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NECESSARY OPTIMALITY CONDITIONS FOR OPTIMAL CONTROL PROBLEMS GOVERNED BY HEMIVARIATIONAL INEQUALITIES

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Abstract. In the present paper an optimal control problem governed by hemivariational inequalities together with its modified one is investigated and several necessary conditions for optimality are established. An illustrative example is also given.

1. Introduction

Let φ and J be functionals defined on Y and $Y \times W$, respectively, where Y and W are finite–dimensional Euclidean spaces such that $W \subset Y^*$. Let Ω be a closed subset of W, A a continuous linear map from Y into Y^*. We shall be concerned with the following optimal control problem :

$$(P) \quad \begin{cases} \text{minimize} & J(y,u) \ , \\ \text{subject to} & u \in Ay + \partial\varphi(y) \ , & (1.1) \\ & u \in \Omega \ , & (1.2) \end{cases}$$

where ∂ is the generalized gradient of F.H. Clarke.

In order to study a lot of mechanical laws P.D. Panagiotopoulos has introduced [6] the nonconvex superpotentials based on the concept of Clarke's generalized gradient. The nonconvex superpotential leads to hemivariational inequalities and to corresponding nonconvex, nonsmooth optimization problems such as aforementioned Problem (P). It should be noted that the constraint (1.1) can be expressed as a hemivariational inequality which is nonmonotone. Namely, since the Clarke derivative of φ at y with respect to v_0 can be expressed in the form $\varphi^0(y;v) = \max \{< \zeta, v > \mid \zeta \in \partial\varphi(y)\}$, it folows that (1.1) is equivalent to the following

$$< u - Ay, v > \leq \ \max \{< \zeta, v > \mid \zeta \in \partial\varphi(y)\} \ (\forall v \in Y) \ ,$$

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which may be rewritten as follows

$$\max \{< \zeta - u + Ay, v > \mid \zeta \in \partial\varphi(y)\} \geq 0 \quad (\forall v \in Y).$$

For the problem governed by a strongly monotone variational inequality it is worth to mention some necessary conditions for optimality of Shuzhong Shi in [8]. The existence of solutions of (P) is studied in [2] by Haslinger and Panagiotopoulos by means of a regularization-approximation method. The investigation of optimality conditions for (P) and its modifications is a complicated problem which remained open until 1989. In 1990 J.V. Outrata [5] found out some necessary conditions for the case J being differentiable in both variables and φ being of a special form.

The aim of this paper is to find necessary optimality conditions for (P) for the general case in which J is nondifferentiable. The paper is organized as follows. In Section 2 we give a necessary optimality condition for the reduced problem (without the constraint (1.2)). Section 3 deals with the whole problem. Section 4 is devoted to the discussion of the lower semicontinuous case and Section 5 gives some necessary conditions for a modified problem. Finally, an illustrative example is derived in Section 6.

2. A necessary optimality condition for the reduced problem of (P)

Let f be a lower semicontinuous function from Y to $\overline{R}$, where $\overline{R} = R \cup \{\pm\infty\}$. For $y_0 \in Y$, the sets of subgradients $\partial f(y_0)$ and $\partial^\infty f(y_0)$ are defined as follows

$$y^* \in \partial f(y_0) \Longleftrightarrow (y^*, -1) \in N_{\mathrm{epi}f}(y_0, f(y_0)) \ ,$$
$$y^* \in \partial^\infty f(y_0) \Longleftrightarrow (y^*, 0) \in N_{\mathrm{epi}f}(y_0, f(y_0)) \ ,$$

which are called the Clarke set of subgradients and the set of singular subgradients of f at y_0, respectively (see [7]). Here, $N_{\mathrm{epi}f}(y_0, f(y_0))$ is the normal cone to epi$f$ at $(y_0, f(y_0))$.

