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ON THE UNIQUENESS OF GLOBAL CLASSICAL SOLUTIONS OF THE CAUCHY PROBLEMS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER*

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Abstract. Some uniqueness theorems for global classical solutions of the Cauchy problems for general nonlinear partial differential equations of first order are established by the method based on the theory of multivalued mappings and differential inclusions.
Key words. The Cauchy problems, Nonlinear partial differential equations of first order, Global classical solutions, Differential inclusions.

1. Introduction

In this paper we consider the Cauchy problems for nonlinear partial differential equations of first order in an n-dimensional space ($n \geq 1$) and establish some uniqueness theorems for global classical solutions. Our method is based on the theory of multivalued functions and of differential inclusions. Indeed we extend some of our uniqueness results for global classical solutions of the Cauchy problems for Hamilton-Jacobi equations [1] to the case of general nonlinear partial differential equations of first order.

As in [1], it must be noted that the theory of nonlinear partial differential equations of first order has attracted much interest in the literature, partly due to its applications in many fields such as classical mechanics, the theory of waves, the theory of optimal control, the theory of differential games, and so on. Through the works of S.H. Benton, V.J.D. Cole, E.D.Conway, M.G. Crandall, B. Doubnov, L.C. Evans, W.H. Fleming, J. Glimm, E. Hopf, S. N. Kružkov,

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P. D. Lax, P. L. Lions, V. P. Maslov,O. Oleinhik, B.L. Rozdestvenskii, A. I. Subbotin, M. Tsuji and others, many fundamental results on global solutions of the Cauchy problems for first-order nonlinear differential equations have been obtained and various kinds of generalized solutions have been introduced.

The paper is organized as follows. In Section 2 we formulate the uniqueness theorems for global classical solutions. Section 3 is devoted to the proof of our uniqueness results. In Section 4 the problem of continuous dependence on the Cauchy data is investigated.

2. Uniqueness of global classical solutions

Let T be a positive number, $\Omega_T = (0,T) \times \mathbb{R}^n = \{(t,x) \in \mathbb{R}^{n+1}, 0 < t < T\}$. We consider the Cauchy problems for general partial differential equations of first order:

$$\frac{\partial u(t,x)}{\partial t} + H(t,x,u(t,x),\nabla_x u(t,x)) = 0, \tag{2.1}$$

$$u(0,x) = u_0(x), \tag{2.2}$$

where H is a function of $(t,x,p,q) \in (0,T) \times \mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n$ and $u_0(x)$ is a known function. A vector $q = (q^1, q^2, ..., q^n)$ is corresponding to $\nabla_x u = (\partial u/\partial x_1, \partial u/\partial x_2, ..., \partial u/\partial x_n)$. We are interested in the uniqueness of global classical solutions for the Cauchy problem (2.1), (2.2).

DEFINITION 2.1. A function u in $C^1(\Omega_T) \cap C([0,T) \times \mathbb{R}^n)$ is called a global classical solution of the Cauchy problem (2.1), (2.2) if u satisfies (2.1) everywhere in Ω_T and (2.2) on $\{t = 0, x \in \mathbb{R}^n\}$.

Further let us denote by $||.||$ and $< .,. >$ the norm and the scalar product in $\mathbb{R}^n$, respectively.

THEOREM 2.1. *Suppose that there exist nonnegative numbers M and N such that for all $p_1, p_2 \in \mathbb{R}^1$, $q_1, q_2 \in \mathbb{R}^n$:*

$$|H(t,x,p_1,q_1) - H(t,x,p_2,q_2)| \leq M|p_1 - p_2| + N(1 + \|x\|)\|q_1 - q_2\|, \tag{2.3}$$

$$\forall\, (t,x) \in \Omega_T.$$

If $u_1(t,x)$ and $u_2(t,x)$ are global classical solutions of the Cauchy problem (2.1),(2.2), then $u_1(t,x) \equiv u_2(t,x)$ in Ω_T.

