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CONNECTEDNESS OF CUBIC METACIRCULANT GRAPHS

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Abstract. In this paper we give necessary and sufficient conditions for cubic metacirculant graphs to be connected.

1. Introduction

Metacirculant graphs were introduced in [1] as an interesting class of vertex-transitive graphs which includes many non-Cayley graphs and which might contain some new non-hamiltonian connected vertex-transitive graphs. But these graphs need not be connected in general. So a natural problem is to find necessary and sufficient conditions for metacirculant graphs to be connected.

We will denote the ring of integers modulo n by Z_n and the multiplicative group of units in Z_n by Z_n^*. Let m and n be two positive integers, $\alpha \in Z_n^*, \mu = \lfloor m/2 \rfloor$ and $S_0, S_1, ..., S_\mu$ be subsets of Z_n satisfying the following conditions:

(1) $0 \notin S_0 = -S_0$,

(2) $\alpha^m S_r = S_r$ for $0 \leq r \leq \mu$,

(3) If m is even, then $\alpha^\mu S_\mu = -S_\mu$.

We define the metacirculant graph $G = MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ to be the graph with vertex-set $V(G) = \{v_y^x | x \in Z_m; y \in Z_n\}$ and edge-set $E(G) = \{v_y^x v_h^{x+r} | 0 \leq r \leq \mu; x \in Z_m; h, y \in Z_n \text{ and } (h-y) \in \alpha^x S_r\}$, where superscripts and subscripts are always reduced modulo m and modulo n , respectively. The subset S_i in the definition is called the i-th symbol of G and the set $V^x = \{v_y^x | y \in Z_n\}$ is called the x-th block of vertices of G .

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The above definition is designed to allow the permutations ρ and τ defined by $\rho(v_y^x) = v_{y+1}^x, \tau(v_y^x) = v_{\alpha y}^{x+1}$ to be automorphisms of G. Thus, G is vertex-transitive. The reader is referred to [1] for basic properties of metacirculant graphs.

In this paper we will restrict ourselves to consider only cubic metacirculant graphs. We will give necessary and sufficient conditions for their connectedness. These conditions may be useful for solving problems concerning connected cubic metacirculants. In particular, we have used one of these conditions in [4] to prove the existence of a Hamilton cycle in every connected cubic metacirculant graph whose number of blocks is divisible by 4. A special case of this result has appeared in [3].

2. The case $S_0 \neq \emptyset$

We assume that all metacirculant graphs $MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ considered in this section are cubic with the condition $S_0 \neq \emptyset$. Components, automorphism groups and Hamilton cycles of these graphs have been considered in [2].

LEMMA 1 [2]. *A metacirculant graph $MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ has $S_0 \neq \emptyset$ and is a cubic graph if and only if one of the following conditions is satisfied:*
1) n is even, $|S_0| = 3, S_i = \emptyset$ for all $i \in \{1, 2, ..., \mu\}$.
2) m is odd, n is even, $|S_0| = 1, |S_i| = 1$ for some $i \in \{1, 2, ..., \mu\}$ and $S_j = \emptyset$ for all $j \neq i$.
3) m is even, n is even, $|S_0| = 1, |S_i| = 1$ for some $i \in \{1, 2, ..., \mu - 1\}$ and $S_j = S_\mu = \emptyset$ for $i \neq j \in \{1, 2, ..., \mu - 1\}$.
4) m is even, n is even, $|S_0| = 1, |S_\mu| = 2$ and $S_i = \emptyset$ for all $i \in \{1, 2, ..., \mu - 1\}$.
5) m is even, $|S_0| = 2, |S_\mu| = 1$ and $S_i = \emptyset$ for all $i \in \{1, 2, ..., \mu - 1\}$.

Let n be a positive integer and S a subset of Z_n such that $0 \notin S = -S$. Then we define the circulant graph $G = C(n, S)$ to be the graph with vertex-set $V(G) = \{v_y | y \in Z_n\}$ and edge-set $E(G) = \{v_y v_h | y, h \in Z_n; (h - y) \in S\}$, where subscripts are always reduced modulo n. The subset S is called the symbol of the circulant graph $C(n, S)$.

LEMMA 2. *Let $G = C(n, S)$ be a circulant graph with the symbol $S = \{\pm s_1, \pm s_2, ..., \pm s_h\}$. Then G is connected if and only if $gcd(s_1, s_2, ..., s_h, n) = 1$.*

PROOF. It is clear that the vertex v_0 is joined by a walk in G only to a vertex v_f of $V(G)$ with f to be a multiple of $e = gcd(s_1, s_2, ..., s_h, n)$. Therefore, if $e > 1$, there exists a vertex of $V(G)$ not joined to v_0 by any walk in G . Thus, G is disconnected.

If $gcd(s_1, s_2, ..., s_h, n) = 1$, then there exist integers $t_1, t_2, ..., t_h, t_{h+1}$ such that $t_1s_1 + t_2s_2 + ... + t_hs_h + t_{h+1}n = 1$. Therefore, for any $y \in Z_n$ we have $yt_1s_1 + yt_2s_2 + ... + yt_hs_h \equiv y (mod\, n)$. We construct now the walks $W_{y,i}, i = 1, 2, ..., h$, as follows. Let $y_1 = 0$ and $y_i = y_{i-1} + yt_{i-1}s_{i-1}$ for $i = 2, 3, ..., h$. Then the initial vertex of $W_{y,i}, i = 1, 2, .., h$, is v_{y_i}. If t_i is positive, then $W_{y,i} = v_{y_i}v_{y_i+s_i}v_{y_i+2s_i}..v_{y_i+yt_is_i}$. If t_i is negative, then $W_{y,i} = v_{y_i}v_{y_i-s_i}v_{y_i-2s_i}..v_{y_i+yt_is_i}$. It is clear that we can join v_0 to v_{y_2} by $W_{y,1}$, v_{y_2} to v_{y_3} by $W_{y,2}$, ..., v_{y_h} to v_y by $W_{y,h}$. Therefore, v_y can be joined to v_0 by a walk in G. It is easy to see now that G is connected.

