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SOLVING A CLASS OF OPTIMAL CONTROL PROBLEMS WHICH ARE LINEAR IN THE CONTROL VARIABLE BY THE METHOD OF ORIENTING CURVES

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Abstract. In this paper we prove that the Method of Orienting Curves given in [16] can be applied for solving a class of linear optimal control problems with state constraints

1. Introduction

The Method of Orienting Curves (MOC) was introduced for the first time in [11] (1987), and then in [2] and [16].

In [11] a class of simple problems with one state and one control variable was considered. There, the state equation is of the form $\dot{x} = u$ while the cost funtional is strongly convex to the control variable and independent of the state variable. In spite of these restrictions, the method was sucessfully used for solving some practical problems, for example, the navigation problem of Zermelo with state constraints [12], Steiner's problem of finding inpolygons of a given convex polygon with minimal circumference [13]. In [2] the MOC was developed for solving a class of regular problems in which the state equation is $\dot{x} = f_1(t,x) + f_2(t,x)u$, and the function L in the cost functional is strictly convex to the control variable and has the general form $L(t,x,u)$.

The general scheme of the Method of Orienting Curves was given in [16], where optimal control problems with state constraints, one state and several control variables were solved whenever the five hypotheses stated in it are fulfilled. Concretely, the method allows constructing the optimal trajectory as

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a path which consists of parts of orienting curves, boundary arcs, and a final curve.

This paper will deal with a class of optimal control problems with one state and one control variable (with state contraints) in which the performance index and the state equation are linear in the control variable. Such a class was considered by several authors (under different hypotheses) and different solving methods were introduced (see, e.g., [4], [7], [8], [9]). Here, we shall use the idea of Method of Region Analysis developed in [8] and [9] to check if the five hypotheses stated in [16] hold. If this is the case, then optimal processes will be constructed by means of Method of Orienting Curves. The results obtained in this paper can be used for solving different practical problems, e.g., inventory problems, optimal control of a hydroelectric power plant, ...

We shall study the problem of determining the control function which minimizes the cost functional

$$J = \int_0^T L(t, x(t), u(t))dt, \qquad L(t,\xi,v) = L_1(t,\xi) + L_2(t,\xi)v, \tag{1.1}$$

under the constraints

$$\begin{aligned} &x(t) = f(t, x(t), u(t)), \quad f(t,\xi,v) = f_1(t,\xi) + f_2(t,\xi)v, \\ &\beta_1 \le u(t) \le \beta_2, \\ &\alpha_1(t) \le x(t) \le \alpha_2(t), \\ &x(0) = x_0, \quad x(T) = x_T. \end{aligned} \tag{1.2}$$

Here x and u are state and control functions, while ξ and v are state and control variables, respectively. We shall suppose that the functions $L_1(.,\xi)$ and $f_1(.,\xi)$ are continuous, $\alpha_1(.)$, $\alpha_2(.)$ are piecewise continuously differentiable, and $L_1(t.,.)$, $f_1(t,.)$, $L_2(.,.)$, $f_2(.,.)$ are continuously differentiable. Besides, we assume

$$\beta_1 < \beta_2, \quad f_2(t,\xi) > 0, \quad \text{and} \quad \alpha_1(t) < \alpha_2(t) \tag{1.3}$$

for $(t,\xi) \in G^{ex}$ where G^{ex} is the *extended state region* , which is any open set in $\mathbb{R}^2$ containing the *state region* $G := \{(t,\xi) \mid t \in [0,T], \ \alpha_1(t) \le x(t) \le \alpha_2(t)\}$.

Moreover, the following will be assumed

$$f(t, \alpha_i(t), \beta_k) \neq \dot{\alpha}_i(t), \text{ a. e. in } [0,T], \quad k = 1,2, \text{ and } i = 1,2. \tag{1.4}$$

In Section 2 the Method of Orienting Curves is introduced and illustrated by an example. Section 3 is left for the verification of the conditions stated in Section 2 for some concrete classes of problems.

2. Method of Orienting Curves

2.1. Orienting and Final Curves

Recall that an optimal process $(x^*(.), u^*(.))$ of Problem (1.1)–(1.4) is said to be normal if it satisfies Pontryagin's maximum principle (see [16, Theorem 2.1]) for $\lambda_0 > 0$. For such a process we can set $\lambda_0 = 1$.