REMARK 2.1. The condition (2.3) is fulfilled, if H, for example, is differentiable with respect to the variables p and q, and its derivatives $\frac{\partial H}{\partial p}, \nabla_q H$ satisfy the constraints:

$$\sup_{\substack{(t,x)\in\Omega_T \\ p\in\mathbb{R}^1, q\in\mathbb{R}^n}} \left| \frac{\partial H(t,x,p,q)}{\partial p} \right| < \infty,$$

$$\sup_{\substack{(t,x)\in\Omega_T \\ p\in\mathbb{R}^1, q\in\mathbb{R}^n}} \|\nabla_q H(t,x,p,q)\|/(1+\|x\|) < \infty.$$

COROLLARY 2.1. (see also [1]) Suppose that the Hamiltonian $H(t,x,q)$ satisfies the condition: There exists a number $N \geq 0$ such that for all $q_1, q_2 \in \mathbb{R}^n$ and $(t,x) \in \Omega_T$,

$$|H(t,x,q_1) - H(t,x,q_2)| \leq N(1+\|x\|)\|q_1 - q_2\|.$$

If u_1 and u_2 in $C^1(\Omega_T) \cap C([0,T) \times \mathbb{R}^n)$ satisfy the equation

$$\frac{\partial u}{\partial t} + H(t,x,\nabla_x u) = 0$$

everywhere in Ω_T with $u_1(0,x) \equiv u_2(0,x)$, $x \in \mathbb{R}^n$, then $u_1(t,x) \equiv u_2(t,x)$ in Ω_T.

THEOREM 2.2. *Suppose that on $\Omega_T \times \mathbb{R}^1 \times \mathbb{R}^n$, $H(t,x,p,q)$ is Lipschitz continuous with respect to $p \in \mathbb{R}^1$ and locally Lipschitz continuous with respect to $q \in \mathbb{R}^n$, i.e. there exist numbers $M \geq 0$, $N(K) > 0$ such that*

$$|H(t,x,p_1,q_1) - H(t,x,p_2,q_2)| \leq M|p_1 - p_2| + N(K)\|q_1 - q_2\|, \tag{2.4}$$

$$\forall\, (t,x) \in \Omega_T,\ p_1,\ p_2 \in \mathbb{R},\ q_1,\ q_2 \in K,$$

where K is any compact set in $\mathbb{R}^n$. If u_1 and u_2 are global quasi-classical solutions of the Cauchy problem (2.1),(2.2) with

$$\sup_{(t,x)\in\Omega_T} \|\nabla_x u_i(t,x)\| < \infty, \quad i = 1,2,$$

then $u_1(t,x) \equiv u_2(t,x)$ *in* Ω_T.

REMARK 2.2. If H is independent of x and p and if $H = H(t,q) \in C^1([0,T] \times \mathbf{R}^n)$,then condition (2.4) is fulfilled.

COROLLARY 2.2. (see also [1]). Suppose that $H(t,x,p,q)$ is independent of $p \in \mathbf{R}^1$, and on $\Omega_T \times \mathbf{R}^n$, $Q(t,x,q) \equiv H(t,x,q)/(1+\|x\|)$ is locally Lipschitz continuous with respect to $q \in \mathbf{R}^n$. If u_1 and u_2 are global classical solutions of the problem (2.1),(2.2) with

$$\sup_{(t,x)\in\Omega_T} \|\nabla_x u_i(t,x)\| < \infty, \quad i = 1,2,$$

then $u_1(t,x) \equiv u_2(t,x)$ in Ω_T.

3. Proof of Theorems 2.1 and 2.2

The proof of Theorems 2.1 and 2.2 is based on the following result.

THEOREM 3.1. *Let* u *be a function in* $C^1(\Omega_T) \cap C([0,T) \times \mathbf{R}^n), u(0,x) \equiv 0$ *on* $\{t = 0, x \in \mathbf{R}^n\}$ *and suppose that there exist non-negative numbers* M, N *such that for any* $(t,x) \in \Omega_T$

$$\left|\frac{\partial u(t,x)}{\partial t}\right| \leq N(1+\|x\|)\|\nabla_x u(t,x)\| + M|u(t,x)|. \tag{3.1}$$

Then $u(t,x) \equiv 0$ *in* Ω_T.

