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KOSZUL HOMOLOGY AND GENERALIZED COHEN-MACAULAY MODULES

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1. Introduction

Throughout this paper A denotes a local ring with maximal ideal $\mathfrak{m}$ and M is a Noetherian A-module with $d = dim M \geq 1$. M is called a *generalized Cohen-Macaulay* module if all local cohomology modules $H^i_{\mathfrak{m}}(M)$, $i < d$, are of finite length. They have been developed during the last fifteen years mainly because of their relations to the theory of Buchsbaum modules (see, e.g. [5]-[8]) and also [2].

In the study of generalized Cohen-Macaulay modules we have the powerful notation of standard system of parameters. Recall that a system of parameters (abbr. s.o.d.) $x_1, \ldots, x_d$ of M is called a *standard s.o.p.* of M if

$$(x_1, \ldots, x_d) H^i_{\mathfrak{m}}(M/(x_1, \ldots, x_j)M) = 0,$$

for all non-negative integers i, j with $i + j < d$ (see, e.g., [8]). It should be mentioned that $x_1, \ldots, x_d$ is a standard s.o.p. of M if and only if it is a unconditioned strong d-sequence [4] and that M is a generalized Cohen-Macaulay module if and only if (one or) every s.o.p. of M contained in $\mathfrak{m}^n$, where n is sufficiently large, is a standard s.o.p. of M [8]. In particular, M is a Buchsbaum module if and only if every s.o.p. of M is a standard s.o.p.. In [8] there are many conditions for a s.o.p. of M to be a standard s.o.p.. One can also characterize standard s.o.p.'s by means of Koszul homology (see[4], Theorem 2.14 and Theorem 6.9). The aim of this paper is to present another similar characterization of standard s.o.p.'s.

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THEOREM 1. *Assume that M is a generalized Cohen-Macaulay module. Let $x_1, \ldots, x_d$ be a s.o.p. of M. Then the following conditions are equivalent:*

(i) $x_1, \ldots, x_d$ *is a standard s.o.p. of* M

(ii) $\ell(H_p(x_1, \ldots, x_r; M)) = \sum_{i=0}^{r-p} \binom{r}{i+p} . \ell(H^i_{\mathfrak{m}}(M))$,

for all $p > 0$ *and* $1 \leq r \leq d$.

(iii) $\ell(H_1(x_1, \ldots, x_d; M)) = \sum_{i=0}^{d=1} \binom{d}{i+1} . \ell(H^i_{\mathfrak{m}}(M))$.

(iv) $\ell(H_p(x_1, \ldots, x_r; M)) = \ell(H_p(x_1^2, \ldots, x_r^2; M))$,

for all $p > 0$ *and* $1 \leq r \leq d$.

(v) $\ell(H_1(x_1, \ldots, x_d; M)) = \ell(H_1(x_1^2, \ldots, x_d^2; M))$.

As consequences of this result we shall obtain new characterizations of Buchsbaum and quasi-Buchsbaum modules, resp. (see Corollary 5 and Proposition 6).

2. Proof of Theorem 1

First, let us recall some basic facts on Koszul homology from [1], Section 1. Let $x_1, \ldots, x_r$ be elements of $A (r > 0)$. We denote by $K_.(x_1, \ldots, x_r; M)$ the Koszul complex generated by $x_1, \ldots, x_r$ over M. Its boundary operator will be denoted by d.

For $r > 1$, let $L_. = K_.(x_1, \ldots, x_{r-1}; M)$ and e be the boundary operator of $L_.$. Since

$$K_. = K_.(x_1, \ldots, x_r; M) \cong K_.(x_1, \ldots, x_{r-1}; M) \otimes K_.(x_r; A),$$

the complex $K_.$ can be treated as follows:

$$K_p = L_{p-1} \otimes L_p,$$

and

$$d_p(u, v) = (e_{p-1}(u), (-1)^{p-1} x_r u + e_p(v)).$$

Let $L.(-1)$ denote the complex shifted by -1, i.e. $(L(-1)) = L_{p-1}$ and the p-th boundary operator is e_{p-1}. Then we have the exact sequence:

$$0 \longrightarrow L. \xrightarrow{i} K. \xrightarrow{j} L.(-1) \longrightarrow 0 \tag{1}$$

where $i_p(v) = (0, v)$ for $v \in L_p$ and $j(u,v) = u$ for $(u, v) \in K_p$. This exact sequence yields the following exact homology sequence

$$\begin{aligned} 0 \longrightarrow H_p(\underline{x}; M)/x_r H_p(\underline{x}, M) &\longrightarrow H_p(\underline{x}, x_r; M) \\ &\longrightarrow 0 :_{H_{p-1}(\underline{x};M)} x_r \longrightarrow 0, \end{aligned} \tag{2}$$

where $p \leq 1$ and $\underline{x}$ denotes the sequence of elements $x_1, \ldots, x_{r-1}$.