It is worth noticing that if f is finite at y_0, then

$$\partial f(y_0) = \{y^* \in Y^* \mid \forall d \in Y \ , \ < y^*, d > \leq f^\uparrow(y_0; d)\} \ ,$$

where $f^{\uparrow}(y_0; d)$ is the Rockafellar derivative of f at y_0 with respect to d defined by

$$f^{\uparrow}(y_0; d) = \inf\ \{\mu \in R \mid (d, \mu) \in T_{\mathrm{epi}f}(y_0, f(y_0))\}, (\inf\ \emptyset = +\infty), \tag{2.1}$$

$T_{\mathrm{epi}f}(y_0, f(y_0))$ is the tangent cone to epi$f$ at $(y_0, f(y_0))$. By Theorem 5.d [3], if $\partial f(y_0) \neq \emptyset$, then

$$f^{\uparrow}(y_0; .) = f^0(y_0; .).$$

We first consider the reduced problem of (P) :

$$(P_1) \begin{cases} \text{minimize } J(y, u)\ , \\ \text{s.t.} \\ \qquad u \in Ay + \partial\varphi(y)\ , \end{cases}$$

where J, A, φ are as in (P).

Denote by F the set–valued map $y \mapsto \partial\varphi(y)$ and $\mathcal{F}$ its graph. Since the map F is upper semicontinuous, the set $\mathcal{F}$ is closed.

A first–order necessary optimality condition for (P_1) can be stated as follows

THEOREM 2.1. *Let (y_0, u_0) be a local solution of (P_1). Suppose that J and φ are locally Lipschitz in $Y \times W$ and Y, respectively. Assume, in addition, that J is regular at (y_0, u_0). Then*

$$J_y^0(y_0, u_0; h) + J_u^0(y_0, u_0; Ah) + J_u^0(y_0, u_0; k) \geq 0 \tag{2.2}$$

for all $(h, k) \in K_{\mathcal{F}}(y_0, u_0 - Ay_0))$, where $K_{\mathcal{F}}(y_0, u_0 - Ay_0)$ stands for the contingent cone to $\mathcal{F}$ at $(y_0, u_0 - Ay_0)$.

PROOF: Putting $v = u - Ay$, $z = y$ one can see that (P_1) is of the following form

$$(P_1') \begin{cases} J_1(z, v) \longrightarrow \inf\ , \\ v \in \partial\varphi(z) \end{cases}$$

where, $J_1(z, v) = J(y, v + Az) = J(y, u)$.

Observe that if (y_0, u_0) is a local solution of (P_1), then (z_0, v_0) is a local solution of (P_1'), where $z_0 = y_0$, $v_0 = u_0 - Ay_0$. It is easily seen that if J is regular at (y_0, u_0), then J_1 is also regular at (z_0, v_0). Note that (P_1') can be written in the form

$$(P_1'')\quad \begin{cases} J_1(z,v) \longrightarrow \inf\,, \\ (z,v) \in \mathcal{F}. \end{cases}$$

We can see that all hypotheses of Theorem 6 in [3] are fulfilled. By this theorem we get

$$J_1^0(z_0, v_0; h, k) \geq 0 \quad (\forall (h,k) \in K_{\mathcal{F}}(z_0, v_0)). \tag{2.3}$$

Since J_1 is regular at (z_0, v_0), it follows that

$$J_1^0(z_0, v_0; h, k) = J_y^0(y_0, u_0; h) + J_u^0(y_0, u_0; Ah + k)$$

which together with (2.3) implies

$$J_y^0(y_0, u_0; h) + J_u^0(y_0, u_0; Ah) + J_u^0(y_0, u_0; k) \geq 0 \quad (\forall (h,k) \in K_{\mathcal{F}}(y_0, u_0 - Ay_0))$$

as was to be shown.