PROOF. Let (t_0, x_0) be an arbitrary point in Ω_T. We have to prove that $u(t_0,x_0) = 0$. For this purpose we define in Ω_T a multivalued function $F : \Omega_T \to \mathbf{R}^n$ in the following way:

$$\begin{aligned} F(t,x) =& \{f | f \in \mathbf{R}^n, ||f|| \leq N(1+||x||), \\ & |\frac{\partial u(t,x)}{\partial t} + < f, \nabla_x u(t,x) > | \leq M|u(t,x)|\}. \end{aligned} \tag{3.2}$$

We consider the differential inclusion

$$\frac{dx(t)}{dt} \in F(t, x(t)), \tag{3.3}$$

subject to the constraint

$$x(t_0) = x_0. \tag{3.4}$$

Now let $X(t_0, x_0)$ be the set of all *absolutely continuous functions* $x(\cdot)$: $[0, T] \longrightarrow \mathbb{R}^n$, which satisfy the initial condition (3.4) and the differential inclusion (3.3) almost everywhere on $[0, T]$. We will show that $X(t_0, x_0)$ is a non-empty compact set in $C([0, T], \mathbb{R}^n)$.

To prove this we now show that the function F defined by (3.2) satisfies all conditions of Theorem П.3, p.206 in [2]. Indeed, we have to verify the following conditions:

(i) for any $(t, x) \in \Omega_T, F(t, x)$ is a non-empty convex set in $\mathbb{R}^n$;

(ii) F is upper semicontinuous in Ω_T.

First we check (i). It is obvious that $F(t, x)$ is a convex closed set in $\mathbb{R}^n$. Further, by condition (3.1), there exists a number $\lambda = \lambda(t, x) \in [0, 1]$ such that

$$\lambda|\partial u(t, x)/\partial t| \leq N(1 + \|x\|)\|\nabla_x u(t, x)\|, \tag{3.5}$$

$$(1 - \lambda)|\partial u(t, x)/\partial t| \leq M|u(t, x)|. \tag{3.6}$$

If $\nabla_x u(t, x) = 0$ it follows that $F(t, x) = \bar{B}_{N(1+\|x\|)} \neq \emptyset$, where we denote by $\bar{B}_r$ the ball $\bar{B}_r^n = \{x \in \mathbb{R}^n : \|x\| \leq r\}$.

If $\nabla_x u(t, x) \neq 0$, we put

$$f = -\frac{\lambda \partial u(t, x)/\partial t}{\|\nabla_x u(t, x)\|^2} . \nabla_x u(t, x).$$

By virtue of (3.5) we have

$$\|f\| = \frac{\lambda|\partial u(t, x)/\partial t|}{\|\nabla_x u(t, x)\|} \leq N(1 + \|x\|).$$

Therefore, from (3.6) we obtain the estimate

$$\begin{aligned}
&|\partial u(t, x)/\partial t + < f, \nabla_x u(t, x) > | = \\
&\qquad = |\partial u(t, x)/\partial t - \frac{\lambda \partial u(t, x)/\partial t}{\|\nabla_x u(t, x)\|^2} < \nabla_x u(t, x), \nabla_x u(t, x) > | = \\
&\qquad = |\partial u(t, x)/\partial t - \lambda \partial u(t, x)/\partial t| = \\
&\qquad = (1 - \lambda)|\partial u(t, x)/\partial t| \leq M|u(t, x)|,
\end{aligned}$$

i.e. $f \in F(t, x)$. Hence $F(t, x) \neq \emptyset$ and $F(t, x)$ is a compact set in $\mathbb{R}^n$.