Assume that M is a generalized Cohen-Macaulay module. Let $x_1, \ldots, x_d$ be a s.o.p. of M. Then we have

$$\ell(H_p(x_1, \ldots, x_r; M)) \leq \sum_{i-0}^{r-p} \binom{r}{p+i} . \ell(H^i_{\mathfrak{m}}(M)),$$

for all $p > 0$ and $1 \leq r \leq d$. Moreover, equality holds if $x_1, \ldots, x_d$ is a standard s.o.p. of M (see [5], Satz 3.7 and [4], the results before 3.15). This gives the implication (i) $\Rightarrow$ (ii) of Theorem 1. The implications (ii)$\Rightarrow$(iii) and (iv) $\Rightarrow$(v) are trivial. Hence, to complete the proof of Theorem 1 one has to show the implications (iii)$\Rightarrow$ (v), (v) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iv).

We need some auxiliary results. Let $\mathfrak{a}$ be an ideal of A. A system of elements $x_1, \ldots, x_n$ is called an *$\mathfrak{a}$-weak M-sequence* if for all $0 \leq i < n$, we have

$$(x_1, \ldots, x_i)M : x_{i+1} \subseteq (x_1, \ldots, x_i)M : \mathfrak{a},$$

where $(x_1, \ldots, x_i)M := 0$ if $i = 0$ (see [8]). The following lemma slightly extends [7], Proposition 2.14 . One can easily prove it by using the exact sequence (2) and by induction on r.

LEMMA 2. *Let $\mathfrak{a}, \mathfrak{b}$ be two $\mathfrak{m}$- primary ideals and let n be a positive integer such that $\mathfrak{b}^n \subseteq \mathfrak{m}\mathfrak{a}$. Let $x_1, \ldots, x_r$ be elements in $\mathfrak{b}$. Assume that for any system of elements $\underline{x} = \{x_2^{n_2}, \ldots, x_r^{n_r}\}$ with $n_i \in \{1, \ldots, n\}, i = 2, \ldots, r$*

$$\mathfrak{a} H_1(x_1, \underline{x}; M) = 0.$$

Then $x_1, x_2^{n_2}, \dots, x_r^{n_r}$ form an $\mathfrak{a}$-weak M-sequence for all $n_i \in \{1, \dots, n\}$.

COROLLARY 3. (cf. [4], Theorem 2.14) *$x_1, \dots, x_d$ is a standard s.o.p. of M if and only if*

$$(x_1, \dots, x_d) H_p(x_1^{b^1}, \dots, x_d^{b^d}; M) = 0$$

for all $b_i \in \{1, 2\}$ and for all $p > 0$ (resp. for $p = 1$).

PROOF. We set $\mathfrak{q} = (x_1, \dots, x_d)A$. By [4], Theorem 2.14 and Corollary 2.15, $x_1, \dots, x_d$ is a standard s.o.p. of M if and only if

$$\mathfrak{q} H_p(x_1, \dots, x_d^{b_d}; M) = 0$$

for all $b_i > 0$ and $p > 0$. Hence we get the if part.

Now assume that $\mathfrak{q} H_1(x_1^{b_1}, \dots, x_d^{b_d}; M) = 0$ for all $b_i \in \{1, 2\}$. By Lemma 2, $x_1^{b_1}, \dots, x_d^{b_d}$ is a $\mathfrak{q}$-weak M-sequence for all $b_i = 1, 2$. Therefore, by Proposition 3.2. of [8], $x_1, \dots, x_d$ is a standard s.o.p. of M.

LEMMA 4. *Assume that M is a generalized Cohen-Macaulay module. Let $x_1, \dots, x_d$ be a s.o.p. of M. Then for $p > 0$, $1 \le r \le d$ and for all positive integers $n_1 \le m_1, \dots, n_r \le m_r$,*

$$\ell(H_p(x_1^{n_1}, \dots, x_r^{n_r}; M)) \le \ell(H_p(x_1^{m_1}, \dots, x_r^{m_r}; M)).$$

Moreover, if equality holds then

$$[o : x_r^{n_r}]_{H_{p-1}(\underline{x}; M)} = [0 : x_r^{m_r}]_{H_{p-1}(\underline{x}; M)},$$

and

$$x_r^{n_r} H_p(\underline{x}; M) = x_r^{m_r} H_p(\underline{x}; M),$$

where $\underline{x}$ denotes the sequence $x_1^{n_1}, \dots, x_{r-1}^{n_{r-1}}$.