THEOREM 1. *Let $G = MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ be a cubic metacirculant graph with $S_0 \neq \emptyset$. Then G is connected if and only if one of the following conditions is satisfied:*
(C1) $m = 1, n$ is even, $S_0 = \{s, -s, n/2\}$ with $gcd(s, n/2) = 1$.
(C2) m is odd, n is even, $S_0 = \{n/2\}, S_i = \{s\}$ for some $i \in \{1, 2, ..., \mu\}$ with $gcd(i, m) = 1$ and $gcd(a, n/2) = 1$, where a is $[s(1 + \alpha^i + \alpha^{2i} + ... + \alpha^{(m-1)i})]$ reduced modulo $n, S_j = \emptyset$ for all $j \neq i$.
(C3) m is even, n is even, $S_0 = \{n/2\}, S_i = \{s\}$ for some $i \in \{1, 2, ..., \mu - 1\}$ with gcd(i,m) = 1 and $gcd(a, n/2) = 1$, where a is $[s(1+\alpha^i+\alpha^{2i}+...+\alpha^{(m-1)i})]$ reduced modulo $n, S_j = S_\mu = \emptyset$ for all $i \neq j \in \{1, 2, ..., \mu - 1\}$.
(C4) $m = 2, n$ is even, $S_0 = \{n/2\}, S_1 = \{s, r\}$ with $gcd(s - r, n/2) = 1$.
(C5) $m = 2, S_0 = \{s, -s\}$ with $gcd(s, n) = 1, S_1 = \{r\}$.

PROOF. Let G be a cubic metacirculant graph $MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ with $S_0 \neq \emptyset$. We say that G belongs to Class t if the parameters $m, n, \alpha, S_0, S_1, ..., S_\mu$ of G satisfy Condition t in Lemma 1 ($t = 1, 2, 3, 4, 5$). Thus, by Lemma 1, G must belong to one of Classes 1-5. We will prove that if G is a graph of Class t, then G is connected if and only if Condition (Ct) in Theorem 1 is satisfied.

Let G be a graph of Class 1. If G is connected, then since $S_i = \emptyset$ for all $i \in \{1, 2, 3, .., \mu\}$, we have $m = 1$. Furthermore, since $|S_0| = 3, 0 \notin S_0 = -S_0 (mod\, n)$, $S_0 = \{s, -s, n/2\}$ with $s \not\equiv 0(mod\, n)$. It is clear that in such a case, G is a circulant graph with the symbol S_0. Therefore, by Lemma 2, $gcd(s, n/2, n) = gcd(s, n/2) = 1$. Thus, Condition (C1) is proved. Conversely, if Condition (C1) is satisfied, it is easy to see that G is a connected graph of Class 1.

Let G be a graph of Class 2. From $|S_0| = 1, 0 \notin S_0 = -S_0(mod\, n)$ it follows that $S_0 = \{n/2\}$. Let G be connected and ρ be the automorphism of G defined in Section 1. We define the graph G/ρ as follows. Vertices of G/ρ are blocks $V^0, V^1, ..., V^{m-1}$ of G and a vertex V^a is adjacent to a vertex V^b if and only if $(b - a) \equiv \pm i(mod \quad m)$, where $i \in \{1, 2, ..., \mu\}$ with $S_i \neq \emptyset$. In other words, G/ρ is a circulant graph with m vertices and the symbol $\{\pm i\}$. From the connectedness of G it follows that G/ρ is connected. By Lemma 2, $gcd(i, m) = 1$.

We define now the graph G^0 as follows. The vertex-set of G^0 is V^0 and two vertices v_y^0 and v_h^0 are adjacent in G^0 if and only if $(h - y) \in \{\pm a, n/2\}$, where a is $[s(1 + \alpha^i + \alpha^{2i} + ... + \alpha^{(m-1)i})]$ reduced modulo n. Thus, G^0 is also a circulant graph with the symbol $S = \{\pm a, n/2\}$. It is not difficult to see that two adjacent vertices of G^0 can be joined by a path in G and the connectedness of G implies the connectedness of G^0. Again by Lemma 2, $gcd(a, n/2, n) = gcd(a, n/2) = 1$. Thus, Condition (C2) holds. Conversely, if Condition (C2) is satisfied, then it is easy to verify that G is a connected graph of Class 2.

Let G be a graph of Class 3. By the same arguments used for graphs in Class 2 we can prove the necessary and sufficient condition (C3) for connectedness of graphs in Class 3.

Now let G be a graph of Class 4. If G is connected, then since $S_i = \emptyset$ for all $i \in \{1, 2, ..., \mu - 1\}$, we must have $m = 2$. From $|S_0| = 1$ and $0 \notin S_0 = -S_0(mod\, n)$, it follows that $S_0 = \{n/2\}$. Assume that $S_1 = \{s, r\}$. We define the graph G^0 as follows. The vertex-set of G^0 is V^0 and two vertices v_y^0 and v_h^0 are adjacent in G^0 if and only if $(h-y) \in \{\pm(s-r), n/2\}$. Thus, G^0 is a circulant graph with the symbol $S = \{\pm(s - r), n/2\}$. It is not difficult to see that two adjacent vertices of G^0 can be joined by a path in G and the connectedness of G implies the connectedness of G^0. By Lemma 2, $gcd(s-r,n/2,n) = gcd(s-r,n/2) = 1$ and Condition (C4) is proved. Conversely, if Condition (C4) is satisfied, then it is not difficult to verify that G is a connected graph of Class 4.

Finally, let G be a graph of Class 5. If G is connected, then we again have $m = 2$ because $S_i = \emptyset$ for all $i \in \{1, 2, ..., \mu - 1\}$. Let $S_0 = \{s, -s\}, S_1 = \{r\}$. As shown in [2], the components of G are isomorphic to the generalized Petersen graph $GP(d, k)$, where $k = \alpha$ and d is $n/gcd(s, n)$. Since G is connected, we must have $gcd(s, n) = 1$ and Condition (C5) is proved. Conversely, if Condition (C5) holds, then G is isomorphic to $GP(n, k)$. Since $GP(n, k)$ is connected, the graph G is also connected.