To verify condition (ii) we observe that the function F is bounded in a neighborhood of any $(t,x) \in \Omega_T$, i.e. there exist numbers $\ell > 0$ and $r > 0$ such that

$$\sup\{\|f\| \quad |f \in F(\tau,y), (\tau,y) \in B^1_\ell(t) \times B^n_r(x) \subset \Omega_T\} < \infty.$$

In addition, it is easily seen that the function F is closed because for any sequence $(t_k,x_k) \in \Omega_T (k=1,2,...), (t_k,x_k) \longrightarrow (t,x) \in \Omega_T$ and for any sequence $f_k \in F(t_k,x_k) \quad (k=1,2,...), f_k \longrightarrow f$ in $\mathbb{R}^n$ we have $f \in F(t,x)$. The function F is closed and locally bounded. Therefore, it is upper semicontinuous in Ω_T.

Thus, we have shown that the function F defined by (3.2) satisfies all conditions of Theorem П.3 in [2]. By virtue of this theorem the set $X(t_0,x_0)$ of solutions of (3.3), (3.4) is a non-empty compact set in $C([0,T],\mathbb{R}^n)$.

Let $x(.) \in X(t_0,x_0)$. We consider the function $\varphi(t) \equiv u(t,x(t))$. Since $u \in C^1(\Omega_T)$ and $x(.)$ is absolutely continuous on $[0,T]$, the function $\varphi(.)$ is absolutely continuous on $[0,T]$, the function $\varphi(.)$ is absolutely on $[\epsilon, T-\epsilon]$ for any $\epsilon \in (0,T/2)$. On the other hand, we have

$$\frac{d\varphi(t)}{dt} = \frac{\partial u(t,x(t))}{\partial t} + < \frac{dx(t)}{dt}, \nabla_x u(t,x(t)) >$$

almost everywhere on $(0,T)$. From the fact that $dx(t)/dt \in F(t,x(t))$ we immediately get

$$\left|\frac{d\varphi(t)}{dt}\right| \le M|u(t,x(t))| = M|\varphi(t)|. \tag{3.7}$$

almost everywhere on $(0,T)$.

Since $\varphi(.) \in C([0,T))$, the last inequality shows that

$$\int_0^{T-\epsilon} \left|\frac{d\varphi(t)}{dt}\right| dt < \infty, \quad \forall \epsilon > 0.$$

Thus $\varphi(.)$ is absolutely continuous on $[0,T-\epsilon]$ and $\varphi(0) = u(0,x(0)) = 0$.

Now we are going to show that $\varphi(t) \equiv 0$ on $[0,T-\epsilon]$. For this we put $\varphi_0(t) = |\varphi(t)|$. It is obvious that $\varphi_0(.)$ is absolutely continuous on $[0,T-\epsilon]$ and $d\varphi_0(t)/dt =$ sign $\varphi(t), d\varphi(t)/dt$ almost everywhere on $[0,T-\epsilon]$.

From (3.7) we get

$$\left|\frac{d\varphi_0(t)}{dt}\right| \le M.\varphi_0(t). \tag{3.8}$$

The function $\varphi_1(t) \equiv \varphi_0(t)e^{-Mt}$ is absolutely continuous on $[0, T-\epsilon]$, and by virtue of (3.8) we have

$$\frac{d\varphi_1(t)}{dt} = \frac{\frac{d\varphi_0(t)}{dt}e^{Mt} - Me^{Mt}\varphi_0(t)}{e^{2Mt}} = \frac{\frac{d\varphi_0(t)}{dt} - M\varphi_0(t)}{e^{Mt}} \leq 0$$

almost everywhere on $[0, T-\epsilon]$. Thus, $\varphi_1(t) \leq \varphi_1(0)$ and $\varphi_0(t) \leq \varphi_0(0)e^{Mt}$. Hence, $|\varphi(t)| \leq |\varphi(0)|e^{Mt} = 0, \quad \forall t \in [0, T-\epsilon]$.

Since ϵ is an arbitrary positive number we obtain that $\varphi(t) = 0$ for all $t \in [0, T)$. In particular, $\varphi(t_0) = u(t_0, x(t_0)) = u(t_0, x_0) = 0$. The proof of Theorem 3.1 is complete.