3. A necessary optimality condition for (P)

To establish necessary condition for (P) we replace the map F in Section 2 by the map $F_1 : y \mapsto \partial\varphi(y) \cap \Omega$. Denote by $\mathcal{F}_1$ the graph of the map F_1. It is easily seen that

$$\mathcal{F}_1 = \mathcal{F} \cap (Y \times \Omega).$$

DEFINITION 3.1 [1]. *Let $Q \subset R^n$ and $y_0 \in Q$. Then a vector $d \in R^n$ is said to be hypertangent to Q at y_0 if there exists a number $\varepsilon > 0$ such that $y + tw \in Q$ for all $y \in (y_0 + \varepsilon B) \cap Q$, $w \in d + \varepsilon B$, $t \in (0, \varepsilon)$, where B stands for the open unit ball.*

We recall [1] that the set Q is said to be regular at y_0 if

$$K_Q(y_0) = T_Q(y_0)$$

THEOREM 3.1. *Suppose that (y_0, u_0) is a local solution of (P) and all hypotheses of Theorem 2.1 are fulfilled. Assume, furthermore that*
(i) $K_{\mathcal{F}}(y_0, u_0 - Ay_0) \cap (Y \times \text{int } K_\Omega(y_0, u_0 - Ay_0)) \neq \emptyset$,
(ii) the set Ω is regular at $u_0 - Ay_0$.

Then,

$$J_y^0(y_0, u_0; h) + J_u^0(y_0, u_0; Ah) + J_u^0(y_0, u_0; k) \geq 0 \tag{3.1}$$

$$(\forall (h, k) \in K_{\mathcal{F}}(y_0, u_0 - Ay_0) \cap (Y \text{ x } K_\Omega(y_0 - Ay_0)).$$

PROOF: Since the set Ω is regular at $u_0 - Ay_0$, we have

$$K_\Omega(u_0 - Ay_0) = T_\Omega(u_0 - Ay_0) \ .$$

Observing that Ω is closed, by virtue of Corollary 1 of Theorem 2.5.8 [1] one gets

$$\text{int } T_\Omega(u_0 - Ay_0) = H_\Omega(u_0 - Ay_0) \ ,$$

where $H_\Omega(u_0 - Ay_0)$ is the set of all hypertangents to Ω at $u_0 - Ay_0$. Hence

$$H_\Omega(u_0 - Ay_0) = \text{int } K_\Omega(u_0 - Ay_0) \ . \tag{3.2}$$

For any $d \in K_{\mathcal{F}}(y_0, u_0 - Ay_0) \cap (Y \times \text{int } K_\Omega(u_0 - Ay_0))$ there exist sequences $\lambda_n \downarrow 0$, $d_n = (d_n^{(1)}, d_n^{(2)}) \longrightarrow (d^{(1)}, d^{(2)}) = d$ such that

$$(y_0, u_0 - Ay_0) + \lambda_n d_n \in \mathcal{F} \tag{3.3}$$

Moreover, by (3.2) we get $d \in Y \times H_\Omega(u_0 - Ay_0)$. Hence, there is a natural number n_0 such that for all $n \geq n_0$,

$$(u_0 - Ay_0) + \lambda_n d_n^{(2)} \in \Omega.$$

From this and (3.3) it follows that for every $n \geq n_0$,

$$(y_0, u_0 - Ay_0) + \lambda_n d_n \in \mathcal{F} \cap (Y \times \Omega) \ ,$$

which means that $d \in K_{\mathcal{F} \cap (Y \times \Omega)}$. Thus,

$$K_{\mathcal{F}}(y_0, u_0 - Ay_0) \cap (Y \times \text{int } K_\Omega(u_0 - Ay_0)) \subset K_{\mathcal{F} \cap (Y \times \Omega)}(y_0, u_0 - Ay_0) \tag{3.4}$$

Since $K_{\mathcal{F}}$, K_{Ω} , $K_{\mathcal{F}\cap(Y\times\Omega)}$ are closed under the assumptions of the theorem, it follows from (3.4) that

$$K_{\mathcal{F}}(y_0, u_0 - Ay_0) \cap (Y \times K_{\Omega}(u_0 - Ay_0)) \subset K_{\mathcal{F}\cap(Y\times\Omega)}(y_0, u_0 - Ay_0) \quad (3.5)$$

Applying Theorem 2.1 we obtain (3.1) for all $(h, k) \in K_{\mathcal{F}\cap(Y\times\Omega)}(y_0, u_0 - Ay_0)$. By virtue of (3.5) the conclusion follows

Theorem 3.1 is a generalization of a necessary optimality condition of [5].

