_{n\to\infty} E^{\mathcal{B}_\infty}[f_n(., x)](\omega)$$

for (ω, x) in $(\Omega \setminus N) \times E$

PROOF. By Lemma 3.1, each $(f_n^k, \mathcal{B}_n)_{n\geq 1}$ is a k-lipschitzean integrand reversed supermartingale and there is a negligible set N^k such that

$$\lim_{n\to\infty} f_n^k(\omega, x) = \uparrow \lim_{n\to\infty} E^{\mathcal{B}_\infty}[f_n^k(., x)](\omega) \tag{3.2.1}$$

for $(\omega, x) \in (\Omega \setminus N^k) \times E$. Let $N_1 = \bigcup_{k=1}^{\infty} N^k$. Then $P(N_1) = 0$. Taking the supremum over k in (3.2.1) we get

$$\begin{aligned}\sup_{k\geq 1} \lim_{n\to\infty} f_n^k(\omega, x) &= \sup_{k\geq 1} \uparrow \lim_{n\to\infty} E^{\mathcal{B}_\infty}[f_n^k(.,x)](\omega) \\ &\lim_{n\to\infty} \sup_{k\geq 1} E^{\mathcal{B}_\infty}[f_n^k(.,x)](\omega) \qquad (3.2.2)\end{aligned}$$

Since $\uparrow f_n^k = f_n$ for each $n \geq 1$, by the monotone convergence theorem for conditional expectation, there is a negligible set N_2^n (see the proof of Theorem 2.3) such that

$$\sup_{k\geq 1} E^{\mathcal{B}_\infty}[f_n^k(.,x)](\omega) = E^{\mathcal{B}_\infty}[f_n(.,x)](\omega) \qquad (3.2.3)$$

for $(\omega, x) \in (\Omega \setminus N_2^n) \times E$. Let $N_2 = \cup_{n=1}^{\infty} N_2^n$ and $N = N_1 \cup N_2$. Then $P(N) = 0$ so (3.2.1), (3.2.2), and (3.2.3) yield

$$\sup_{k\geq 1} \lim_{n\to\infty} f_n^k(\omega, x) = \lim_{n\to\infty} E^{\mathcal{B}_\infty}[f_n^k(.,x)](\omega)$$

for (ω, x) in $(\Omega \setminus N) \times E$.

REMARK. Analogous results for lower semi-continuous integrand martingales and amarts were obtained by Choukairi ([12]) when E is a separable Banach space.

Let us focus our attention to the lower semi-continuous integrand reversed martingales. First we give an easy and useful lemma which is a consequence of Doob's theorem for positive reversed martingales.

LEMMA 3.3. *Let $(f_n)_{n\geq 1}$ be a sequence of positive random variables. Let $(\mathcal{B}_n)_{n\geq 1}$ be a decreasing sequence of σ- subalgebras of $\mathcal{F}$ and $\mathcal{B}_\infty = \cap_{n=1}^{\infty} \mathcal{B}_n$. Then*

$$\liminf_{n\to\infty} E^{\mathcal{B}_n} f \geq E^{\mathcal{B}_\infty}(\liminf_{n\to\infty} f_n) \text{ a.s.}$$

PROOF. For each integer $m \geq 1$, put $g_m = \inf_{n\geq m} f_n$. Then $E^{\mathcal{B}_n} f_n \geq E^{\mathcal{B}_n} g_m$ a.s. for $n \geq m$. So we have

$$\liminf_{n\to\infty} E^{\mathcal{B}_n} f_n \geq \lim_{n\to\infty} E^{\mathcal{B}_n} g_m = E^{\mathcal{B}_\infty} g_m \text{ a.s.}$$

because $(E^{\mathcal{B}_n} g_m)_{n\geq 1}$ is a positive reversed martingale for each $m \geq 1$, which converges a.s. to $E^{\mathcal{B}_\infty} g_m$ by Doob's theorem, see eg. Neveu ([30], Corollaire

V-3-12). Hence by taking the supremum over m in (3.3.1) and by using the monotone convergence theorem for conditional expectation, we get

$$\begin{aligned}\liminf_{n\to\infty} E^{\mathcal{B}_n} f_n &\geq \sup_{m\geq 1} E^{\mathcal{B}_\infty} g_m \\ &= E^{\mathcal{B}_\infty}(\sup_{m\geq 1} g_m) \\ &= E^{\mathcal{B}_\infty}(\liminf_{n\to\infty} f_n) \text{ a.s.}\end{aligned}$$

Before giving the main result of this section, recall the following definition of Mosco-convergence for a sequence $(\varphi_n)_{n\geq 1}$ of lower semi-continuous convex functions on Banach space F. Let φ be a lower semi-continuous function defined on F. $(\varphi_n)_{n\geq 1}$ M_S^W-converges to φ_∞ if a) for any sequence $(x_n)_{n\geq 1}$ in F which converges to x_∞ for $\sigma(F, F')$ topology, we have $\liminf_{n\to\infty} \varphi_n(x_n) \geq \varphi_\infty(x_\infty)$, b) for every y in F, there is a sequence $(y_n)_{n\geq 1}$ in F which converges to y such that $\limsup_{n\to\infty} \varphi_n(y_n) \leq \varphi_\infty(y)$.

We begin with some lemmas.

LEMMA 3.4. *Let $(\mathcal{B}_n)_{n\geq 1}$ be a decreasing sequence of complete σ-subalgebras of $\mathcal{F}$ and $\mathcal{B}_\infty = \cap_{n=1}^\infty \mathcal{B}_n$. Let S be a topological Suslin space and f be a positive $\mathcal{F} \otimes \mathcal{B}(S)$-measurable integrand defined on $\Omega \times S$. Let $\varphi_n = E^{\mathcal{B}_n} f$ for $n \geq 1$. Then $(\varphi_n)_{n\geq 1}$ is an integrand reversed martingale. Moreover, if S is metrizable, f is lsc, $\mathcal{F} \otimes \mathcal{B}(S)$-measurable integrand and there exists a $(\mathcal{B}_\infty, \mathcal{B}(S))$- measurable mapping $\overline{u}$ from Ω to S such that $f(., \overline{u}(.))$ is integrable, then $(\varphi_n)_{n\geq 1}$ is a lsc integrand reversed martingale.*

PROOF. Since $(\mathcal{B}_n)_{n\geq 1}$ is decreasing, by Theorem 2.1, for any $n \geq 1$,any A in $\mathcal{B}_{n+1}$ and any $(\mathcal{B}_{n+1}, \mathcal{B}(S)$-measurable mapping u from Ω to S, we have

$$\int_A E^{\mathcal{B}_{n+1}} f(\omega, u(\omega))P(d\omega) = \int_A f(\omega, u(\omega))P(d\omega) = \int_A E^{\mathcal{B}_n} f(\omega, u(\omega))P(d\omega)$$

This proves that $(E^{\mathcal{B}_n} f, \mathcal{B}_n)_{n\geq 1}$ is an integrand reversed martingale. If S is metrizable, f is a lsc, $\mathcal{F} \otimes \mathcal{B}(S)$-measurable integrand and if there is a $(\mathcal{B}_\infty, \mathcal{B}(S))$-measurable mapping $\overline{u}$ from Ω to S such that $f(., \overline{u}(.))$ is integrable, then $E^{\mathcal{B}_n} f(., \overline{u}(.))$ is $\mathcal{B}_n$-measurable and integrable. Hence by Theorem

2.3, $E^{\mathcal{B}_n} f$ is lower semi-continuous on S for each $n \geq 1$, so that $(E^{\mathcal{B}_n} f, \mathcal{B}_n)_{n \geq 1}$, is a lsc integrand reversed martingale.