For our further investigation we give a local version of Pontryagin's maximum principle without state constraints which can be written as follows:

$$\begin{aligned}
&p(z) = q, \ x(z) = y, \\
&\dot{p}(t) = -p(t)[f_{1\xi}(t, x(t)) + f_{2\xi}(t, x(t))u(t)] + L_{1\xi}(t, x(t)) + L_{2\xi}(t, x(t))u(t), \\
&\dot{x}(t) = f_1(t, x(t)) + f_2(t, x(t))u(t), \\
&u(t) \in \underset{v\in[\beta_1,\beta_2]}{\arg\max}\, H(t, x(t), v, p(t), 1), \text{ a. e.,}
\end{aligned} \tag{2.1}$$

where $(z, y) \in G$ and $q \in \mathbb{R}$.

The set of all solutions of the system (2.1) for some certain parameters z, y, q will be denoted by $S_{z,y,q}$.

Let $(x^*(.), u^*(.))$ be a normal optimal process of Problem (1.1)–(1.4). A subinterval $I \subset [0,T]$ will be called a *contact interval* of $x^*(.)$ with $\alpha_i(.)$, i = 1 or 2, if $x^*(t) = \alpha_i(t)$ for all $t \in I$, and every interval I' with $I' \supset I$ and $I' \setminus I \neq \emptyset$ contains at least one $t \in I' \setminus I$ with $x^*(t) \neq \alpha_i(t)$. It is possible that some contact intervals have only one point.

From now on, S^* stands for the set of all triple $(x^*(.), u^*(.), p^*(.))$ where $(x^*(.), u^*(.))$ is a normal optimal process of Problem (1.1)–(1.4), and $p^*(.)$ is the corresponding solution of the adjoint equation in Pontryagin's maximum principle. Moreover, $x^*(.)$ has only finitely many contact intervals with $\alpha_i(.)$, i $= 1, 2$.

Denote by B_1 and B_2 the *lower* or *upper boundary* of the state region G, i.e.,

$$B_i = \mathrm{gr}\alpha_i = \{(t,\xi) \mid t \in [0,T],\ \xi = \alpha_i(t)\},\ i = 1,2.$$

The sets

$$B_1^{ex} = B_1 \cup \{(0,\xi) \mid \alpha_1(0) \le \xi < x_0\} \ \cup\ \{(T,\xi) \mid \alpha_1(T) \le \xi < x_T\},$$
$$B_2^{ex} = B_2 \cup \{(0,\xi) \mid x_0 < \xi \le \alpha_2(0)\} \ \cup\ \{(T,\xi) \mid x_T < \xi \le \alpha_2(T)\}$$

will be called the *extended lower* and *extended upper boundary* of G, respectively. It is possible that G is unbounded, i.e., α_1 or α_2 can be equal to infinite.

Let (z, y) be an arbitrary point of G.

DEFINITION 2.1. If for some $q \in \mathbb{R}$ there exists $(x(.), u(.), p(.)) \in S_{z,y,q}$ such that

$$x(T) = x_T \text{ and } (t, x(t)) \in G \text{ for all } t \in [z,T],$$

then $x(.)$ is said to be a *final function* and its graph is called a *final curve* through (z, y).

DEFINITION 2.2. If for some $q \in \mathbb{R}$ there is a triple $(x(.), u(.), p(.)) \in S_{z,y,q}$ such that there exist ρ and ν with

$$\begin{aligned} &0 \le z < \nu < \rho \le T,\\ &(t, x(t)) \in G \text{ for } t \in [z,\rho],\\ &(\nu, x(\nu)) \in B_i \text{ for i} = 1 \text{ or } 2\ ,\\ &(\rho, x(\rho)) \in B_j^{ex} \text{ for } j \neq i, \end{aligned} \tag{2.2}$$

then $x(.)$ is said to be an *orienting function* and its graph is called an *orienting curve* through the *initial point* $(z, y) \in G$.

In this case, $(\rho, x(\rho))$ is the *terminal point* of this orienting curve, if, additionally to (2.2), the following holds:

$$\text{for all } \delta > 0,\ \text{there exists } t \in (\rho, \rho+\delta) \text{ with } (t, x(t)) \notin G. \tag{2.3}$$

$(\nu, x(\nu))$ is called the *transfer point* of this curve if additionally to (2.2)–(2.3)

$$(t, x(t)) \notin B_i \text{ for all } t \in (\nu, \rho).$$

DEFINITION 2.3. Let $q \in \mathbb{R}$ and $(x(.), u(.), p(.)) \in S_{z,y,q}$. Suppose $i \in \{1, 2\}$, and

$$\begin{aligned} & z \in [z_1, z_2] \subset [0, T], \ z_1 < z_2, \\ & (z_k, x(z_k)) \in B_i^{ex} \cup \{(0, x_0), (T, x_T)\}, \ \mathrm{k} = 1, 2, \\ & (t, x(t)) \in G, \quad t \in [z_1, z_2], \end{aligned} \tag{2.4}$$

then $x(.)$ is said to be a *barrier function* through (z, y).