PROOF. The case $r = 1$ is trivial because

$$H_1(x_r^{n_r}; M) = 0 :_M x_r^{n_r} \subseteq 0 :_M x_r^{m_r} = H_1(x_r^{m_r}; M).$$

Let $r \geq 2$. Since Koszul homology does not depend on the order of $x_1, \ldots, x_r$, by induction one may assume that $n_1 = m_1, \ldots, n_{r-1} = m_{r-1}$ and $m_r \geq n_r$. Consider the following exact sequences:

$$0 \longrightarrow H_p(\underline{x}; M)/x_r^{n_r} H_p(\underline{x}; M) \longrightarrow H_p(\underline{x}, x_r^{n_r}; M) \longrightarrow [0 : x_r^{n_r}]_{H_{p-1}(\underline{x};M)} \longrightarrow 0,$$

and

$$0 \longrightarrow H_p(\underline{x}; M)/x_r^{m_r} H_p(\underline{x}; M) \longrightarrow H_p(\underline{x}, x_r^{m_r}; M) \longrightarrow [0 : x_r^{m_r}]_{H_{p-1}(\underline{x};M)} \longrightarrow 0.$$

Since $m_r \geq n_r$, we have

$$x_r^{m_r} H_p(\underline{x}; M) \subseteq x_r^{n_r} H_p(\underline{x}; M),$$

and

$$[o : x_r^{n_r}]_{H_{p-1}(\underline{x};M)} \subseteq [0 : x_r^{m_r}]_{H_{p-1}(\underline{x};M)}.$$

Hence $\ell(H_p(\underline{x}, x_r^{n_r}; M)) \leq \ell(H_p(\underline{x}, x_r^{m_r}; M))$.

If

$$\ell(H_p(x_1^{n_1}, \ldots, x_r^{n_r}; M)) = \ell(H_p(x_1^{m_1}, \ldots, x_r^{m_r}; M)),$$

then we must have

$$\ell(H_p(\underline{x}, x_r^{n_r}; M)) = \ell(H_p(\underline{x}, x_r^{m_r}; M)).$$

Since M is a generalized Cohen-Macaulay module, all Koszul homology modules $H_p(\underline{x}; M)$ are of finite length ($p > 0$). Hence the above equality implies

$$[o : x_r^{n_r}]_{H_{p-1}(\underline{x};M)} = [0 : x_r^{m_r}]_{H_{p-1}(\underline{x};M)}$$

and

$$x_r^{n_r} H_p(\underline{x}; M) = x_r^{m_r} H_p(\underline{x}; M).$$

Now we can conclude the proof of Theorem 1 as follows: (iii) $\Rightarrow$ (v): By Lemma 4 and Remark 1 we have

$$\sum_{i=0}^{d-1} \binom{d}{i+1} . \ell(H_{\mathfrak{m}}^i(M)) = \ell(H_1(x_1, \ldots, x_d; M)) \leq \ell H_1(x_1^2, \ldots, x_d^2; M))$$

$$\leq \sum_{i=0}^{d-1} \binom{d}{i+1} \ell(H^i_{\mathfrak{m}}(M)),$$

which implies (v). Similarly we also get (ii)$\Rightarrow$(iv).

(v) $\Rightarrow$(i): We denote by $\underline{x}$ the sequences $x_1^{n_1}, \ldots, x_{d-1}^{n_{d-1}}$. By Lemma 4 we have for all $n_1, \ldots, n_{d-1} \in \{1, 2\}$:

$$\ell(H_1(x_1, \ldots, x_d; M)) \leq \ell(H_1(\underline{x}, x_d; M)) \leq \ell(H_1(\underline{x}, x_d^2; M))$$
$$\leq \ell(H_1(x_1^2, \ldots, x_d^2; M)).$$

Hence

$$\ell(H_1(\underline{x}, x_d; M)) = \ell(H_1(\underline{x}, x_d^2; M)).$$

We set $\overline{M} = M/(\underline{x})M$. Using the equality of Lemma 4 and Nakayama's lemma we get

$$(3) \quad 0 :_{\overline{M}} x_d = 0 :_{\overline{M}} x_d^2 \text{ and } x_d H_1(\underline{x}; M) = 0.$$