By combining Corollary 3.2 and Lemma 3.4, we can prove the main theorem in this section.

THEOREM 3.5. *Let f be a positive lsc, $\mathcal{F} \otimes \mathcal{B}(E)$-measurable integrand. Assume that there is a $(\mathcal{B}_\infty, \mathcal{B}(E))$-measurable mapping $\overline{u}$ such that $f(., \overline{u}(.))$ is integrable. Then there is a negligible set N such that*

$$\sup_{k \in \mathbb{N}^*} \lim_{n \to \infty} \inf_{y \in E} [kd(x, y) + E^{\mathcal{B}_n} f(\omega, y)] = E^{\mathcal{B}_\infty}[f(., x)](\omega) \tag{3.5.1}$$

for all (ω, x) in $(\Omega \setminus N) \times E$.

If E is a separable Banach space and if $f(\omega, .)$ is convex on E for any ω in Ω, then $(E_n^{\mathcal{B}} f)_{n \geq 1}$ satisfies the following variational inequality. For any sequence $(x_n)_{n \geq 1}$ which converges weakly to x_∞, one has

$$\liminf_{n \to \infty} E^{\mathcal{B}_n}[f(., x_n)](\omega) \geq E^{\mathcal{B}_\infty}[f(., x_\infty)] \text{ a.s.} \tag{3.5.2}$$

PROOF. By Lemma 3.4, $(E^{\mathcal{B}_n} f)_{n \geq 1}$ is a lsc integrand reserved martingale. Hence we can apply Corollary 3.2 which shows that

$$\sup_{k \in \mathbb{N}^*} \lim_{n \to \infty} \inf_{y \in E} [kd(x, y)] + E^{\mathcal{B}_n} f(\omega, y)] =$$

$$\lim_{n \to \infty} E^{\mathcal{B}_\infty}[f(., x)](\omega) = E^{\mathcal{B}_\infty}[f(., x)](\omega) \text{ a. s.}$$

since $\mathcal{B}_\infty = \cap_{n=1}^\infty \mathcal{B}_n$.

Since $f(\omega, .)$ is convex on E for any ω in Ω, $E^{\mathcal{B}_n} f(\omega, .)$ is convex too. See also Castaing-Valadier ([11], Theorem VIII-36) for the convexity of $E^{\mathcal{B}_\infty} f(., x)$. Hence $f(\omega, .)$ and $E^{\mathcal{B}_\infty} f(\omega, .)$ are weakly lower semi-continuous on E. If $(x_n)_{n \geq 1}$ is a sequence in E which converges to x_∞ for $\sigma(E, E')$ topology we have

$$\liminf_{n \to \infty} f(\omega, x_n) \geq f(\omega, x_\infty)$$

By property of conditional expectation of lsc random integrand and Lemma 3.4, we get

$$\liminf_{n\to\infty} E^{\mathcal{B}_n}[f(.,x_n)](\omega) \geq E^{\mathcal{B}_\infty}[\liminf_{n\to\infty} f(.,x_n)](\omega)$$
$$\geq E^{\mathcal{B}_\infty}[f(.,x_\infty)](\omega) a.s.$$

REMARK. It is known that 3.5.1 implies that $(E^{\mathcal{B}_n} f)_{n\geq 1}$ epi-converges a.s. to $E^{\mathcal{B}_\infty} f$. See Hess ([20], Proposition 2.5). We refer to Attouch ([1]) for variational convergence of integrand. In particular, note that (3.5.1) and (3.5.2) imply that $(E^{\mathcal{B}_n} f)_{n\geq 1}$ Mosco-converges a.s. to $E^{\mathcal{B}_\infty} f$. Moreover if E is reflexive, $(E^{\mathcal{B}_n} f)^*_{n\geq 1}$ Mosco-converges a.s. to $(E^{\mathcal{B}_\infty} f)^*$ where φ^* denotes the dual of a lsc convex function φ. For the decription of $(E^{\mathcal{B}_\infty} f)^*$, see Castaing-Valadier ([11], Theorem VII-40).

APPLICATIONS. LAW OF LARGE NUMBERS.

Let F be a Banach space with separable dual. Let $(X_n)_{n\geq 0}$ be a sequence of integrably bounded multifunction from Ω into the set wck(F) of nonempty convex weakly compact subsets of F. Suppose that $(X_n)_{n\geq 0}$ is independent and identically distributed. See Hess ([19]) for details concerning iid random sets. Define

$$S_k(\omega) = \sum_{i=0}^{k-1} X_i(\omega)$$

for $k \geq 1$ and ω in Ω. It will be proved in a forthcoming paper that there is a decreasing sequence $(\mathcal{B}_n)_{n\geq 0}$ of σ-subalgebras of $\mathcal{F}$ such that for each n, $\frac{1}{n} S_n = E^{\mathcal{B}_n} X_0$.

By Valadier ([37]) the conditional expectation $E^{\mathcal{B}_n} X_0$ of X_0 with respect to $\mathcal{B}_n$ is an integrably bounded multifunction from Ω to wck(E) such that $\int_A E^{\mathcal{B}_n} X_0 dP = \int_A X_0 dP$ for A in $\mathcal{B}_n$. Moreover we have

$$S^1_{E^{\mathcal{B}_n} X_0} = \{E^{\mathcal{B}_n} f : f \in S^1_{X_0}\}$$

where $S^1_{X_0}$ (resp. $S^1_{E^{\mathcal{B}_n} X_0}$) is the set of $\mathcal{F}$-measurable (resp. $\mathcal{B}_n$)-measurable selection of X_0 (resp. $E^{\mathcal{B}_n} X_0$).

Now put $f_n(\omega, x') = E^{\mathcal{B}_n}[\delta^*(x', X_0(.)](\omega)$ for (ω, x') in $\Omega \times B'$ and for $n \geq 0$, where $\delta^*(., X_0(\omega))$ is the support function of $X_0(\omega)$. Then by properties of conditional expectation for convex weakly compact random sets, ([37]), one checks easily that $(f_n)_{n\geq 1}$ is a lsc integrand reversed martingale.