If, additionally to (2.4),

$$(z, y) \in B_j, \ j \neq i,$$

i.e., $y = x(z) = \alpha_j(z)$, then $(z, \alpha_j(z))$ is called a *narrow pass point* .

We refer to [16] for more details and the reasons why such a function is called a barrier function as well as $(z, \alpha_j(z))$ a narrow pass point. We are now in a position to recall the algorithm which tells us how an optimal process $(x^*(.), u^*(.))$ of a control problem belonging to the class (1.1)–(1.4) can be constructed by means of the Method of Orienting Curves.

2.2. Algorithm

Step 1. Begin in $(0, x_0)$. Set $l = 0, t_0 = 0$, and $x^*(t_0) = x_0$.

Step 2. Consider $(t_l, x^*(t_l))$.

– If there is a final curve through $(t_l, x^*(t_l))$, go to Step 5.

– If there is an orienting curve through $(t_l, x^*(t_l))$, go to Step 3.

– If neither of the above cases appears, then it is guaranteed that (under the assumption $S^* \neq \emptyset$) $x^*(t_l) = \alpha_{i_l}(t_l)$, $i_l = 1$ or 2, and there exists $t_l' > t_l$ such that $(z, \alpha_{i_l}(z))$ is a narrow pass point for all $z \in [t_l, t_l']$. Then go to Step 4.

Step 3. Let $x(.)$ be an orienting function through $(t_l, x^*(t_l))$ with $(t_{l+1}, x(t_{l+1}))$ as its transfer point. Then we have $x^*(t) = x(t)$ for all $t \in [t_l, t_{l+1}]$. Set $l := l + 1$ and go to Step 2.

Step 4. Determine t_{l+1} with:

– $(z, \alpha_{i_l}(z))$ is a narrow pass point for all $z \in [t_l, t_{l+1}]$,

– Either $t_{l+1} = T$ or there exists an orienting curve or a final curve through $(t_{l+1}, \alpha_{i_l}(t_{l+1}))$.

Then $x^*(t) = \alpha_{i_l}(t)$ for all $t \in [t_l, t_{l+1}]$. Set $l := l + 1$ and go to Step 2.

Step 5. Let $x(.)$ be a final function through $(t_l, x^*(t_l))$. Then $x^*(t) = x(t)$ for all $t \in [t_l, T]$. STOP.

By the previous algorithm, the optimal state function $x^*(.)$ (and hence, $(x^*(.), u^*(.), p^*(.)))$ can be determined completely. It is worth mentioning that the assertions in all steps hold for every $x^*(.)$. This implies that S^* has at most one element, i.e., Problem (1.1)–(1.4) possesses at most one normal process which has finitely many contact intervals with the boundary of the state region G. Finally, if all three cases mentioned in Step 2 do not appear, then $S^* = \emptyset$.

In order to illustrate the way how the previous algorithm operates, we give an example. For the sake of simplicity, we omit here the conditions under which the Method of Orienting Curves can be applied. It will be considered in the next section.

2.3 Example

Consider

$$\int_0^{14} x(t)u(t)dt \longrightarrow \min, \tag{2.5}$$

$$\begin{aligned} &\dot{x}(t) = b(t) + u(t), \\ &-4 \le u(t) \le 0, \\ &0 \le x(t) \le 10, \\ &x(0) = 5, \quad x(14) = 9, \end{aligned} \tag{2.6}$$

where $b(t) = -0.05t^2 + 0.8t + 1$.

We use the algorithm stated above to contruct optimal process $(x^*(.), u^*(.))$ of Problem (2.5)–(2.6).