Theorem 1 is completely proved.

3. The case $S_0 = \emptyset$

All metacirculant graphs $G = MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ considered in this section are assumed to be cubic and to have $S_0 = \emptyset$. Because of this only the following cases can happen:

(1) m is even, $S_i = \emptyset$ for all $i \in \{1, 2, ..., \mu - 1\}$ and $|S_\mu| = 3$.

(2) $m > 2$ is even, $|S_i| = 1$ for some $i \in \{1, 2, ..., \mu - 1\}$ and $|S_\mu| = 1$.

First we consider Case (1).

LEMMA 3. *Let $G = MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ be a cubic metacirculant graph such that m is even, $S_0 = S_1 = ... = S_{\mu-1} = \emptyset$ and $S_\mu = \{k, r, s\}$ with k, r, s to be pairwise distinct modulo n. Then G is connected if and only if $m = 2$ and $gcd(r - k, s - k, n) = 1$.*

PROOF. By the definition of metacirculant graphs, it is trivial that such a graph G is disconnected if $m > 2$ is even. So, if G is connected, we must have $m = 2$. We define now the graph G^0. The vertex-set of G^0 is V^0 and two vertices v_y^0 and v_h^0 are adjacent in G^0 if and only if $(h - y) \in \{\pm(r - k), \pm(s - k)\}$. Thus, G^0 is a circulant graph with the symbol $S = \{\pm(r - k), \pm(s - k)\}$. It is not difficult to see that two adjacent vertices of G^0 can be joined by a path in G

and the connectedness of G implies the connectedness of G^0. By Lemma 2, $gcd(r-k, s-k, n) = 1$. Conversely, if $m = 2$ and $gcd(r-k, s-k, n) = 1$, then G^0 is connected by Lemma 2. Therefore G is connected.

Now we consider Case (2).

LEMMA 4. *Let $G = MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ be a cubic metacirculant graph such that $m > 2$ is even, $S_0 = \emptyset$, $S_i = \{s\}$ for some $i \in \{1, 2, ..., \mu-1\}$, $S_j = \emptyset$ for all $i \neq j \in \{1, 2, ..., \mu-1\}$ and $S_\mu = \{k\}$ $(s, k \in Z_n)$. Then*

(a) If G is connected, then either i is odd and $gcd(i, m) = 1$ or i is even, μ is odd and $gcd(i, m) = 2$.

(b) If i is odd and $gcd(i, m) = 1$, then G is isomorphic to the cubic metacirculant graph $G' = MC(m, n, \alpha', S'_0, S'_1, ..., S'_\mu)$ with $\alpha' = \alpha^i, S'_0 = \emptyset, S'_1 = \{s\}, S'_2 = ... = S'_{\mu-1} = \emptyset$ and $S'_\mu = \{k\}$.

(c) If i is even, μ is odd, $gcd(i, m) = 2$ and $i = 2^t i'$ with $t \geq 1$ and i' odd, then G is isomorphic to the cubic metacirculant graph $G'' = MC(m, n, \alpha'', S''_0, S''_1, S''_2, ..., S''_\mu)$ with $\alpha'' = \alpha^{i'}$, $S''_0 = S''_1 = ... = S''_{2^t-1} = \emptyset$, $S''_{2^t} = \{s\}, S''_{s^t+1} = ... = S''_{\mu-1} = \emptyset$, $S''_\mu = \{k\}$.

PROOF. (a) Let ρ be the automorphism of the graph G defined in Section 1, i.e., $\rho(v_y^x) = v_{y+1}^x$ for all $v_y^x \in V(G)$. We define now the graph G/ρ as follows. Vertices of G/ρ are blocks $V^0, V^1, .., V^{m-1}$ of G. A vertex V^a is adjacent to a vertex V^b if and only if $(b-a) \in \{\pm i, \mu\}$, where $S_i = \{s\}$ is a nonempty symbol of G with $i \in \{1, 2, ..., \mu-1\}$. Thus, G/ρ is a circulant graph of order m and with the symbol $S = \{\pm i, \mu\}$. Since G is connected, the graph G/ρ is also connected. By Lemma 2, $gcd(i, \mu, m) = gcd(i, \mu) = 1$. Therefore, either i is odd and $gcd(i, m) = 1$ or i is even, μ is odd and $gcd(i, m) = 2$. Assertion (a) is proved.

(b) Let i be odd and $gcd(i, m) = 1$. Then $0, i, 2i, 3i, ..., (m-1)i$ are all distinct integers modulo m. Moreover, since i is odd, we have

$$\mu i \equiv \mu (mod \quad m). \tag{3.1}$$

Let $\varphi : V(G) \longrightarrow V(G') : v_y^{xi} \longrightarrow v_y^x$. It is not difficult to verify that φ is a bijection from $V(G)$ on $V(G')$. Let $v_y^x v_h^{x+r}$ be an edge of G and $x \equiv$

$x'i(mod \quad m)$. By definition, we have either $r = i$ and $(h-y) \equiv \alpha^x s(mod\, n)$ or $r = \mu$ and $(h-y) \equiv \alpha^x k(mod\, n)$.

If $r = i$ and $(h-y) \equiv \alpha^x s(mod\, n)$, then $\varphi(v_y^x)\varphi(v_h^{x+i}) = \varphi(v_y^{x'i})\varphi(v_h^{x'i+i})$ $= v_y^{x'} v_h^{x'+1}$. We have $(h-y) = \alpha^x s = \alpha^{x'i} s = (\alpha^i)^{x'} s(mod\, n)$. Thus, $\varphi(v_y^x)\varphi(v_h^{x+i})$ is an edge of G'. Let now $r = \mu$ and $(h-y) \equiv \alpha^x k(mod\, n)$. By (3.1) we have $r = \mu \equiv \mu i(mod \quad m)$. Therefore, $\varphi(v_y^x)\varphi(v_h^{x+\mu}) = \varphi(v_y^{x'i})$ $\varphi(v_h^{x'i+\mu i}) = v_y^{x'} v_h^{x'+\mu}$. From $(h-y) = \alpha^{x'i} k = (\alpha^i)^{x'} k(mod\, n)$, it follows that $\varphi(v_y^x)\varphi(v_h^{x+\mu})$ is an edge of G'.