By the previous arguments, it is clear that the Mosco convergence of the law of large numbers for convex weakly compact iid random sets is equivalent to the Mosco convergence of the reversed multivalued martingale $E^{\mathcal{B}_n} X_0$ in the case where E' is strongly separable, and the Mosco convergence of the sequence of lsc random integrands $(h_n)_{n\geq 1}$ defined on $\Omega \times E'$ by

$$h_n(\omega, x') = \frac{1}{n} \sum_{i=0}^{n-1} \delta^*(x', X_i(\omega))$$

for (ω, x') in $\Omega \times E'$, is equivalent to the Mosco convergence of the lsc integrand reversed martingale (f_n).

We refer to Hess ([19]) and Hiai ([22]) and the references therein for the Mosco convergence of the law of large numbers for closed convex iid random sets .

We refer to Attouch-Wets ([2]) and Hess ([20]) for recent results of epiconvergence for lsc pairwise independent and identically distributed integrands via the Etemadi's strong law of large numbers.

It is known that the classical theorem of the strong law of large numbers for iid random variables is a consequence of three classical theorems, namely

- Doob's theorem for reversed martingales
- Birkhoffs ergodic theorem
- Komlos' theorem.

See Valadier ([39]) for the proofs and the references concerning the previous implications.

The reduction formula $\frac{1}{n} S_n = E^{\mathcal{B}_n} X_0$, $n \geq 1$, offers many nice applications for the variational convergence problems related to the law of large numbers mentioned above. It is worthwhile to note that Theorem 3.5 allows to obtain the Mosco convergence of integral functionals $(I_{\varphi_n})_{n\geq 1}$ and $(I_{\varphi_n})^*_{n\geq 1}$ with $\varphi_n = E^{\mathcal{B}_n} f$ and $\varphi_n^* = (E^{\mathcal{B}_n} f)^*$ for $n \in \mathbb{N}^* \cup \{\infty\}$ to I_{φ_∞} and $I_{\varphi_\infty^*}$ respectively.

See Salvadori ([31], Theorem 3.1) and ([32], Theorem 1) for details. So to illustrate the possibilities of applications of Theorem 3.5, we give without proof the following variational result.

PROPOSITION 3.6. *Let E be a reflexive separable Banach space. Let $(\mathcal{B}_n)_{n\geq 1}$ be a decreasing sequence of complete σ-subalgebras of $\mathcal{F}$ and $\mathcal{B}_\infty = \bigcap_{n=1}^{\infty} \mathcal{B}_n$. Let $f : \Omega \times E \to [0,+\infty]$ be a random lsc convex integrand such that $f(\omega, 0) = 0$ for all ω in Ω. Put, for each n in $\mathbb{N}^* \cup \{\infty\}$,*

$$\varphi_n = E^{\mathcal{B}_n} f$$
$$I_{\varphi_n}(u) = \int_\Omega \varphi_n(\omega, u(\omega))P(d\omega), \forall\, u \in L^1_E(\mathcal{B}_\infty)$$
$$I_{\varphi_n^*}(u) = \int_\Omega \varphi_n^*(\omega, u(\omega))P(d\omega), \forall\, u \in L^\infty_{E'}(\mathcal{B}_\infty)$$

Then I_{φ_n} M_S^W-converges to I_{φ_∞} in $L^1_E(\mathcal{B}_\infty)$ and $I_{\varphi_n^*} M_\tau^{w^*}$- converges to $I_{\varphi_\infty^*}$ in $L^\infty_{E'}(\mathcal{B}_\infty)$. In particular, for any sequence $(u_n)_{n\geq 1}$ which converges to u_∞ for $\sigma(L^1_E(\mathcal{B}_\infty), L^\infty_{E'}(\mathcal{B}_\infty))$ topology, we have

$$\liminf_{n\to\infty} \int_\Omega E^{\mathcal{B}_n} f(\omega, u_n(\omega))P(d\omega) \geq \int_\Omega E^{\mathcal{B}_\infty} f(\omega, u_\infty(\omega))P(d\omega)$$

4 - Upper semi-continuous integrand reversed submartingales.

Let $(\mathcal{B}_n)_{n\geq 1}$ be a decreasing sequence of complete σ-subalgebras of $\mathcal{F}$ and $\mathcal{B}_\infty = \bigcap_{n=1}^{\infty} \mathcal{B}_n$.

DEFINITION 4.1. A sequence $(f_n)_{n\geq 1}$ of integrands defined on $\Omega\times E$ with values in $\overline{\mathbb{R}}$ is a *usc integrand reversed submartigale* if $(f_n)_{n\geq 1}$ satisfies the following conditions.

(i) For each $n \geq 1, f_n$ *is* $\mathcal{B}_n \otimes \mathcal{B}(E)$-measurable and for any ω in $\Omega, f_n(\omega, .)$ is upper semi-continuous on E.

(ii) For each $n \geq 1$ and for any $(\mathcal{B}_\infty, \mathcal{B}(E))$-measurable mapping u from Ω to E, $f_n(., u(.))$ is integrable.

(iii) For each $n \geq 1$, there is a positive integrable random variable α_n such that

$$f_n(\omega, x) \leq \alpha_n(\omega)$$

for all $(\omega, x) \in \Omega \times E$.

(iv) For each $n \geq 1$, for any A in $\mathcal{B}_{n+1}$ and any $(\mathcal{B}_{n+1}, \mathcal{B}(E))$-measurable mapping u from Ω to E, one has

$$\int_A f_{n+1}(\omega, u(\omega))P(d\omega) \leq \int_A f_n(\omega, u(\omega))P(d\omega)$$

The previous definition shows that the sequence $(-f_n, \mathcal{B}_n)_{n\geq 1}$ is a lower semi-continuous integrand reversed supermartingale given in 2.4. By (i), (ii), (iii) and Theorem 2.3, for each $n \geq 1$, the conditional expectation $E^{\mathcal{B}_{n+1}} f_n$ is a $\mathcal{B}_{n+1} \otimes \mathcal{B}(E)$-measurable integrand such that $E^{\mathcal{B}_{n+1}} f(\omega, .)$ is upper semi-continuous on E for any ω in Ω. Moreover (iv) means that $f_{n+1} \leq E^{\mathcal{B}_{n+1}} f_n$ a.s. In view of a convergence result given below, we need integrability condition (ii) as in Doob's reversed integrable submartingale convergence theorem. See Dubley ([15]), Theorem 10.6.4) and Neveu ([30], Corollaire V-3-13).

Here is the main result in this section which is a variational convergence version for integrand reversed submartingales.

THEOREM 4.2. *Let $(f_n)_{n\geq 1}$ be a usc integrand reversed submartingale. Then there is a negligible set N such that*

$$\inf_{k\in \mathbb{N}^*} \lim_{n\to\infty} \sup_{y\in E} [f_n(\omega, y) - kd(x, y)] \leq E^{\mathcal{B}_\infty}[f_n(., x)](\omega)$$

for all $(\omega, x, n) \in (\Omega \backslash N) \times E \times \mathbb{N}^*$.