4. The lower semicontinuous case

More and more technical and physical problems have been recently formulated in the form of variational and hemivariational inequalities. Such inqualities often lead to nonsmooth optimization problems whose objective functions may be discontinuous.

In this section we shall deal with the case of J being lower semicontinuous, not necessarily locally Lipschitz.

Denote by $\mathcal{F}_2$ the graph of the map $F_1 : y \mapsto Ay + \partial\varphi(y)$.

THEOREM 4.1. *Let (y_0, u_0) be a local solution of (P_1). Suppose that J is lower semicontinuous in $Y \times W$ and finite at (y_0, u_0). Assume further that either the problem (P_1) is calm at (y_0, u_0) in the sense that there is no $(y_k, u_k) \longrightarrow (y_0, u_0)$ with $(y_k, u_k) \notin \mathcal{F}_2$ such that*

$$\frac{J(y_k, u_k) - J(y_0, u_0)}{distt_{\mathcal{F}_2}(y_k, u_k)} \longrightarrow -\infty ,$$

where dist *stands for the function of distance, or there is no nonzero $z \in N_{\mathcal{F}_2}(y_0, u_0)$ with $-z \in \partial^{\infty} J(y_0, u_0)$. Then,*

$$N_{\mathcal{F}_2}(y_0, u_0) \cap [-\partial J(y_0, u_0)] \neq \emptyset . \quad (4.1)$$

PROOF: The problem (P_1) can be rewritten in the form

$$\begin{cases} J(y, u) \longrightarrow \inf , \\ (y, u) \in \mathcal{F}_2. \end{cases}$$

It is easily seen that $\mathcal{F}_2$ is closed. By Corollary 5.2.1 [7] there exists $(y^*, u^*) \in N_{\mathcal{F}_2}(y_0, u_0)$ with $-(y^*, u^*) \in \partial J(y_0, u_0)$, which implies (4.1).

REMARK: To find necessary optimality conditions for (P) we replace $\mathcal{F}_2$ in Theorem 4.1 by $\mathcal{F}_2 \cap (Y \times \Omega)$.

From Theorem 4.1 we can see that if J is locally Lipschitzian, then $\partial^\infty J(y_0, u_0) = \{(0,0)\}$, which implies (4.1).

5. The modified problem of (P).

We now consider the following problem

$$(P_2) \begin{cases} J(y,u) \longrightarrow \inf , \\ Bu \in Ay + \partial\varphi(y) , & (5.1) \\ u \in \Omega . & (5.2) \end{cases}$$

Here $J, \varphi, A, Y, W, \Omega$ are as in (P) (not necessarily $W \subset Y^*$), B is a map from W into Y^*.

Since the set $\partial\varphi(y)$ is closed, the constraint (5.1) is equivalent to the following

$$\mathrm{dist}_{\mathcal{F}}((y, Bu - Ay)) = 0 . \tag{5.3}$$

Hence, we may rewrite (P_2) in the form

$$(P_2') \quad \begin{cases} J(y,u) \longrightarrow \inf , \\ \text{subject to} \\ \bar{h}(y,u) = 0, \\ (y,u) \in Y \times \Omega , \end{cases}$$

where

$$\bar{h}(y,u) = \mathrm{dist}_{\mathcal{F}}((y, Bu - Ay))$$

Denote by Q the feasible set of the problem.