From the exact sequence (1) we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_.(\underline{x}; M) & \xrightarrow{i} & K_.(\underline{x}, x_d; M) & \xrightarrow{j} & K_.(\underline{x}; M)(-1) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle \cdot x_d} & & \\ 0 & \longrightarrow & K_.(\underline{x}; M) & \longrightarrow & K_.(\underline{x}, x_d^2; M) & \longrightarrow & K_.(\underline{x}; M)(-1) & \longrightarrow & 0. \end{array}$$

By the exact sequence (2) for $p = 1$ and by (3), this gives the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(\underline{x}; M) & \xrightarrow{i_*} & H_1(\underline{x}, x_d; M) & \xrightarrow{j_*} & 0 :_{\overline{M}} x_d & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle f_*} & & \uparrow{\scriptstyle \cdot x_d} & & \\ 0 & \longrightarrow & H_1(\underline{x}; M) & \longrightarrow & H_1(\underline{x}, x_d^2; M) & \longrightarrow & 0 :_{\overline{M}} x_d^2 & \longrightarrow & 0. \end{array}$$

Since $x_d(0 :_{\overline{M}} x_d^2) = x_d(0 :_{\overline{M}} x_d) = 0$, we have a uniquely determined homomorphism

$$g: \quad H_1(\underline{x}, x_d^2; M) \to H_1(\underline{x}; M)$$

with $i_* g = f_*$. Thus, the second row of the last diagram splits, i.e.

$$H_1(\underline{x}, x_d^2; M) \cong H_1(\underline{x}; M) \oplus [0 :_{\overline{M}} x_d^2].$$

By (3) this implies

$$x_d H_1(\underline{x}, x_d^2; M) = 0.$$

Of course,

$$x_d H_1(\underline{x}, x_d; M) = 0 \quad \text{([1], Proposition 1.5)}.$$

Since we can change the order of $x_1, \ldots, x_d$, from the above equalities we get that

$$(x_1, \ldots, x_d) H_1(x_1^{n_1}, \ldots, x_d^{n_d}; M) = 0,$$

for all $n_1, \ldots, n_d \in \{1, 2\}$. Hence by Corollary 3, $x_1, \ldots, x_d$ form a standard s.o.p. of M as required.

Using Theorem 1 and Proposition 3.2 of [8] we obtain a new characterization of Buchsbaum modules. Recall that a finite generating set S of a primary ideal $\mathfrak{a}$ is called an M-basis of $\mathfrak{a}$ if every d element subset of S forms a s.o.p. of M (see [6], Proposition 1.9 for the existence of M-bases of $\mathfrak{a}$).

COROLLARY 5. *M is a Buchsbaum module if and only if $\ell(H_1(x_1, \ldots, x_d; M)) < \infty$ and (one of) the equivalent conditions of Theorem 1 hold for all d elements $x_1, \ldots, x_d$ of an M-basis of the maximal ideal $\mathfrak{m}$.*

Finally we want to give a characterization of quasi-Buchsbaum modules by means of Koszul homology. Recall that M is called a quasi-Buchsbaum module if $\mathfrak{m} H^i_{\mathfrak{m}}(M) = 0$ for all $i < d$ (see [3]).

PROPOSITION 6. *Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. The following conditions are equivalent:*

(i) $\mathfrak{a} H^i_{\mathfrak{m}}(M) = 0$ for all $i < d$.

(ii) There is a s.o.p. $x_1, \ldots, x_d$ of M contained in $\mathfrak{a}^2$ such that

$$\mathfrak{a} H_1(x_1, \ldots . x_d; M) = 0.$$

(iii) $\mathfrak{a} H_p(x_1, \ldots . x_r; M) = 0$ for all sub-s.o.p. $x_1, \ldots, x_r$ of M contained in $\mathfrak{a}^2$ and for all $p > 0$.

PROOF. To prove (i) $\Rightarrow$ (iii) let $x_1, \ldots, x_d$ be a s.o.p. of M contained in $\mathfrak{a}^2$. From the implication (iii) $\Rightarrow$ (ii) of [6], Proposition 3.1 it follows that $x_1, \ldots, x_d$

is a d-sequence. Hence, by [4], Theorem 1.14 $\mathfrak{a}H_p(x_1, \ldots, x_r; M) = 0$ for all $p > 0$. The implication (iii)$\Rightarrow$ (ii) is trivial. If $x_1, \ldots, x_d$ satisfies the condition (ii), then by Lemma 2 ($\mathfrak{b} = \mathfrak{a}^2$), $x_1, \ldots, x_d$ form an $\mathfrak{a}$-weak M-sequence. Hence (i) follows from the implication (i) $\Rightarrow$(iii) of the above mentioned Proposition 13 of [6].

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