PROOF. We can suppose that $d(x, y) \leq 1$ for (x, y) in $E \times E$. For (k, n) in $\mathbb{N}^* \times \mathbb{N}^*$, put

$$f_n^k(\omega, x) = \sup_{y\in E}[f_n(\omega, y) - kd(x, y)]$$

for (ω, x) in $\Omega \times E$. Then

$$|f_n^k(\omega, x) - f_n^k(\omega, y)| \leq kd(x, y), \forall(\omega, x, y) \in \Omega \times E \times E$$

$$\downarrow \lim_{k\to\infty} f_n^k(\omega, x) = f_n(\omega, x), \forall(\omega, x) \in \Omega \times E$$

$$\alpha_n(\omega) \geq f_n^k(\omega, x) \geq f_n(\omega, x), \forall(\omega, x), \in \Omega \times E$$

Moreover $f_n^k(.,x)$ is $\mathcal{B}_n$-measurable. For any $(\mathcal{B}_\infty, \mathcal{B}(E))$-measurable mapping u from Ω to E, we have

$$\alpha_n(\omega) \geq f_n^k(\omega, u(\omega)) \geq f_n(\omega, u(\omega)) \tag{4.2.1}$$

for ω in Ω. By condition (ii) and (iii) of Definition 4.1. and by (4.2.1), $f_n^k(., u(.))$ is integrable. According to Corollary 2.2, $E^{\mathcal{B}_\infty} f_n^k$ is k-lipschitzean. By uniqueness of conditional expectation and by the dominated convergence theorem, we have a.s.

$$\downarrow \lim_{k\to\infty} E^{\mathcal{B}_\infty} f_n^k = E^{\mathcal{B}_\infty} f_n \tag{4.2.2}$$

since f_n^k satisfies (4.2.1), (see the proof of Theorem 2.3)

First we prove that $(f_n^k, \mathcal{B}_n)_{n\geq 1}$ is a k-lipschitzean integrand reversed submartingale. Let A be in $\mathcal{B}_{n+1}$ and let u be a $(\mathcal{B}_{n+1}, \mathcal{B}(E))$-measurable mapping from Ω to E. By a result due to Hiai-Umegaki ([21], Theorem 2.2)

$$\int_A f_{n+1}^k(\omega, u(\omega))P(d\omega)$$

$$\begin{aligned}
&= \sup_{v\in\mathcal{L}_E(\mathcal{B}_{n+1})} \int_A f_{n+1}(\omega, v(\omega)) - kd(u(\omega), v(\omega))P(d\omega) \\
&\leq \sup_{v\in\mathcal{L}_E(\mathcal{B}_n)} \int_A f_n(\omega, v(\omega)) - kd(u(\omega), v(\omega))P(d\omega) \\
&= \int_A f_n^k(\omega, u(\omega))P(d\omega)
\end{aligned}$$

In particular, for each x in E, $(f_n^k(.,x))_{n-1}$ is an integrable reversed submartingale. By Doob's theorem, see Dudley ([15], Theorem 10.6.4) and Neveu ([30], Corollaire V-3-13, $f_n^k(.,x)$ converges a.s. to a $\mathcal{B}_\infty$-random variable $g_\infty^k(.,x)$ such that $g_\infty^k(.,x) \leq E^{\mathcal{B}_\infty}[f_n^k(.,x)]$ a.s. for all $n \geq 1$. Let D be countable dense subset of E. Then for each x in D, there is a negligible set N_x^k such that

$$\lim_{n\to\infty} f_n^k(\omega, x) \leq E^{\mathcal{B}_\infty}[f_n^k(.,x)](\omega)$$

for all (ω, n) in $(\Omega\backslash N_x^k) \times \mathbf{N}^*$. Let $N = \bigcup_{\substack{k\geq 1\\ x\in D}} N_x^k$

Then P(N) = 0 and

$$\lim_{n\to\infty} f_n^k(\omega,x) \le E^{\mathcal{B}_\infty}[f_n^k(.,x)](\omega) \tag{4.2.3}$$

for all (ω, x, n, k) in $(\Omega\backslash N) \times D \times \mathbb{N}^* \times \mathbb{N}^*$. Since both sides of (4.2.3) are k-lipschitzean, we have

$$\lim_{n\to\infty} f_n^k(\omega,x) \le E^{\mathcal{B}_\infty}[f_n^k(.,x)](\omega)$$

for all (ω, x, n, k) in $(\Omega\backslash N) \times E \times \mathbb{N}^* \times \mathbb{N}^*$.
Taking the infimum over k, we get

$$\inf_{k\in\mathbb{N}^*} \lim_{n\to\infty} f_n^k(\omega,x) \le E^{\mathcal{B}_\infty}[f_n(.,x)](\omega)$$

since $\downarrow \lim_{k\to\infty} E^{\mathcal{B}_\infty} f_n^k = E^{\mathcal{B}_\infty} f_n$ a.s. by (4.2.2).

5. Lower semi-continuous additive random sequences

We introduce the notion of lower semi-continuous integrand superadditive random sequences.

DEFINITION. Let T be a measure preserving transformation of Ω into itself. A sequence $(f_m)_{m\ge1}$ of $\mathcal{F}\otimes\mathcal{B}(E)$-measurable integrands from $\Omega\times E$ to $]-\infty,+\infty]$ is *a lower semi-continuous superadditive random sequence* if $(f_m)_{m\ge1}$ has the following properties.

(i) For each $m \ge 1, f_m(\omega, .)$ is lower semi-continuous on E for any ω in Ω.

(ii) For all positive integers m and n, and for all (ω, x) in $\Omega \times E$

$$f_{m+n}(\omega,x) \ge f_m(\omega,x) + f_n(T^m(\omega,x))$$

where $T^j = ToTo\dots\ T$ to j terms with $T^0 = \mathit{Identity\ on}\ \Omega$.

(iii) For each $m \ge 1$, there is a $(\mathcal{J}, \mathcal{B}(E))$-measurable mapping u_m from Ω to E such that $f_m(., u_m(.))$ is integrable where $\mathcal{J}$ is the σ-algebra of invariant sets A in $\mathcal{F}$, that is $A = T^{-1}(A)$.

If in (i), $f_m(\omega, .)$ is continuous on E for any ω in Ω, we say that $(f_m)_{m\ge1}$ is a Caratheodory superadditive random sequence.

In the rest of this section, (E,d) is a compact metric space, $(\Omega, \mathcal{F}, P)$ is $([0,1], \mathcal{T}_\lambda([0,1], \lambda)$ where λ is the Lebesgue measure on [0,1] and $\mathcal{T}_\lambda([0,1])$ is the σ-algebra of Lebesgue sets in [0,1].

We begin with a useful lemma.

LEMMA 5.1. *If f is a real continuous function defined on $\Omega \times E$ and u is a $(\mathcal{J}, \mathcal{B}(E))$-measurable mapping from Ω to E, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), u(\omega)) = E^{\mathcal{J}}[f(.,u(.))](\omega) \quad a.s. \tag{5.1.1}$$

PROOF. First observe that if f is a linear combination of elements in $\mathcal{C}(\Omega) \otimes \mathcal{C}(E)$, then (5.1.1) is true. Indeed it is enough to check (5.1.1) when f is $g \times h$ with g in $\mathcal{C}(\Omega)$ and h in $\mathcal{C}(E)$.