Conversely, let $v_y^x v_h^{x+r}$ be an edge of G'. Then we have either $r = 1$ and $(h-y) = (\alpha^i)^x s$ or $r = \mu$ and $(h-y) = (\alpha^i)^x k$. If $r = 1$ and $(h-y) = (\alpha^i)^x s$, then $\varphi^{-1}(v_y^x)\varphi^{-1}(v_h^{x+1}) = v_y^{xi} v_h^{(x+1)i} = v_y^{xi} v_h^{xi+i}$. From $(h-y) = (\alpha^i)^x s = (\alpha^{ix})s$, it follows that $\varphi^{-1}(v_y^x)\varphi^{-1}(v_h^{x+1})$ is an edge of G. Let now $r = \mu$ and $(h-y) = (\alpha^i)^x k$. by (3.1) we have $r = \mu \equiv \mu i(mod \quad m)$. Therefore $\varphi^{-1}(v_y^x)\varphi^{-1}(v_h^{x+\mu}) = v_y^{xi} v_h^{(x+\mu)i} = v_y^{xi} v_h^{xi+\mu}$. Since $(h-y) = (\alpha^i)^x k = (\alpha^{ix})k$, $\varphi^{-1}(v_y^x)\varphi^{-1}(v_h^{x+\mu})$ is again an edge of G. Thus, φ is an isomorphism from G on G' and the assertion (b) is proved.

(c) Let i be even, μ be odd and $gcd(i,m) = 2$. Then $0, i, 2i, 3i, ..., (\mu-1)i$ are all distinct even integers modulo m. Let $i = 2^t i'$ with $t \geq 1$ and i' odd. Then $\mu i' \equiv \mu(mod \quad m)$. Let $\psi : V(G) \longrightarrow V(G') : v_y^{xi} \longrightarrow v_y^{x2^t}$ and $v_y^{xi+\mu} \longrightarrow v_y^{x2^t+\mu}$. As in (b) we can now verify that ψ is a bijection from $V(G)$ onto $V(G')$ which preserves adjacency. The detailed verifications are left to the reader.

The proof of Lemma 4 is now complete.

Let us now recall and give some definitions needed for the proofs of further results.

Let $W_1 = v_1 v_2 v_3 ... v_{r-2} v_{r-1} v_r$ be a walk of a graph $\Gamma = (V, E)$. Then the inverse walk of W_1 denoted by W_1^{-1} is $W_1^{-1} = v_r v_{r-1} v_{r-2} ... v_3 v_2 v_1$. Let $W_2 = v_r v_{r+1} ... v_{h-1} v_h$ be another walk of Γ such that the initial vertex v_r of W_2 coincides with the terminal vertex of W_1. Then the concatenation of W_1 and W_2 denoted by $W_1 * W_2$ is defined to be the walk $W_1 * W_2 = v_1 v_2 v_3 ... v_{r-2} v_{r-1} v_r v_{r+1}$ $... v_{h-1} v_h$.

Let $G = MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ be a metacirculant graph. Then an edge e of G is called an S_i-edge if it is $v_y^x v_h^{x+i}$ with $(h-y) \in \alpha^x S_i$ or $v_y^x v_h^{x-i}$ with

$(y-h) \in \alpha^{x-i}S_i$. If all edges of a walk W are S_i-edges, then W is called an S_i-walk.

Now let $G = MC(m,n,\alpha,S_0,S_1,...,S_\mu)$ be a cubic metacirculant graph such that $m > 2$ is even, $S_0 = S_1 = ... = S_{i-1} = \emptyset, S_i = \{s\}, S_{i+1} = ... = S_{\mu-1} = \emptyset, S_\mu = \{k\}$, and let $v_y^x v_h^u$ be an edge of G. Then the subscript h is completely determined by x, y, u. From now on we will write the edge $v_y^x v_h^u$ simply as $v_y^x v^u$ whenever h is clearly determined by x, y, u. Let $W = v_y^x v^{u(1)} v^{u(2)} ... v^{u(r)} v_h^x$ be a walk in G that starts and terminates at vertices of the same block V^x. Then the difference $(h-y)$ reduced modulo n is called the change (in subscripts) of W and denoted by $ch(W)$. If W has the form $W = W_1 * Q * Q^{-1} * W_2$, then it is clear that $ch(W)$ and $ch(W')$, where $W' = W_1 * W_2$, are the same. From now on we will not distinguish these walks and will write simply $W = W'$. If a walk W is an S_i-walk and has the form $W = v_y^x v^{x+i} v^{x+2i} ... v_h^{x+ui}$, then we say that it has a positive orientation. Similarly, if $W = v_y^x v^{x-i} v^{x-2i} ... v_h^{x-ui}$, then we say that it has a negative orientation. From a vertex v_y^x we can go along an S_i-edge to v_z^{x+i} or v_w^{x-i}. When we go from v_y^x to v_z^{x+i}, we say that we go in the positive direction. When we go from v_y^x to v_w^{x-i}, we say that we go in the negative direction.

LEMMA 5. *Let $G = MC(m,n,\alpha,S_0,S_1,...,S_\mu)$ be a cubic metacirculant graph such that $m > 2$ is even, $S_0 = \emptyset, S_1 = \{s\}, S_2 = ... = S_{\mu-1} = \emptyset, S_\mu = \{k\}$, If a walk W of G joins two vertices of V^0, then $ch(W)$ is a multiple in Z_n of $[k - s(1+\alpha+\alpha^2+...+\alpha^{\mu-1})]$ reduced modulo n.*

PROOF. We will prove this lemma by induction on the number of S_μ-edges contained in a walk W joining v_a^0 to v_d^0.