Begin in $(0,5)$. There is an orienting curve described by $(x_1(.), u_1(.), p_1(.)) \in S_{0,5,10}$. The orienting function is

$$x_1(t) = \begin{cases} -\frac{1}{60}t^3 + 0.4t^2 + t + 5, \text{ for } t \in [0, t_1], \\ -\frac{1}{60}t^3 + 0.4t^2 - 3t + c_1, \text{ for } t \in [t_1, 14] \end{cases} \tag{2.7}$$

where $c_1 \approx 15.384, t_1 \approx 2.596$. Its transfer point is $(t_1, 10)$ and its terminal point is $(14, \rho_1)$, $\rho_1 \approx 6.051$. Hence,

$$x^*(t) = x_1(t), \qquad t \in [0, t_1]. \tag{2.8}$$

For all $z \in (t_1, t_2)$, where $t_2 \approx 4.0$, $(z, 10)$ is a narrow pass point, the corresponding barrier function $x(.)$ is given by $(x(.), u(.), p(.)) \in S_{z,10,10}$. For example, if $z = 3$, then

$$x(t) = \begin{cases} -\frac{1}{60}t^3 + 0.4t^2 + t + 3.85, & t \in [0, 3], \\ -\frac{1}{60}t^3 + 0.4t^2 - 3t + 15.85, & t \in [3, 14]. \end{cases}$$

Hence,

$$x^*(t) = 10, \text{ for all } t \in [t_1, t_2]. \tag{2.9}$$

Through $(t_2, 10)$, there is an orienting function $x_2(.)$ with $(x_2(.), u_2(.), p_2(.)) \in S_{t_2,10,10}$, and

$$x_2(t) = -\frac{1}{60}t^3 + 0.4t^2 - 3t + c_2, \tag{2.10}$$

where $c_2 \approx 16.667$. Its transfer point and terminal point are $(t_3, 10)$ and $(14, \rho_2)$, where $t_3 = 10$ and $\rho_2 \approx 7.334$. respectively. Therefore,

$$x^*(t) = x_2(t) \text{ for all } t \in [t_2, t_3]. \tag{2.11}$$

Analogously as above, for all $z \in (t_3, t_4)$, where $t_4 \approx 13.282$, $(z, 10)$ is a narrow pass point corresponding to the barrier function $x(.)$ with $(x(.), u(.), p(.)) \in S_{z,10,10}$. Hence,

$$x^*(t) = 10 \text{ for } t \in (t_3, t_4). \tag{2.12}$$

There is final function $x_3(.)$ through $(t_4, 10)$ with $(x_3(.), u_3(.), p_3(.)) \in S_{t_4,10,10}$, and

$$x_3(t) = -\frac{1}{60}t^3 + 0.4t^2 - 3t + c_3, \quad t \in [t_4, 10], \tag{2.13}$$

where $c_3 \approx 18.333$. Hence,

$$x^*(t) = x_3(t), \text{ for all } t \in [t_4, 14]. \tag{2.14}$$

Thus, the optimal state function is determined completely by (2.7)– (2.14) (see Figure 1). The optimal control function $u^*(.)$ is given by