By Birkhoff-Kingman's ergodic theorem, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g(T^j(\omega)) = E^{\mathcal{J}} g(\omega) \quad a.s.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), u(\omega)) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g(T^j(\omega))h(u(\omega))$$
$$= E^{\mathcal{J}}[g(.)h(u(.))](\omega) \tag{5.1.2}$$

because $h \circ u$ is $\mathcal{J}$-measurable.

It is well known that there is a sequence $(f_p)_{p\geq 1}$ in $\mathcal{C}(\Omega \times E)$ such that each f_p is linear combination of elements in $\mathcal{C}(\Omega) \otimes \mathcal{C}(E)$ which converges uniformly to f. It is obvious that for ω in Ω and for positive integers n and p,

$$\left|\frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), u(\omega)) - \frac{1}{n} \sum_{j=0}^{n-1} f_p(T^j(\omega), u(\omega))\right|$$

$$\leq \frac{1}{n} \sum_{j=0}^{n-1} \left|f(T^j(\omega), u(\omega)) - f_p(T^j(\omega), u(\omega))\right|$$
$$\leq \frac{1}{n} \sum_{j=0}^{n-1} ||f - f_p|| = ||f -$$

where $||\varphi|| = \sup_{(\omega,x)\in\Omega\times E} |\varphi(\omega,x)|$ for φ in $\mathcal{C}(\Omega\times E)$. So we have

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega),u(\omega)) \leq \frac{1}{n}\sum_{j=0}^{n-1} f_p(T^j(\omega),u(\omega)) + ||f-f_p||$$

It follows that, for any fixed $p \geq 1$,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega),u(\omega)) \leq E^{\mathcal{J}}[f_p(.,u(.))](\omega) + ||f-f_p|| \qquad (5.1.3)$$

a.s. by applying (5.1.2) to each f_p. Since $(f_p)_{p\geq 1}$ converges uniformly to f when p goes to $\infty, E^{\mathcal{J}} f_p(.,u(.))$ converges a.s. to $E^{\mathcal{J}} f(.,u(.))$ by the dominated convergence theorem for conditional expectation. Therefore (5.1.3) yields

$$\limsup_{n\to\infty}\sum_{j=0}^{n-1} f(T^j(\omega),u(\omega)) \leq E^{\mathcal{J}}[f(.,u(.))](\omega) \quad a.s. \qquad (5.1.4)$$

Analogously we have

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega),u(\omega)) \geq E^{\mathcal{J}}[f(.,u(.))](\omega) \quad a.s. \qquad (5.1.5)$$

Hence by (5.1.4) and (5.1.5) we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega),u(\omega)) = E^{\mathcal{J}}[f(.,u(.))](\omega)$$

a.s. as stated.

By using Lemma 5.1, we obtain the following result.

PROPOSITION 5.2. *If f is a Carathéodory integrand defined on $\Omega\times E$ and u is a $(\mathcal{J},\mathcal{B}(E))$-measurable mapping from Ω to E such that $f(.,u(.))$ is integrable, then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega),u(\omega)) = E^{\mathcal{J}}[f(.,u(.))](\omega) \quad a.s. \qquad (5.2.1)$$

PROOF. Since f is a Caratheodory integrand, by Scorza-Dragoni's theorem, see ([5], Theorem 3.1) and ([6]), for every $\epsilon > 0$, there is a compact set K_ϵ in Ω with $P(\Omega\backslash K_\epsilon) \leq \epsilon$ and $f_{|K_\epsilon\times E}$ is continuous, so that Proposition 5.2. is an easy consequence of Lemma 5.1.

The main result of this section is the following theorem.

THEOREM 5.3. *Let* $f : \Omega \times E \to [0, +\infty]$ *be a* $\mathcal{J} \otimes \mathcal{B}(E)$*-mesurable integrand such that for any* ω *in* Ω, $f(\omega, .)$ *is lower semi-continuous on* E. *Assume that there is a* $(\mathcal{F}, \mathcal{B}(E))$*-mesurable mapping* $\overline{u}$ *from* Ω *to* E *such that* $f(., \overline{u}(.))$ *is integrable. Let* u *be a* $(\mathcal{J}, \mathcal{B}(E))$*-mesurable mapping from* Ω *to* E. *Then there is a negligible* N *such that*

$$\sup_{k \in \mathbb{N}^*} \liminf_{n \to \infty} \inf_{y \in E} \left[\frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), y) + kd(u(\omega), y)\right] \geq E^{\mathcal{J}}[f(., u(.))](\omega) \tag{5.3.1}$$

for all $\omega \notin N$.

PROOF. For (k, n) in $\mathbb{N}^* \times \mathbb{N}^*$ and (ω, x) in $\Omega \times E$, put

$$f_n(\omega, x) = \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), x)$$

$$f_n^k(\omega, x) = \inf_{y \in E} [f_n(\omega, y) + kd(x, y)]$$

$$f^k(\omega, x) = \inf_{y \in E} [f(\omega, y) + kd(x, y)]$$

Then we have

$$f_n^k(\omega, x) \geq \frac{1}{n} \sum_{j=0}^{n-1} f^k(T^j(\omega), x) \tag{5.3.2}$$

for all (ω, x) in $\Omega \times E$. Without loss of generality we can suppose that $d \leq 1$. Since we have

$$0 \leq f^k(\omega, x) \leq f^k(\omega, \overline{u}(\omega)) + kd(x, \overline{u}(\omega)) \leq f(\omega, \overline{u}(\omega)) + k$$

for all (ω, x) in $\Omega \times E$, $f^k(., u(.))$ is integrable. Since each integrand f^k is a Caratheodory mapping, we can apply Proposition 5.2 to each f^k. Hence there is a negligible set N_k such that

$$\lim_{k \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f^k(T^j(\omega), u(\omega)) = E^{\mathcal{J}}[f^k(., u(.))](\omega$$

for $\omega \notin N_k$. By combining (5.3.2) and (5.3.3) we get

$$\liminf_{n\to\infty} \inf_{y\in E} \Big[\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega), y) + kd(u(\omega), y)\Big] \geq E^{\mathcal{J}}[f^k(.,u(.))](\omega) \tag{5.3.4}$$

for $\omega \notin N$. Let $N = \bigcup_{k\geq 1}^{\infty} N_k$. Then N is negligible and (5.3.4) is valid for $\omega \notin N$. Since $f^k(.,u(.)) \uparrow f(.u(.))$, then by (5.3.4) and by the monotone convergence theorem for conditional expectation, we obtain

$$\sup_{k\in\mathbf{N}^*} \liminf_{n\to\infty} \inf_{y\in E} \Big[\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega), y) + kd(u(\omega), y)\Big] \geq E^{\mathcal{J}}[f(.,u(.)](\omega)$$

for $\omega \notin N$.

The following lemma gives a half epi-convergence result for additive random sequence.