Proof of Theorem 2.1. We consider the function $u = u_1 - u_2$. Then $u(0, x) \equiv 0, x \in \mathbf{R}^n$. Besides that, from condition (2.3) we have

$$\begin{aligned}
&|\frac{\partial u(t,x)}{\partial t}| = |H(t,x,u_1(t,x),\nabla_x u_1(t,x)) - H(t,x,u_2(t,x),\nabla_x u_2(t,x))| \\
&\leq M|u_1(t,x) - u_2(t,x)| + N(1+\|x\|)\|\nabla_x u_1(t,x) - \nabla_x u_2(t,x)\| = \\
&= M|u(t,x)| + N(1+\|x\|)\|\nabla_x u(t,x)\|.
\end{aligned}$$

Now it follows from Theorem 3.1 that $u(t, x) \equiv 0$ in Ω_T. This proves Theorem 2.1.

Proof of Theorem 2.2. Consider the function $u = u_1 - u_2$. Then $u(0, x) \equiv 0, x \in \mathbf{R}^n$. Let

$$k = \max_{i=1,2}\{\sup_{(t,x)\in\Omega_T} \|\nabla_x u_i(t,x)\|\}.$$

Since H satisfies (2.4), there exist numbers $M \geq 0, N \geq 0$ such that

$$|H(t,x,p_1,q_1) - H(t,x,p_2,q_2)| \leq M|p_1 - p_2| + N\|q_1 - q_2\|,$$

$$\forall\ (t,x) \in \Omega_T,\ p_1,p_2 \in \mathbf{R}^1,\ q_1,q_2 \in \bar{B}_k.$$

Hence

$$\left|\frac{\partial u(t,x)}{\partial t}\right| = |H(t,x,u_1(t,x),\nabla_x u_1(t,x)) - H(t,x,u_2(t,x),\nabla_x u_2(t,x))|$$
$$\leq M|u(t,x)| + N\|\nabla_x u(t,x)\|, \quad \forall\ (t,x) \in \Omega_T.$$

Applying Therem 3.1 to the function u we obtain that $u(t,x) \equiv 0$ in Ω_T, which proves Theorem 2.2.

4. The continuous dependence of solutions on initial conditions

THEOREM 4.1. *Suppose that H satisfies the condition (2.3) in Theorem 2.1. If $u_i \in C^1(\Omega_T) \cap C([0,T) \times \mathbf{R}^n)(i = 1,2)$ satisfy everywhere in Ω_T the equation (2.1) and the Cauchy data*

$$u_i(0,x) = \varphi_i(x), \quad x \in \mathbf{R}^n, \varphi_i \in C(\mathbf{R}^n),\ i = 1,2,$$

then

$$|u_1(t,x) - u_2(t,x)| \leq e^{Mt} \sup_{\|y\|+1\leq(1+\|x\|)e^{Nt}} |\varphi_1(y) - \varphi_2(y)|.$$

The proof of Theorem 4.1 follows directly from the following lemma which is similar to Theorem 3.1.

LEMMA 4.1. *Let u be a function in $C^1(\Omega_T) \cap C([0,T) \times \mathbf{R}^n)$. Suppose that there exist numbers $M \geq 0, N \geq 0$ such that for any $(t,x) \in \Omega_T$,*

$$\left|\frac{\partial u(t,x)}{\partial t}\right| \leq N(1+\|x\|)\|\nabla_x u(t,x)\| + M|u(t,x)|. \tag{4.1}$$

Then

$$|u(t,x)| \leq e^{Mt} \sup_{\|y\|+1\leq(1+\|x\|)e^{Nt}} |u(0,y)|. \tag{4.2}$$

PROOF OF LEMMA 4.1. Repeatting the proof of Theorem 3.1, we get that the function φ is absolutely continuous on $[0, T-\epsilon]$, and almost everywhere on $[0, T-\epsilon]$ we have

$$\left|\frac{d\varphi(t)}{dt}\right| \leq M|\varphi(t)|.$$

Thus, $|\varphi(t)| \leq e^{Mt}|\varphi(0)|$.