$$u^*(t) = \dot{x}^*(t) - b(t), \text{ for all } t \in [0, 14].$$

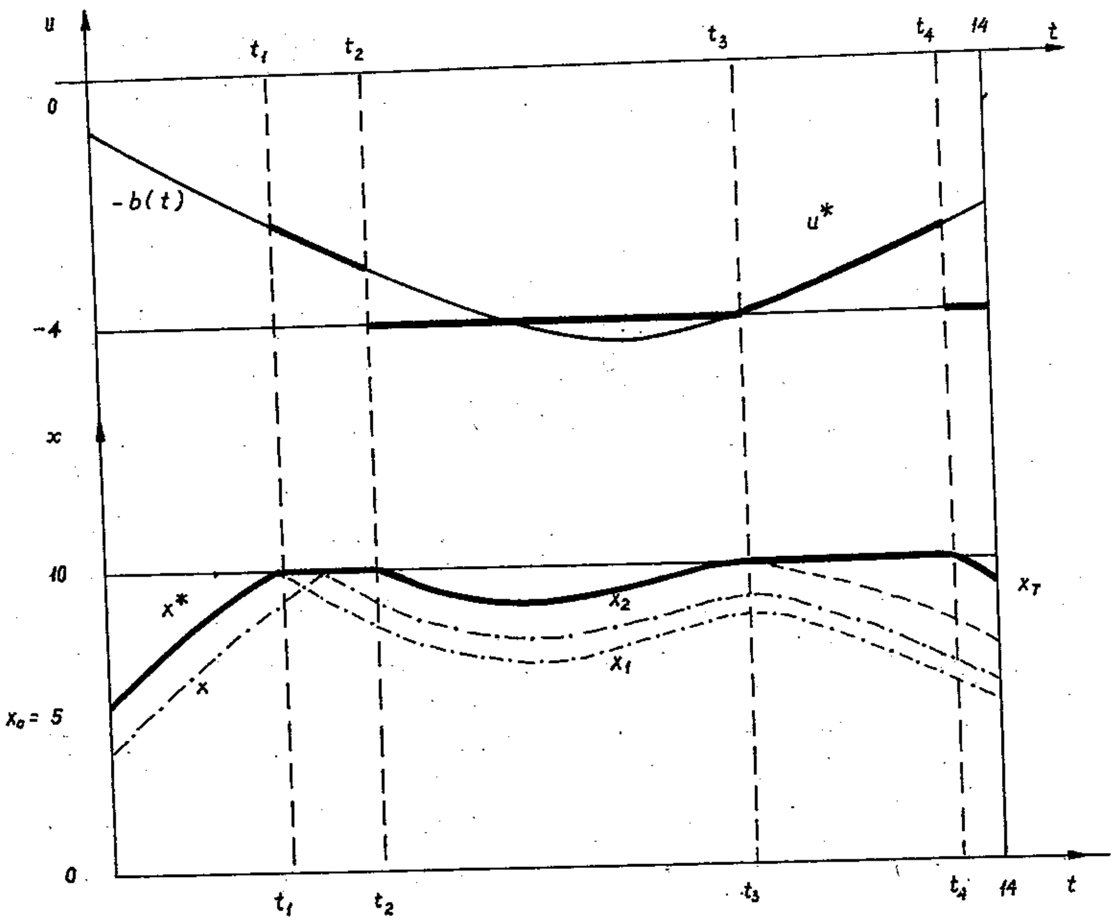

Figure 1.

2.4. Conditions for Application of the Method of Orienting Curves

We have just stated the main idea of the Method of Orienting Curves as well as the way how the method works. But when can the method be applied or under what conditions does the method work? As it was proved in [16] (see also [17]), one can use the algorithm of MOC to determine a normal optimal process of Problem (1.1)–(1.4) provided that the four following hypotheses are satisfied:

(H1) Let $(x_k(.), u_k(.), p_k(.)) \in S_{z,y,q_k}$, and let $(t, x_k(t)) \in G^{ex}$ for all $t \in [z, z'] \subset [0, T]$, k = 1, 2. If $x_1(z') = x_2(z')$, then $x_1(t) = x_2(t)$ for all $t \in [z, z']$.

(H3) Let $(x^*(.), u^*(.), p^*(.)) \in S^*$ and $x^*(t) = \alpha_i(t)$ for all $t \in [t_1, t_2] \subset [0, T]$, i = 1 or 2. Suppose that $z \in [t_1, t_2]$ and $q = p^*(z)$ or $q = p^*(z+0)$. Then there exist $(x(.), u(.), p(.)) \in S_{z,x^*(z),q}$ and t_1' and t_2' with $t_1 \le t_1' \le z \le t_2' \le t_2$ such that

$$(t, x(t)) \in G \text{ for all } t \in [t_1', t_2'],$$

and, for k = 1, 2,

$$t_k' = t_k \text{ or } x(t_k') = \alpha_j(t_k') \text{ for } j \neq i.$$

(H4) Let ω, q be arbitrary real numbers. Then, for almost all t and for all x with $(t, x) \in G^{ex}$, there exists at most one v^* such that

$$H(t, x, v^*, q, 1) = \max_{v \in [\beta_1, \beta_2]} H(t, x, v, q, 1),$$
$$f(t, x, v^*) = \omega.$$

(H5) Let i = 1, 2. For almost all $t \in [0, T]$, if $q_1 \neq q_2$ and

$$v^* \in \operatorname*{argmax}_{v \in [\beta_1, \beta_2]} H(t, \alpha_i(t), v, q_1, 1) \cap \operatorname*{argmax}_{v \in [\beta_1, \beta_2]} H(t, \alpha_i(t), v, q_2, 1)$$

then $f(t, \alpha_i(t), v^*) \neq \dot{\alpha}_i(t)$.

REMARK 3.1. In fact, it was proved in [16] that the MOC can be applied for any class of problems which satisfy five hypotheses (H1)–(H5). In a recent paper (see [17]), however, we proved that (H2) follows from (H1) and (H4).

3. Verification of (H1), (H3)–(H5) for some concrete problem classes

We now check if the four hypotheses (H1), (H3)–(H5) hold for some problems of the class (1.1)–(1.4). For this aim, we observe first that (H4) follows from (1.3) and Lemma 3.3 in [16] while (H5) is a direct consequence of (1.4) and the fact that the function H is linear in the control variable. We need the following functions for the verification of (H1) and (H3).