LEMMA 5.4. *Let $f : \Omega \times E \to \mathbf{R}$ be a Carathéodory integrand such that the function $\omega \to \sup_{x\in E} |f(\omega, x)|$ is integrable. Then there is a negligible set N such that*

$$\sup_{k\in\mathbf{N}^*} \limsup_{n\to\infty} \inf_{y\in E} \Big[\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega), y) + kd(x, y)\Big] \leq E^{\mathcal{J}}[f(.,x)](\omega) \tag{5.4.1}$$

for $(\omega, x) \in (\Omega\backslash N) \times E$.

PROOF. By our assumption, we can identify f with an element of $L^1_{C(E)}([0,1],\lambda)$ so that the conditional expectation $\varphi := E^{\mathcal{J}} f$ of f belongs to $L^1_{C(E)}(\mathcal{J})$. For any integer $k \geq 1$, put

$$\varphi^k(\omega, x) = \inf_{y\in E} [\varphi(\omega, y) + kd(x, y)], \quad (\omega, x) \in \Omega \times E.$$

Then $\varphi^k(., x)$ is $\mathcal{J}$-measurable for any x in E.

Let D be a countable dense subset of E. Then for $k \geq 1$, x in D, there is a $\mathcal{J}$-measurable mapping $v_{k,x}$ from Ω to E such that

$$kd(x, v_{k,x}(\omega)) + \varphi(\omega, v_{k,x}(\omega)) = \varphi^k(\omega, x)$$

for ω in Ω by an easy application of measurable selection theorem. See eg. Castaing-Valadier ([11], Theorem III-9). Then by Proposition 5.2, there is a negligible set $N_{k,x}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), v_{k,x}(\omega)) = E^{\mathcal{J}}[f(., v_{k,x}(.)](\omega)$$

for ω in $\Omega\backslash N_{k,x}$. Put $N = \bigcup_{k\geq 1} \bigcup_{x\in D} N_{k,x}$.

Then $P(N) = 0$ and for x in D, k in $\mathbb{N}^*$, ω in $\Omega\backslash N$, we have

$$\begin{aligned} &\limsup_{n\to\infty} \inf_{y\in E} \left[\frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), y) + kd(x, y)\right] \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), v_{k,x}(\omega)) + kd(x, v_{k,x}(\omega)) \\ &= kd(x, v_{k,x}(\omega)) + E^{\mathcal{J}}[f(., v_{k,x}(.))](\omega) = \varphi^k(\omega, x) \end{aligned}$$

Hence we get

$$\limsup_{n\to\infty} \inf_{y\in E} \left[\frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), y) + kd(x, y)\right] \leq \varphi^k(\omega, x) \tag{5.4.2}$$

for (ω, x) in $\Omega\backslash N \times D$. Since both sides in (5.4.2) is k-lipschitzean, (5.4.2) is valid for (ω, x) in $\Omega\backslash N \times E$ and $k \geq 1$. By taking the supremum over k in (5.4.2), we obtain

$$\begin{aligned} \sup_{k\in\mathbb{N}^*} \limsup_{n\to\infty} \inf_{y\in E} &\left[\frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), y) + kd(x, y)\right] \\ &\leq E^{\mathcal{J}}[f(., x)](\omega) \end{aligned} \tag{5.4.1}$$

for (ω, x) in $(\Omega\backslash N) \times E$, since $\sup_{k\in\mathbb{N}^*} \varphi^k = E^{\mathcal{J}} f$.

The following proposition is concerned with the product of a superaddit

PROPOSITION 5.5. *Let f be a positive continuous function defined on $\Omega \times E$. Let $(g_n)_{n\geq 1}$ be a positive superadditive random sequence. Assume that for ω in $\Omega, \sup\limits_{n\geq 1}\frac{1}{n}E^{\mathcal{J}}g_n(\omega) < \infty$. Then there is a negligible set N such that*

$$\sup_{k\in\mathbb{N}^*} \limsup_{n\to\infty} \inf_{y\in E} \left[(\frac{1}{n}g_n(\omega) \times \frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega), y)) + kd(x,y)\right]$$

$$\leq \sup_{n\geq 1} \frac{1}{n} E^{\mathcal{J}} g_n(\omega) \times E^{\mathcal{J}}[f(.,x)](\omega).$$

for all (ω, x) in $(\Omega\backslash N) \times E$.

PROOF. Let $\varphi := \left(\sup\limits_{n\geq 1} E^{\mathcal{J}}\ g_n\right) \times E^{\mathcal{J}} f$. For any integer $k \geq 1$, put

$$\varphi^k(\omega, x) = \inf_{k\in E} [\varphi(\omega, y) + kd(x,y)],\ (\omega, x) \in \Omega \times E.$$

Then $\sup\limits_{k\geq 1} \varphi^k = \varphi$. Let D be a countable dense subset of E. Let $k \geq 1$ and x in D. By the proof of Lemma 5.4, there is a $(\mathcal{J}, \mathcal{B}(E)))$-measurable mapping $v_{k,x}$ such that

$$\varphi^k(\omega, x) + kd(x, v_{k,x}(\omega) + \varphi(\omega, v_{k,x}(\omega))$$

for all ω in Ω. By equality (5.1.1) of Lemma 5.1, there is a negligible set $N_{k,x}$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega), v_{k,x}(\omega)) = E^{\mathcal{J}} f(., v_{k,x}(.))(\omega)$$

for all ω in $\Omega\backslash N_{k,x}$, and by Kingman's theorem. there is a negligible set N_1 such that

$$\lim_{n\to\infty} \frac{1}{n}\ g_n(\omega) = \sup_{n\geq 1} \frac{1}{n}E^{\mathcal{J}} g_n(\omega)$$

for all ω in $\Omega\backslash N_1$. Let $N = \bigcup\limits_{\substack{k\geq 1\\ x\in D}} N_{k,x} \cup N_1$.

Then P(N) = 0. Note that for any ω in $\Omega\backslash N_1$

$$\limsup_{n\to\infty}\Big(\frac{1}{n}g_n(\omega)\times\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega),v_{k,x}(\omega))\Big)$$

$$=\lim_{n\to\infty}\frac{1}{n}g_n(\omega)\times\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{1-n} f(T^j(\omega),v_{k,x}(\omega))$$

since $\lim_{n\to\infty}\frac{1}{n}g_n(\omega)=\sup_{n\geq 1}\frac{1}{n}E^{\mathcal{J}}g_n(\omega)<\infty$ and

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega),v_{k,x}(\omega))\leq m:=\max_{(\omega,x)\in\Omega\times E} f(\omega,x)<\infty$$