Further,

$$|u(t_0, x_0)| = |u(t_0, x(t_0))| = |\varphi(t_0)| \leq$$
$$\leq e^{Mt_0}|\varphi(0)| = e^{Mt_0}|u(0, x(0))|. \tag{4.3}$$

If we can show that

$$\|x(0)\| \leq (1 + \|x_0\|)e^{Nt_0} - 1, \tag{4.4}$$

then the estimate (4.2) will be proved by virtue of (4.3). In turn, the estimate (4.4) follows from the following

LEMMA 4.2. *Let $x(.) : [0,T] \longrightarrow \mathbb{R}^n$ be absolutely continuous on $[0,T]$, $x(t_0) = x_0$, and almost everywhere on $[0,T]$*

$$\left\|\frac{dx(t)}{dt}\right\| \leq N(1 + \|x(t)\|),$$

Then

$$\|x(t)\| \leq (1 + \|x_0\|)e^{N|t-t_0|} - 1. \tag{4.5}$$

PROOF OF LEMMA 4.2. For every $\epsilon > 0$, put

$$m_\epsilon(t) = (1 + \|x_0\| + \epsilon)e^{N|t-t_0|} - 1.$$

The function $m_\epsilon(.)$ is absolutely continuous, positive on $[0,T]$ and

$$\frac{dm_\epsilon(t)}{dt} = \begin{cases} N(1 + m_\epsilon(t)), & t > t_0, \\ -N(1 + m_\epsilon(t)), & t < t_0. \end{cases}$$

To prove (4.5) we have only to show that

$$\|x(t)\| < m_\epsilon(t), \quad \forall\, t \in [0,T], \quad \forall\, \epsilon > 0. \tag{4.6}$$

Since $m_\epsilon(t_0) > \|x_0\| = \|x(t_0)\|$ there exists a number $\delta > 0$ such that for all $t \in [0,T] \cap (t_0 - \delta, t_0 + \delta)$,

$$m_\epsilon(t) > \|x(t)\|.$$

Assume that (4.6) is false. Then there exists $t' \in [0,T]$ such that $m_\epsilon(t') \leq \|x(t')\|$.

i) If $t' \geq t_0$, putting $t_1 = \inf\{t \in (t_0, T]; m_\epsilon(t) \leq \|x(t)\|\}$, we have

$$\|x(t_1)\| = m_\epsilon(t_1), \qquad m_\epsilon(t) > \|x(t)\|, \quad \forall t \in [t_0, t_1),$$

and almost everywhere on (t_0, t_1)

$$\begin{aligned} \frac{dm_\epsilon(t)}{dt} &= N(1 + m_\epsilon(t)) > N(1 + \|x(t)\|) \\ &\geq \|\frac{dx(t)}{dt}\| \geq \frac{d}{dt}\|x(t)\|, \end{aligned}$$

(Since $\|x(.)\|$ is absolutely continuous on $[0, T]$ it follows that $\frac{d}{dt}\|x(t)\| \leq \|\frac{dx(t)}{dt}\|$). On the other hand,

$$\int_{t_0}^{t_1} \frac{dm_\epsilon(t)}{dt} dt < \int_{t_0}^{t_1} \frac{d\|x(t)\|}{dt} dt$$

if and only if $m_\epsilon(t_1) - m_\epsilon(t_0) = \|x(t_1)\| - m_\epsilon(t_0) < \|x(t_1)\| - \|x(t_0)\|$. Hence we get a contradiction.

ii) If $t' \leq t_0$, putting $t_2 = \sup\{t \in [0, t_0); m_\epsilon(t) \leq \|x(t)\|\}$ and proceeding analogously as in i) we also come to a contradiction. This proves Lemma 4.2.

The uniqueness of global quasi-classical solutions of the Cauchy problems for nonlinear partial equations of first order will be studied in a forthcoming paper by the method used here.

References

[1] T. D. Van and N. D. T. Son, *On the uniqueness of global classical solutions of the Cauchy problem for Hamilton-Jacobi equations*, Acta Math. Vietnam. **17** (1992), p. 161-167.

[2] A. I. Subbotin, *Minimax inequalities and Hamilton-Jacobi equations (Russian)*, Nauka, Moscow (1991).

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