Assume first that W contains no S_μ-edges. Denote $P(v_y^0, v_h^0) = v_y^0 v^1 v^2 v^3 ... v^{2\mu-2} v^{2\mu-1} v_h^0$. By exchanging the role of a and d and by deleting subwalks of the type $Q * Q^{-1}$ if necessary, we may assume that

$$W = P(v_a^0, v_{b(1)}^0) * P(v_{b(1)}^0, v_{b(2)}^0) * ... * P(v_{b(j)}^0, v_d^0). \tag{3.2}$$

For the path $P(v_y^0, v_h^0)$ we have

$$h \equiv y + s + \alpha s + ... + \alpha^{2\mu-3}s + \alpha^{2\mu-2}s + \alpha^{2\mu-1}s \equiv$$
$$y - (1+\alpha^{\mu})[k - s(1+\alpha+\alpha^2+...+\alpha^{\mu-1})](mod \quad n). \qquad (3.3)$$

Let p be $[k - s(1+\alpha+\alpha^2+...+\alpha^{\mu-1})]$ reduced modulo n. Then from (3.3) we see that $ch(P(v_y^0, v_h^0))$ is a multiple of p. From (3.2) it follows that $ch(W)$ is also a multiple of p.

Assume now that W contains an S_μ-edge. By deleting subwalks of the type $Q * Q^{-1}$ if necessary, we may assume that $W = W_1 * W_2 * W_3$, where W_1 and W_3 are walks joining v_a^0 to v_b^0 and v_c^0 to v_d^0 respectively and have no S_μ-edges, W_2 has one of the following forms:

1) $W_2 = v_b^0 v^1 v^2 ... v^j v^{j+\mu} v^{j+\mu+1} ... v^{2\mu-1} v_c^0$,

2) $W_2 = v_b^0 v^1 v^2 ... v^j v^{j+\mu} v^{j+\mu-1} v^{j+\mu-2} ... v^1 v_c^0$,

3) $W_2 = v_b^0 v^{2\mu-1} v^{2\mu-2} ... v^j v^{j+\mu} v^{j+\mu-1} v^{j+\mu-2} ... v^1 v_c^0$,

4) $W_2 = v_b^0 v^{2\mu-1} v^{2\mu-2} ... v^j v^{j+\mu} v^{j+\mu+1} v^{j+\mu+2} ... v^{2\mu-1} v_c^0$,

and only v_b^0 and v_c^0 are vertices of V^0 contained in W_2.

If the subwalk of W_2 from v_b^0 to v^j has length $l \leq \mu$, then in these cases we respectively have

1) $c \equiv b + s + \alpha s + ... + \alpha^{j-1}s + \alpha^j k + \alpha^{j+\mu}s + ... + \alpha^{2\mu-1}s$

$\equiv b + (\alpha^j - 1 - \alpha^\mu)[k - s(1+\alpha+\alpha^2+...+\alpha^{\mu-1})](mod \quad n)$.

2) $c \equiv b + s + \alpha s + ... + \alpha^{j-1}s + \alpha^j k - \alpha^{j+\mu-1}s - \alpha^{j+\mu-2}s - ... - s$

$\equiv b + \alpha^j[k - s(1+\alpha+\alpha^2+...+\alpha^{\mu-1})](mod \quad n)$.

3) $c \equiv b - \alpha^{2\mu-1}s - \alpha^{2\mu-2} - ... - \alpha^j s + \alpha^j k - \alpha^{j+\mu-1}s - \alpha^{j+\mu-2}s - ... - \alpha s - s$

$\equiv b + (1 + \alpha^\mu - \alpha^{j+\mu})[k - s(1+\alpha+\alpha^2+...+\alpha^{\mu-1})](mod \quad n)$.

4) $c \equiv b - \alpha^{2\mu-1}s - \alpha^{2\mu-2} - ... - \alpha^j s + \alpha^j k + \alpha^{j+\mu}s + \alpha^{j+\mu+1}s + \alpha^{j+\mu+2}s + ... + \alpha^{2\mu-2}s + \alpha^{2\mu-1}s$

$\equiv b - \alpha^{j+\mu}[k - s(1+\alpha+\alpha^2+...+\alpha^{\mu-1})](mod \quad n)$.

In all cases, $ch(W_2) \equiv (c-b)(mod\, n)$ is a multiple of p. The same result can be obtained if $\mu < l < m$. On the other hand, $ch(W_1)$ and $ch(W_3)$ have been proved to be multiples of p. Therefore, $ch(W) \equiv (ch(W_1) + ch(W_2) + ch(W_3))(mod\, n)$ is also a multiple of p. Thus, the assertion of Lemma 5 is true when the number of S_μ-edges contained in W is 0 and 1.

Assume that the assertion of Lemma 5 is true when the number of S_μ-edges contained in W is r. We shall prove that it is also true when the number of S_μ-edges contained in W is $(r+1)$. We represent W as the concatenation of two subwalks W_1 and W_2 of W, i.e.,$W = W_1 * W_2$, such that W_1 contains r S_μ-edges and W_2 contains only one S_μ -edge. Let W_1 terminate at some v_y^x and let $Q = v_y^x v^{x+1} v^{x+2} ... v^{2\mu-1} v_h^0$. Then $W = W_1 * W_2 = (W_1 * Q) * (Q^{-1} * W_2)$, the subwalk $W_1 * Q$ joins two vertices of V^0 and contains only r S_μ-edges, the subwalk $Q^{-1} * W_2$ joins two vertices of V^0 and contains only one S_μ-edge. By the induction hypothesis, $ch(W_1 * Q)$ and $ch(Q^{-1} * W_2)$ are multiples of p. Therefore, $ch(W)$ is also a multiple of p because $ch(W) \equiv (ch(W_1 * Q) + ch(Q^{-1} * W_2))(mod\, n)$. Thus, the assertion of Lemma 5 is true for every walk in G joining two vertices of V^0.