Hence

$$\limsup_{n\to\infty}\Big(\frac{1}{n}g_n(\omega)\times\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega),v_{k,x}(\omega))\Big)$$
$$=\Big(\sup_{n\geq 1}\frac{1}{n}E^{\mathcal{J}}g_n(\omega)\Big)\times E^{\mathcal{J}}f(.;v_{k,x}(.))(\omega)$$

for all ω in $\Omega\backslash N$. Consequently, it follows that for x in $D, k\geq 1,\omega$ in $\Omega\backslash N$,

$$\limsup_{n\to\infty}\inf_{y\in E}\Big[(\frac{1}{n}g_n(\omega)\times\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega),y))+kd(x,y)\Big]$$
$$\leq\limsup_{n\to\infty}\Big[kd(x,v_{k,x}(\omega)+(\frac{1}{n}g_n(\omega)\times\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega),v_{k,x}(\omega)))\Big]$$
$$\leq kd(x,v_{k,x}(\omega))+\big(\sup_{n\geq 1}\frac{1}{n}E^{\mathcal{J}}g_n(\omega)\big)\times E^{\mathcal{J}}f(.,v_{k,x}(.))(\omega)$$
$$=\varphi^k(\omega,x).$$

Finally we have

$$\limsup_{n\to\infty}\inf_{y\in E}\Big[(\frac{1}{n}g_n(\omega)\times\frac{1}{n}\sum_{j=0}^{n-1} f(T^j(\omega$$

for all (ω, x) in $(\Omega\backslash N) \times E$, because both sides of the preceding inequality are k-lipschizean. By taking the supremum over k in this inequality, we get

$$\sup_{k\geq 1} \limsup_{n\to\infty} \inf_{y\in E} \Big[\Big(\frac{1}{n}g_n(\omega) \times \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), y)\Big) + kd(x,y)\Big]$$
$$\leq \varphi(\omega, x) := \big(\sup_{n\geq 1} \frac{1}{n} E^{\mathcal{J}} g_n(\omega)\big) \times E^{\mathcal{J}}[f(., x)](\omega)$$

for all (ω, x) in $(\Omega\backslash N) \times E$ since $\sup_{k\geq 1} \varphi^k = \varphi$

REMARK. The crucial fact in the proof of Lemma 5.1 is the equality

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), u(\omega)) = E^{\mathcal{J}} f(., u(.))(\omega) \tag{5.1.1}$$

a.s. for any $(\mathcal{J}, \mathcal{B}(E))$-measurable mapping u from Ω to E when f is continuous on the compact metric space $\Omega \times E$, or more generally f is a Caratheodory integrand, see Proposition 5.2.

EXERCISE 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, E a Suslin metrizable space, f a Carathéodory integrand, u a $(\mathcal{J}, \mathcal{B}(E))$-measurable mapping from Ω to E such that f(.,u(.) is integrable. Prove or disprove the equality

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), u(\omega)) = E^{\mathcal{J}} f(., u(.))(\omega) \quad a.s.$$

So the validity of Lemma 5.1 in this general case is an open problem. This fact has been kindly communicated to us by G. Michaille.

EXERCISE 2. Let $(\Omega.\mathcal{F}, P)$ be a probability space, E a Suslin metrizable space, $(f_n)_{n\geq 1}$ a positive lower semi-continuous superadditive random sequence, u a $(\mathcal{J}, \mathcal{B}(E))$-measurable mapping from Ω to E.

Prove or disprove the inequality

$$\limsup_{n\to\infty} \frac{1}{n} f_n(\omega, u(\omega)) \leq \sup_{n\geq 1} \frac{1}{n} E^{\mathcal{J}} f_n(., u(.))(\omega) \quad a.s.$$

Now let us focus our attention to the case when $(\Omega, \mathcal{F}, P)$ is an arbitrary complete probability space and E is a separable Banach space and T is ergodic.

There is an analogous result to Lemma 5.3.

LEMMA 5.6. *Assume that E is a separable Banach space. Let $f : \Omega \times E \to [0, +\infty]$ be a $\mathcal{F} \otimes \mathcal{B}(E)$-measurable integrand such that for any ω in Ω, $f(\omega, .)$ is lower semi-continuous on E. Assume that there is an $(\mathcal{F}, \mathcal{B}(E))$-measurable and integrable mapping $\overline{u}$ from Ω to E such that $f(., \overline{u}(.))$ is integrable. Assume further that T is ergodic. Then there is a negligible set N such that*

$$\sup_{k\in \mathbf{N}^*} \liminf_{n\to\infty} \inf_{y\in E} [\frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), y) + k||x-y||] \geq \int f(\omega, x) P(d\omega)$$

for all (ω, x) in $(\Omega \backslash N) \times E$.

PROOF. For (k, n) in $\mathbf{N}^* \times \mathbf{N}^*$ and (ω, x) in $\Omega \times E$, put

$$f_n(\omega, x) = \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), x)$$

$$f_n^k(\omega, x) = \inf_{y\in E} [f_n(\omega, y) + k||x-y||]$$

$$f^k(\omega, x) = \inf_{y\in E} [f(\omega, y) + k||x-y||]$$

Then we have

$$f_n^k(\omega, x) \geq \frac{1}{n} \sum_{j=0}^{n-1} f^k(T^j(\omega), x)$$

for all (ω, x) in $\Omega \times E$. Since we have

$$\begin{aligned} 0 \leq f^k(\omega, x) &\leq f^k(\omega, \overline{u}(\omega)) + k||x - \overline{u}(\omega)|| \\ &\leq f(\omega, \overline{u}(\omega)) + k||x - \overline{u}(\omega)|| \end{aligned}$$

for all (ω, x) in $\Omega \times E$, $f^k(., x)$ is integrable for any x in E. Therefore we can apply Birkhoff-Kingman's ergodic theorem to the additive random sequence $(g_n^k(., x))_{n\geq 1}$, where $g_n^k(\omega, x) = \sum_{j=0}^{n-1} f^k(T^j(\omega), x)$ for (ω, x) in $\Omega \times E$. Hence, for any x in E, there is a negligible set N_x^k such that

$$\lim_{n\to\infty} \frac{1}{n} g_n^k(\omega, x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f^k(T^j(\omega), x) \tag{5.6.1}$$

$$= \int f($$

for all ω in $\Omega \backslash N_n^k$ since T is ergodic by our assumption. Recall that

$$|f^k(\omega, x) - f^k(\omega, y)| \leq k||x - y||$$

for all (ω, x, y) in $\Omega \times E \times E$, and $\sup_{k \geq 1} f^k = f$. Let D be a countable dense subset of E. By (5.6.1) there is a negligible set $N^k = \bigcup_{x \in D} N_x^k$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f^k(T^j(\omega), x) = \int f^k(\omega, x) P(d\omega) \tag{5.6.2}$$

for (ω, x) in $(\Omega \backslash N^k) \times D$, so that (5.6.2) is valid for (ω, x) in $(\Omega \backslash N^k) \times E$ because $x \to \int f^k(\omega, x) P(d\omega)$ and $\lim_{n \to \infty} \frac{1}{n} g_n^k(\omega, .)$ are k-lipschitzean. Hence we get

$$\begin{aligned} \liminf_{n \to \infty} f_n^k(\omega, x) &\geq \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f^k(T^j(\omega), x) \\ &= \int f^k(\omega, x) P(d\omega) \end{aligned} \tag{5.6.3}$$

for all (ω, x) in $(\Omega \backslash N^k) \times E$. By the monotone convergence theorem we have $\sup_{k \geq 1} \int f^k(\omega, x) P(d\omega) = \int f(\omega, x) P(d\omega)$ for all $x \in E$. Let $N = \bigcup_{k=1}^{\infty} N^k$. Then $P(N) = 0$.