THEOREM 5.1. *Let (y_0, u_0) be a local solution of (P_2). Assume that J and φ are locally Lipschitz in $Y \times W$ and Y, respectively; the map B is continuously differentiable in a neighbourhood of u_0. Suppose, in addition, that (y_0, u_0) is a regular point for h relative to $Y \times \Omega$ in the sense of Ioffe [4], i.e. there are $k > 0$ and neighbourhoods V of y_0 , U of u_0 such that for all $(y, u) \in (V \times U) \cap (Y \times \Omega)$,*

$$dist_Q(y, u) \leq kh(y, u)$$

Then

$$(0,0) \in \partial J(y_0, u_0) + \mathcal{A}^T N_{\mathcal{F}}(y_0, Bu_0 - Ay_0) + \{0\} \times N_\Omega(u_0), \tag{5.4}$$

where $\mathcal{A}^T$ is the transpose of

$$\mathcal{A} = \begin{pmatrix} I & -A \\ 0 & B'(u_0) \end{pmatrix}.$$

PROOF: For $r > 0$ we set

$$M_r(y, u) = J(y, u) + r(h(y, u) + \text{dist}_{Y\times\Omega}(y, u)) \tag{5.5}$$

Since the distance function is Lipschitz and (y_0, u_0) is a regular point in the sense of Ioffe [4], it follows from the reduction theorem [4] that for sufficiently large r, M_r attains its local minimum at (y_0, u_0). Hence

$$(0,0) \in \partial J(y_0, u_0) + r[\partial h(y_0, u_0) + \partial(\text{dist}_{Y\times\Omega}(y_0, u_0))]. \tag{5.6}$$

By Proposition 2.4.2 and the chain rule of [1] we have

$$\partial \bar{h}(y_0, u_0) \subset (G'(y_0, u_0))^T N_{\mathcal{F}}(y_0, Bu_0 - Ay_0) , \tag{5.7}$$

where $G(y, u) = (y, Bu - Ay)$.

It is well-known from the nonsmooth analysis that

$$r\partial(\text{dist}_{Y\times\Omega}(y_0, u_0)) \subset N_{Y\times\Omega}(y_0, u_0) \tag{5.8}$$

and

$$N_{Y\times\Omega}(y_0, u_0) = \{0\} \times N_\Omega(u_0) \tag{5.9}$$

Substituting from (5.7)–(5.9) in (5.6) yields (5.4). The proof is complete.

In the case where (y_0, u_0) is not necessarily regular in the sense of Ioffe we get the following

THEOREM 5.2. *Let (y_0, u_0) be a local solution of (P_2). Suppose that J and φ are locally Lipschitz in $Y \times W$ and Y, respectively. Assume furthermore that the map B is continuously differentiable in a neighbourhood of u_0. Then there exist scalars $\lambda \geq 0$ and μ, not all zero, such that*

$$(0,0) \in \lambda \partial J(y_0, u_0) + \mu \mathcal{A}^T N_{\mathcal{F}}(y_0, Bu_0 - Ay_0) + \{0\} \times N_\Omega(u_0) \, .$$

PROOF: Since the function of distance is Lipschitz and the map B is continuously differentiable in a neighbourhood of u_0, the function h is locally Lipschitz. Since (P_2) can be rewritten in the form (P_2'), applying the Lagrange multiplier rule ([1], Th.6.1.1) yield scalars $\lambda \geq 0$ and μ, not all zero, such that

$$(0,0) \in \lambda \partial J(y_0, u_0) + \mu \partial \bar{h}(y_0, u_0) + N_{Y \times \Omega}(y_0, u_0). \tag{5.10}$$

It is obvious that

$$N_{Y \times \Omega}(y_0, u_0) = \{0\} \times N_\Omega(u_0). \tag{5.11}$$

By Proposition 2.4.2 and the chain rule of [1] we get

$$\partial \bar{h}(y_0, u_0) \subset \mathcal{A}^T(N_{\mathcal{F}}(y_0, Bu_0 - Ay_0). \tag{5.12}$$

Hence, a combination of (5.10)-(5.12) yields the conclusion.