LEMMA 6. *Let $G = MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ be a cubic metacirculant graph such that $m > 2$ is even, $S_0 = \emptyset, S_1 = \{s\}, S_2 = ... = S_{\mu-1} = \emptyset, S_\mu = \{k\}$. Then G is connected if and only if $gcd(p, n) = 1$, where p is $[k - s(1 + \alpha + \alpha^2 + ... + \alpha^{\mu-1})]$ reduced modulo n.*

PROOF. Let $gcd(p, n) > 1$. By Lemma 5, among the vertices of V^0, v_0^0 is joined by a walk in G only to a vertex v_f^0 with f to be a multiple of p. Since $gcd(p, n) > 1$, there exists a vertex in V^0 which is not joined to v_0^0 by any walk in G. So G is disconnected.

Conversely, let $gcd(p, n) = 1$. Denote $R(v_y^0, v_h^0) = v_y^0 v^\mu v^{\mu-1} v^{\mu-2} ...$ $... v^2 v^1 v_h^0$. It is easy to see that the change of $R(v_y^0, v_h^0)$ is exactly p. So, we can join v_0^0 to v_p^0 by $R(v_0^0, v_p^0), v_p^0$ to v_{2p}^0 by $R(v_p^0, v_{2p}^0), v_{2p}^0$ to v_{3p}^0 by $R(v_{2p}^0, v_{3p}^0), ...$ Therefore, every vertex of V^0 can be joined to v_0^0 by a walk in G because $gcd(p, n) = 1$. Now we can easily see that G is connected.

LEMMA 7. *Let $G = MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ be a cubic metacirculant graph such that $m > 2$ is even , $\mu = m/2$ is odd, $S_0 = S_1 = ... = S_{2^t-1} = \emptyset$ with $t \geq 1, S_{2^t} = \{s\}$, $S_{2^t+1} = ... = S_{\mu-1} = \emptyset, S_\mu = \{k\}$. If a walk W of G joins two vertices of V^0, then $ch(W)$ is a multiple in Z_n of $[k(1 + \alpha + \alpha^2 + ... + \alpha^{2^t-1}) - s(1 + \alpha + \alpha^2 + ... + \alpha^{\mu-1})]$ reduced modulo n.*

PROOF. Let G be such a cubic metacirculant graph and let W be a walk in G joining a vertex v_a^0 to a vertex v_d^0. We will prove this lemma by induction on the number of S_μ-edges contained in a walk W joining v_a^0 to v_d^0.

We note that the number of S_μ-edges in W must be even because only S_μ-edges join vertices of blocks with even superscripts to vertices of blocks with odd superscripts. We divide the proof into several steps.

(a) Denote $P(v_y^0, v_h^0) = v_y^0 v^{2^t} v^{2.2^t} v^{3.2^t} ... v^{(\mu-1)2^t} v_h^0$. Then

$$h \equiv (y + s + \alpha^{2^t} s + \alpha^{2.2^t} s + ... + \alpha^{(\mu-1)2^t} s)(mod\ n). \tag{3.4}$$

From $gcd(2^t, m) = 2$ it follows that $0, 2^t, 2.2^t, 3.2^t, ..., (\mu - 1)2^t$ are all even integers modulo m. Therefore,

$$s + \alpha^{2^t} s + \alpha^{2.2^t} s + ... + \alpha^{(\mu-1)2^t} s \equiv$$

$$s + \alpha^2 s + \alpha^4 s + ... + \alpha^{2\mu-2} s \equiv$$

$$s(1 + \alpha + \alpha^2 + ... + \alpha^{(\mu-1)})(1 - \alpha + \alpha^2 - ... + \alpha^{(\mu-1)})(mod\ n). \tag{3.5}$$

By the definition of metacirculant graphs we have $\alpha^\mu k \equiv -k(mod\ n) \leftrightarrow (\alpha^\mu + 1)k \equiv 0(mod \quad n)$. Therefore, we have

$$\begin{aligned} 0 &\equiv k((\alpha^\mu + 1)(1 + \alpha^2)(1 + \alpha^{2^2})...(1 + \alpha^{2^{t-1}}) \\ &\equiv k(1 + \alpha + \alpha^2 + ... + \alpha^{2^t - 1})(1 - \alpha + \alpha^2 - ... + \alpha^{(\mu-1)})(mod\ n). \end{aligned} \tag{3.6}$$

From (3.4),(3.5),(3.6) it follows that $h \equiv y - k(1 + \alpha + \alpha^2 + ... + \alpha^{2^t-1})(1 - \alpha + \alpha^2 - ... + \alpha^{(\mu-1)}) + s(1 + \alpha + \alpha^2 + ... + \alpha^{\mu-1})(1 - \alpha + \alpha^2 - ... + \alpha^{(\mu-1)}) \equiv y - k(1 - \alpha + \alpha^2 - ... + \alpha^{(\mu-1)})[k(1 + \alpha + \alpha^2 + ... + \alpha^{2^t-1}) - s(1 + \alpha + \alpha^2 + ... + \alpha^{(\mu-1)})](mod\ n)$.

Let q be $[k(1 + \alpha + \alpha^2 + ... + \alpha^{2^t-1}) - s(1 + \alpha + \alpha^2 + ... + \alpha^{(\mu-1)})]$ reduced modulo n. From the above calculations we see that $ch(P(v_y^0, v_h^0))$ is a multiple of q.

(b) Assume that a walk W joining a vertex v_a^0 to a vertex v_d^0 of V^0 contains no S_μ-edges. Then by exchanging the role of a and d and by deleting subwalks of the type $Q * Q^{-1}$ if necessary, we may assume that W has the form $W = P(v_a^0, v_{b(1)}^0) * P(v_{b(1)}^0, v_{b(2)}^0) * ... * P(v_{b(j)}^0, v_d^0)$. Therefore from (a) it follows that $ch(W)$ is a multiple of q.

(c) By the same arguments used above we can also prove that if a walk Z joining a vertex v_y^x to a vertex v_h^x of V^x contains no S_μ-edges, then $ch(Z)$ is a multiple of q.