By taking the supremum over k in (5.6.3) we get

$$\sup_{k \in \mathbb{N}^*} \liminf_{n \to \infty} f_n^k(\omega, x) \geq \sup_{k \geq 1} \int f^k(\omega, x) P(d\omega) = \int f(\omega, x) P(d\omega)$$

for all (ω, x) in $(\Omega \backslash N) \times E$.

REMARKS.

1) The lemma is valid if we replace E by a Suslin metrizable space (S, d) with $d \leq 1$.

2) Positive value assumption for f can be relaxed. Namely, the preceding lemma is still valid if we suppose that f is a $\mathcal{F} \otimes \mathcal{B}(E)$-measurable integrand such that for any ω in $\Omega, f(\omega, .)$ is lower semi-continuous on E and there is an integrable function $\overline{u} : \Omega \to E$ and a positive integrable random variable m

such that $f(.,\overline{u}(.))$ is integrable and $f(\omega,x)+m(\omega)\geq 0$ for all (ω,x) *in* $\Omega\times E$. The details are left to the reader.

LEMMA 5.7. *Assume that E is a separable reflexive Banach space. Let $f:\Omega\times E\to\,]-\infty,+\infty]$ be a $\mathcal{F}\otimes\mathcal{B}(E)$- measurable integrand such that for any ω in $\Omega, f(\omega,.)$ is convex lower semi-continuous on E and there is an integrable function $\overline{u}:\Omega\to E$ and a positive integrable random variable m such that $f(.,\overline{u}(.))$ is integrable and $f(\omega,x)+m(\omega)\geq 0$ for all (ω,x) in $\Omega\times E$. Assume that T is ergodic. Then there is a negligible set N such that*

$$\sup_{k\in\mathbb{N}^\bullet}\limsup_{n\to\infty}\inf_{y\in E}\left[\frac{1}{n}\sum_{j=0}^{n-1}f(T^j(\omega),y)+k||x-y||\right] = \int f(\omega,x)P(d\omega) \tag{5.7.1}$$

for all $(\omega,x)\in(\Omega\backslash N)\times E$.

PROOF. We imitate some arguments given by Hess [20]. Put $\varphi(x)=\int f(\omega,x)P(d\omega)$ for $x\in E$. Then φ is convex lower semi-continuous on E. For any integer $k\geq 1$, put

$$\varphi^k(x)=\inf_{y\in E}[\varphi(y)+k||x-y||]\quad x\in E.$$

Then $\sup_k\varphi^k=\varphi$.

Let D be a countable dense subset of E. Then for $k\geq 1$, x in D, there is $v_{k,x}$ in E such that

$$k||x-v_{k,x}||+\varphi(v_{k,x})=\varphi^k(x)$$

since E is reflexive and the function $y\to\ k||x-y||+\varphi(y)\ \ (y\in E)$ is inf - $\sigma(E,E')$ compact. Since T is ergodic by Birkhoff-Kingman's ergodic theorem, there is a negligible set $N_{k,x}$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}f(T^j(\omega),v_{k,x})=\int f(\omega,v_{k,x})P($$

for all ω in $\Omega\backslash N_{k,x}$. Put $N = \bigcup_{k\geq 1} \bigcup_{x\in D} N_{k,x}$. Then P(N) = 0 and for x in D, k in $\mathbb{N}^*, \omega$ in $\Omega\backslash N$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \inf_{y\in E} \left[\frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), y) + k\|x-y\|\right] \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), v_{k,x}) + k\|x - v_{k,x}\|) \\
&= k\|x - v_{k,x}\| + \varphi(v_{k,x}) = \varphi^k(\omega, x)
\end{aligned}$$

Hence we get

$$\limsup_{n\to\infty} \inf_{y\in E} \left[\frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), y) + k\|x-y\|\right] \leq \varphi^k(x) \tag{5.7.2}$$

for (ω, x) in $\Omega\backslash N \times D$. Since both sides in (5.7.2) are k-lipschitzean, (5.7.2) is valid for (ω, x) in $\Omega\backslash N \times E$ and $k \geq 1$. By taking the supremum over k in (5.7.2), we have

$$\sup_{k\in\mathbb{N}^*} \limsup_{n\to\infty} \inf_{y\in E} \left[\frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), y) + k\|x-y\|\right] \leq \varphi(x) \tag{5.7.3}$$

for (ω, x) in $(\Omega\backslash N) \times E$, since $\sup_{k\in\mathbb{N}^*} \varphi^k = \varphi$.

Then (5.7.1) follows from Lemma 5.6 and (5.7.3).

By combining Lemma 5.6 and Lemma 5.7 we obtain the following epiconvergence result.

THEOREM 5.8. *Assume that the hypotheses of Lemma 5.7 are satisfied. Then there is a negligible set N such that*

$$\operatorname{epilim}_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(\omega), x) = \int f(\omega, x) P(d\omega)$$

for all $(\omega, x) \in (\Omega\backslash N) \times E$.

COMMENTS. Parametrized lipschitzean approximation for lower semicontinuous random integrands has been implicitely used in many places. Castaing ([17]) used it to prove the existence of Caratheodory selection for multifunctions with nonempty convex closed values in a separable Banach spaces which are globally measurable and separately lower semi-continuous via Scorza-Dragoni's theorem, in the same vein Larhrissi ([24]) used it to prove the existence of separately $\mathcal{BP}(E)$-measurable (i.e. measurable with to the σ-algebra of the sets with Baire property) and separately continuous for such kind of multifunctions. Other uses of parametrized lipschitzean approximations can be found in Castaing-Clauzure ([13]) concerning the lower semi-continuity of integral functionals, in Castaing ([10]) and Valadier ([38]) concerning the compactness of Young measures, and also in Moreau - Valadier ([25]) concerning the derivation of vector measures. The reader is referred to Buttazo ([4]), Clauzure ([13]), Castaing ([9]), Dynkin-Estignev ([16]), Thibault ([33], [34]) for the use of the parametrized lipschitzean approximations concerning the integral representations and the conditional expectations for lsc integrands. Recently Gavioli ([18]) and Moussaoui ([27]) used it to state the lipschizean approximation for upper semi-continuous multifunctions with convex closed bounded values in a reflexive Banach space which allows to give many interesting applications to the study of the sweeping process (or Moreau process) and the existence theorems in differential inclusions. Choukairi ([12]) used it to obtain epi convergence for lsc integrand martingales and amarts. Recently Hess ([20]) used it in the study of epiconvergence for normal lsc integrands and particularly for normal lsc independent integrands. To our knowledge, lipschitzean approximation for lsc functions defined on a metric space is very classical. See e.g. Bott - MacShane ([3]) and Natanson ([29]). More recently, Jalby ([23]) obtains a lipschitzean approximation result for random lsc vector functions and also by this way, the condition expectation for this class of lsc vector functions.

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