6. Example

Let us consider the following functions

$$\varphi(y) = \begin{cases} 0 & , \text{ if } y \leq -1, \\ (y+1)^2 & , \text{ if } y \in (-1, 0), \\ (y-1)^2 & , \text{ if } y \in [0, 1), \\ 0 & , \text{ if } y \geq 1 \, , \end{cases}$$

$$f_1(y, a) = y^2 - ay \, , \quad a, y \in R \, .$$

The graphs of $\varphi(y)$ and $\partial \varphi(y)$ are as follows

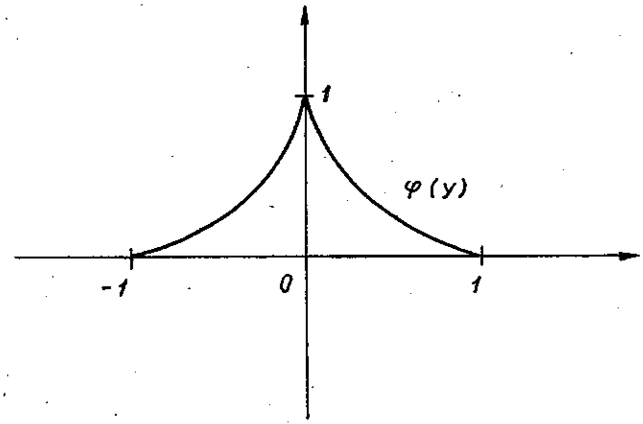

Fig. 1

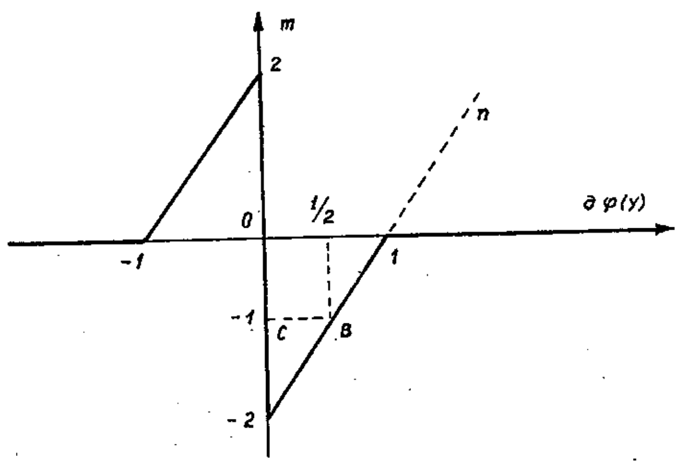

Fig. 2

Putting $f(y,a) = f_1(y,a) + \varphi(y)$ we get

$$\partial_y f(y,a) = 2y - a + \partial\varphi(y).$$

Hence

$$0 \in \partial_y f(y,a) \Longleftrightarrow a \in 2y + \partial\varphi(y)$$

We now consider the following problem

$$(I)\qquad \begin{cases} J(y,a) = (y+1)^2 + (a+5)^2 \longrightarrow \inf\ , \\ \text{subject to} \\ a \in 2y + \partial\varphi(y)\ . \end{cases}$$

To study Problem (I) we investigate the following auxiliary problem

$$(II)\qquad \begin{cases} (y+1)^2 + (a+5)^2 \longrightarrow \inf\ , \\ \text{subject to} \\ f(y,a) = y^2 - ay + \varphi(y) \longrightarrow \inf\ . \end{cases}$$

Observe that each solution of (II) is also a solution of (I).