(d) Assume now that a walk W joining v_a^0 to v_d^0 contains only two S_μ-edges. Let W_2 be the subwalk of W such that it contains both S_μ-edges of W and only the endvertices v_b^0 and v_c^0 are vertices from V^0. Let W_1 and W_3 be the subwalks of W joining respectively v_a^0 to v_b^0 and v_c^0 to v_d^0. Then it is clear that $W = W_1 * W_2 * W_3$. Since W_1 and W_3 contain no S_μ-edges, by (b) $ch(W_1)$ and $ch(W_3)$ are multiples of q. Therefore, $ch(W)$ is a multiple of q if and only if $ch(W_2)$ is a multiple of q.

(e) We construct the subwalks Z_1, Z_2, Z_3 and Z_4 of W_2 as follows. The subwalk Z_1 starts at v_b^0 and terminates with the first S_μ-edge $v_y^{x2^t} v_{y'}^{x2^t+\mu}$ contained in W_2. Start Z_2 at $v_{y'}^{x2^t+\mu}$ and go along W_2 until reaching a vertex $v_z^{x2^t+\mu}$ which is the last vertex of W_2 with superscript $x2^t+\mu$. This vertex $v_z^{x2^t+\mu}$ is the terminal vertex of Z_2. The walk Z_3 starts at $v_z^{x2^t+\mu}$ and terminates with the second S_μ-edge $v_w^{u2^t+\mu} v_{w'}^{u2^t}$ contained in W_2. Finally, start Z_4 at $v_{w'}^{u2^t}$ and terminate it at v_c^0.

Thus, by the construction $W_2 = Z_1 * Z_2 * Z_3 * Z_4$. Moreover, Z_2 is a walk joining two vertices of the same block $V^{x2^t+\mu}$ and having no S_μ-edges. $Ch(Z_2)$ is a multiple of q by (c). So $ch(W_2)$ is a multiple of q if and only if $[ch(Z_1) + ch(Z_3) + ch(Z_4)]$ reduced modulo n is a multiple of q.

(f) By the construction, Z_4 is an S_{2^t}-walk. All edges but the last one of Z_1 and of Z_3 are also S_{2^t}-edges. The orientations of S_{2^t}-portions of Z_1 and Z_3 and the orientation of Z_4 may be positive or negative. But we can verify that in all cases $[ch(Z_1) + ch(Z_3) + ch(Z_4)]$ reduced modulo n is always a multiple of q. Here we will demonstrate the calculations only for the case when the S_{2^t}-portions of Z_1 and Z_3 and Z_4 have a positive orientation. The verifications of all remaining cases are left to the reader.

Let

$$Z_1 = v_b^0 v^{2^t} v^{2.2^t} v^{3.2^t} \dots v_y^{x2^t} v_{y'}^{x2^t+\mu},$$

$$Z_3 = v_z^{x2^t+\mu} v^{(x+1)2^t+\mu} v^{(x+2)2^t+\mu} \dots v^{u2^t+\mu} v_{w'}^{u2^t},$$

$$Z_4 = v_{w'}^{u2^t} v^{(u+1)2^t} v^{(u+2)2^t} \dots v^{(\mu-1)2^t} v_c^0.$$

Then

$$[ch(Z_1) + ch(Z_3) + ch(Z_4)] \equiv$$
$$(s + \alpha^{2^t} s + \alpha^{2.2^t} s + ... + \alpha^{(x-1)2^t} s + \alpha^{x2^t} k) +$$
$$(\alpha^{x2^t+\mu} s + \alpha^{(x+1)2^t+\mu} s + ... + \alpha^{(u-1)2^t+\mu} s + \alpha^{u2^t+\mu} k) +$$
$$(\alpha^{u2^t} s + \alpha^{(u+1)2^t} s + ... + \alpha^{(\mu-1)2^t} s) \equiv$$
$$(s + \alpha^{2^t} s + \alpha^{2.2^t} s + ... + \alpha^{(\mu-1)2^t} s) +$$
$$(\alpha^{x2^t} k + \alpha^{x2^t+\mu} s + \alpha^{(x+1)2^t+\mu} s + ... + \alpha^{(u-1)2^t+\mu} s + \alpha^{u2^t+\mu} k$$
$$-\alpha^{(u-1)2^t} s - \alpha^{(u-2)2^t} s - ... - \alpha^{x2^t} s)(mod\ n)$$

By the calculations in (a), the first sum $(s + \alpha^{2^t} s + \alpha^{2.2^t} s + ... + \alpha^{(\mu-1)2^t} s)$ is a muliple of q. Consider the second sum. We have

$$(\alpha^{x2^t} k + \alpha^{x2^t+\mu} s + \alpha^{(x+1)2^t+\mu} s + ... + \alpha^{(u-1)2^t+\mu} s + \alpha^{u2^t+\mu} k - \alpha^{(u-1)2^t} s -$$
$$-\alpha^{(u-2)2^t} s - ... - \alpha^{x2^t} s)$$
$$\equiv \alpha^{x2^t}(1-\alpha)(1 + \alpha^{2^t} + \alpha^{2.2^t} + ... + \alpha^{(u-x-1)2^t})$$
$$[k(1 + \alpha + \alpha^2 + ... + \alpha^{2^t-1}) - s(1 + \alpha + \alpha^2 + ... + \alpha^{(\mu-1)})](mod\ n).$$

i.e., the second sum is also a multiple of q. Thus, $[ch(Z_1) + ch(Z_3) + ch(Z_4)]$ reduced modulo n is a multiple of q in this case.

(g) From (d)-(f) it follows that if a walk W joining two vertices v_a^0 and v_d^0 of V^0 contains only two S_μ-edges, then $ch(W)$ is a multiple of q.