To find solutions of (II) we first solve the following problem

$$(III)\qquad y^2 - ay + \varphi(y) \longrightarrow \inf .$$

a) $0 < a < 2$. The graph of $f(y,a)$ is as follows

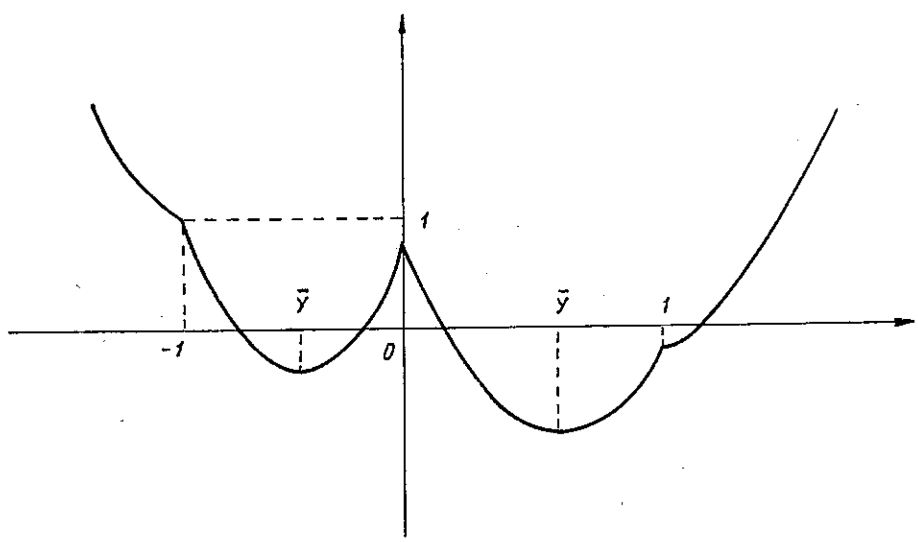

Fig. 3

The solutions of (III) are

$$\overline{y} = \frac{a-2}{4} \quad : \quad \text{local minimum}\ ,$$
$$\overline{y} = \frac{a+2}{4} \quad : \quad \text{global minimum}\ .$$

b) $a = 2$.

$$f(y,a) = \begin{cases} y^2 - 2y & , \ y \le -1, \\ 2y^2 + 1 & , \ y \in (-1, 0), \\ 2y^2 - 4y + 1 & , \ y \in [0, 1), \\ y^2 - 2y & , \ y \ge 1 \ . \end{cases}$$

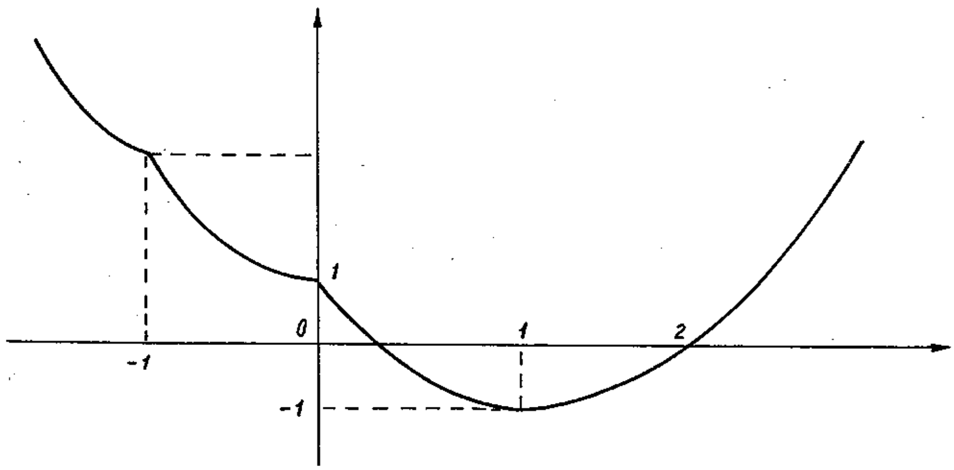

Fig. 4

In this case, $\overline{y} = 0$ and $\overline{y} = 1$ ($\overline{y} = 1$ is the global minimum).