(h) Suppose now that the induction hypothesis is true when the number of S_μ-edges contained in a walk joining two vertices of V^0 is $2r$. Let W be a walk joining two vertices v_a^0 and v_d^0 of V^0 and having $2(r+1)$ S_μ-edges. We represent W as the concatenation $W_1 * W_2$ of two walks W_1 and W_2 such that W_1 contains $2r$ S_μ-edges and W_2 contains only two S_μ-edges. Let v_y^x be the terminal vertex of W_1. Then x must be even. Let Q be any walk joining v_y^x to any vertex of V^0 such that Q contains no S_μ-edges. Such a walk Q can be always found, for example, $Q = v_y^x v^{x+2^t} v^{x+2.2^t} ... v^{(\mu-1)2^t} v_h^0$. Then $W = W_1 * W_2 = (W_1 * Q) * (Q^{-1} * W_2)$, the walk $(W_1 * Q)$ joins two vertices of V^0 and has $2r$ S_μ-edges, and the walk $(Q^{-1} * W_2)$ also joints two vertices of V^0 and has only two S_μ-edges. By the induction hypothesis $ch(W_1 * Q)$ and $ch(Q^{-1} * W_2)$ are multiples of q. Since $ch(W) \equiv [ch(W_1 * Q) + ch(Q^{-1} * W_2)](mod\, n)$, $ch(W)$ is also a multiple of q. Thus, the induction hypothesis is true for a walk joining two vertices of V^0 and having $2(r+1)$ S_μ-edges.

LEMMA 8. *Let $G = MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ be a cubic metacirculant graph such that $m > 2$ is even , $\mu = m/2$ is odd, $S_0 = S_1 = ... = S_{2^t-1} = \emptyset$ with $t \geq 1, S_{2^t} = \{s\}$, $S_{2^t+1} = ... = S_{\mu-1} = \emptyset, S_\mu = \{k\}$. Then G is connected if and only if $gcd(q, n) = 1$, where q is $[k(1+\alpha+\alpha^2+...+\alpha^{2^t-1}) - s(1+\alpha+\alpha^2+...+\alpha^{\mu-1})]$ reduced modulo n.*

PROOF. Let $gcd(q, n) > 1$. By Lemma 7, among the vertices of V^0, v_0^0 is joined by a walk in G only to a vertex v_f^0 with f to be a multiple of q in Z_n. Since $gcd(q, n) > 1$, there exists a vertex in V^0 which is not joined to v_0^0 by any walk in G. So G is disconnected.

Conversely, let $gcd(q, n) = 1$. First we construct the walk $R(v_0^0)$ as follows. Start $R(v_0^0)$ with the S_μ-edge $v_0^0 v_k^\mu$ and proceed it in the negative direction until reaching the first vertex v_y^j with the superscript j in $\{0, 1, ..., 2^t - 1\}$. We note that since μ is odd, j must be odd. Therefore, $gcd(j, 2^t) = 1$. Proceed now with the S_μ-edge $v_y^j v^{j+\mu}$ and then turn in the negative direction until reaching the following vertex v_z^u with u in $\{0, 1, ..., 2^t - 1\}$. It is easy to see that $u \equiv 2j (mod \quad 2^t)$. Similarly, proceed now with the S_μ-edge $v_z^u v^{u+\mu}$ and then turn in the negative direction until reaching the following vertex v_w^r with r in $\{0, 1, ..., 2^t - 1\}$. As before, it is not difficult to see that $r \equiv 3j (mod \quad 2^t)$. Continue this procedure until reaching first a vertex v_h^0 of V^0. This vertex v_h^0 is then the terminal vertex of $R(v_0^0)$.

Now from $R(v_0^0)$ we construct for every $e \in Z_n$ the walk $R(v_e^0)$ by replacing every vertex v_y^x of $R(v_0^0)$ by v_{y+e}^x. From $gcd(j, 2^t) = 1$, it follows that $j, 2j, 3j, ...$ reduced modulo 2^t run through all numbers $0, 1, 2, ..., 2^t - 1$. Therefore, it is not difficult to see that $ch(R(v_e^0))$ is exactly q. We can join v_0^0 to v_q^0 by $R(v_0^0)$, v_q^0 to v_{2q}^0 by $R(v_q^0)$, v_{2q}^0 to v_{3q}^0 by $R(v_{2q}^0)$,... From $gcd(q, n) = 1$ it follows that $0, q, 2q, 3q, ..., (n-1)q$ are all integers modulo n. Thus, we can join v_0^0 to every vertex of V^0 by a walk in G. Now it is not difficult to see that G is connected.

From Lemmas 3, 4, 6, 8 we immediately obtain the following.

THEOREM 2. *Let $G = MC(m, n, \alpha, S_0, S_1, ..., S_\mu)$ be a cubic metacirculant graph with $S_0 = \emptyset$. Then G is connected if and only if one of the following conditions is met :*

(D1) $m = 2, S_1 = \{k, r, s\}$ with k, r, s to be pairwise distinct modulo n and $gcd(r - k, s - k, n) = 1$.

(D2) $m > 2$ is even , $S_1 = ... = S_{i-1} = \emptyset$ with i odd and $gcd(i, m) = 1$, $S_i = \{s\}$, $S_{i+1} = ... = S_{\mu-1} = \emptyset, S_\mu = \{k\}$ and $gcd(p, n) = 1$. where p is $[k - s(1 + \alpha^i + \alpha^{2i} + ... + \alpha^{(\mu-1)i})]$ reduced modulo n.

(D3) $m > 2$ is even , $\mu = m/2$ is odd, $S_1 = ... = S_{i-1} = \emptyset$ with i even and $gcd(i, m) = 2, S_i = \{s\}$, $S_{i+1} = ... = S_{\mu-1} = \emptyset$, $S_\mu = \{k\}$ and $gcd(q, n) = 1$, where $i = 2^t i'$ with $t \geq 1$ and i' odd and q is $[k(1 + \alpha^{i'} + \alpha^{2i'} + ... + \alpha^{(2^t-1)i'}) - s(1 + \alpha^{i'} + \alpha^{2i'} + ... + \alpha^{(\mu-1)i'})]$ reduced modulo n.

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