c) $a > 2$

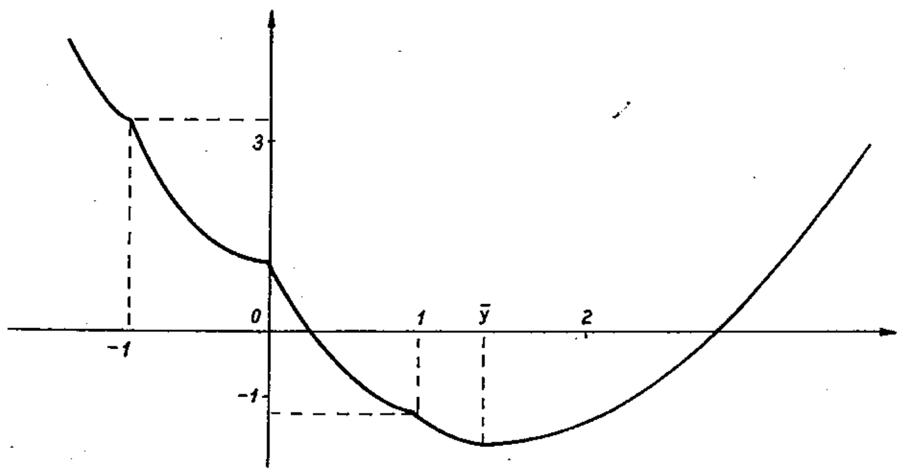

Fig. 5

$\overline{y} = \frac{a}{2}$ is the global minimum.

d) $a = 0$

$$f(y,a) = \begin{cases} y^2 & , \ y \le -1, \\ 2y^2 + 2y + 1 & , \ y \in (-1,0), \\ 2y^2 - 2y + 1 & , \ y \in [0,1), \\ y^2 & , \ y \ge 1. \end{cases}$$

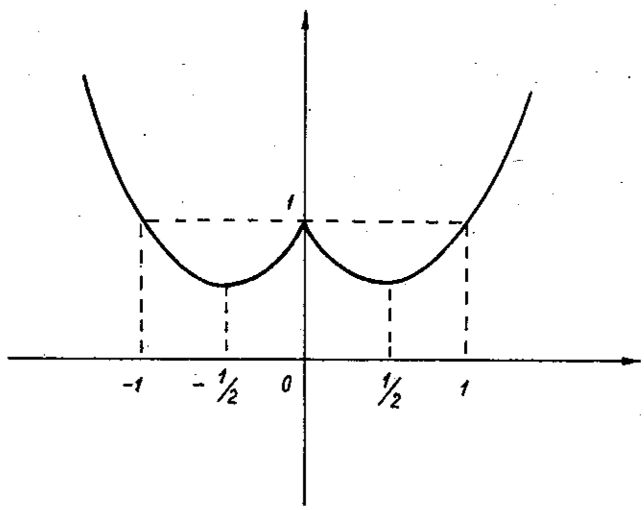

Fig. 6

$\overline{y} = \pm\frac{1}{2}$ are the global minima.

e) $a \in [-2,0)$. We get $\overline{y} = \frac{a\pm 2}{4}$ and for $a \in (-\infty,-2)$, $\overline{y} = \frac{a}{2}$.

Thus,

$$\begin{cases} \overline{y} = \frac{a\pm2}{4} & \text{with } a \in [-2,2] \ , \\ \overline{y} = \frac{a}{2} & \text{with } |a| > 2 \ . \end{cases} \tag{6.1}$$

It follows from (6.1) that Problem (I) possesses a solution $(0,-2)$.

We can see that the necessary condition

$$J_y^0(y_0,a_0;h) + J_a^0(y_0,a_0;Ah) + J_a^0(y_0,a_0;k) \ge 0 \ , \quad (\forall (h,k) \in K_{\mathcal{F}}(y_0,a_0 - Ay_0)), \tag{6.2}$$

is fulfilled at $(y_0, a_0) = (0, -2)$, where $\mathcal{F}$ is the graph of the map $y \mapsto \partial\varphi(y)$. Namely,

$$K_{\mathcal{F}}(y_0, a_0 - Ay_0) = K_{\mathcal{F}}(0, -2) ,$$

and $K_{\mathcal{F}}(0,-2)$ consists of two half–lines Am, An (see Fig.2).

For the point $B(\frac{1}{2}, -1) \in An$, $h = \frac{1}{2}$, $k = 1$, and (6.2) becomes

$$2.(0+1).\frac{1}{2} + 2.(-2+5).2.\frac{1}{2} + 2.(-2+5).1 = 13 > 0 .$$

For the point $C(0,-1)$, $h = 0$, $k = 1$ and, (6.2) becomes

$$2.(0+1).0 + 2.(-2+5).2.0 + 2.(-2+5).1 = 6 > 0 .$$

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