Let $h(.,.)$ be the function from G^{ex} to $\mathbb{R}$ which is defined by

$$h(t,\xi) := L_{1\xi} - \frac{L_2 f_{1\xi}}{f_2} - \frac{1}{f_2^2}[(L_{2\xi}f_2 - L_2 f_{2\xi})f_1 + L_{2t}f_2 - L_2 f_{2t}]_{|(t,\xi)} \tag{3.1}$$

where all the functions on the right side depend on (t,ξ). Next, we abbreviate $L(t,x(t),u(t))$ by $L(t,x,u)$, $f(t,x(t),u(t))$ by $f(t,x,u)$ and so on.

For $(x(.),u(.),p(.)) \in S_{z,y,q}$ and $(x^*(.),u^*(.),p^*(.)) \in S^*$, denote by $\phi(.)$ and $\phi^*(.)$ the functions defined by

$$\phi(t) := H_v(t,x,u,p,1) = p(t)f_2(t,x) - L_2(t,x), \tag{3.2}$$

$$\phi^*(t) := H_v(t,x^*,u^*,p^*,1) = p^*(t)f_2(t,x^*) - L_2(t,x^*), \tag{3.3}$$

respectively.

Obviously, $\phi(.)$ is continuous on the interval in which it is defined. Moreover, because $p^*(.)$ is continuous on the left on $[0,T]$ (see [5]), $\phi^*(.)$ is continuous on the left on this interval. Define

$$G^{ex+} := \{(t,\xi) \in G^{ex} |\ h(t,\xi) > 0\},$$
$$G^{ex-} := \{(t,\xi) \in G^{ex} |\ h(t,\xi) < 0\},$$
$$G^{exo} := \{(t,\xi) \in G^{ex} |\ h(t,\xi) = 0\}.$$

3.1. Problem (1.1)–(1.4) of the First Type

A problem of Class (1.1)–(1.4) is called of the *first type* if $G \subset G^{ex+}$ or $G \subset G^{ex-}$.

THEOREM 3.1. *For Problem (1.1)–(1.4) of the first type, the hypotheses (H1) and (H3) are satisfied.*

In order to prove Theorem 3.1, we need the following lemmas.

LEMMA 3.1. *Suppose that* $q \in I\!R$, $(z,y) \in G$, *and* $(x(.), u(..), p(.)) \in S_{z,y,q}$. *Suppose further that* $[z, z'] \subset [0,T]$ *and* $(t, x(t)) \in G^{ex}$ *for all* $t \in [z, z']$. *Then*

$$\phi(t) = f_2(t,x)\exp(-\int_z^t f_\xi(\tau,x,u)d\tau)[\ c_z + \int_z^t h(\tau,x)\exp(\int_z^\tau f_\xi(\iota,x,u)d\iota)d\tau], \tag{3.4}$$

$$u(t) = \beta_1 \text{ a. e. in } \{t \in [z,z'] \mid \phi(t) < 0\},$$
$$u(t) = \beta_2 \text{ a. e. in } \{t \in [z,z'] \mid \phi(t) > 0\},$$

where $c_z := H_v(z, x(z), u(z), p(z), 1)/f_2(z, x(z))$.

LEMMA 3.2. *Suppose that* $[z_1, z_2] \subset [0,T]$ *and* $(x(.), u(.), p(.)) \in S_{z,y,q}$.

(i) *If* $(t, x(t)) \in G^{ex}+$ *for all* $t \in [z_1, z_2]$ *then there exists* $z \in [z_1, z_2]$ *such that* $\phi(t) < 0$ *for* $t \in (z_1, z)$ *and* $\phi(t) > 0$ *for* $t \in (z, z_2)$.

(ii) *If* $(t, x(t)) \in G^{ex}-$ *for all* $t \in [z_1, z_2]$ *then there exists* $z \in [z_1, z_2]$ *such that* $\phi(t) > 0$ *for all* $t \in (z_1, z)$ *and* $\phi(t) < 0$ *for all* $t \in (z, z_2)$.

Here z may be equal to z_1 or z_2, i.e., (z_1, z) or (z, z_2) may be empty. Moreover, if $z_1 < z < z_2$ then $\phi(z) = 0$.

Lemma 3.1 is a consequence of (2.1) and Theorem 2 in [8]. Lemma 3.2 can be proved by the same way as in the proof of Theorem 3 in [8].

REMARK 3.2. Note that for any triple $(x(.), u(.), p(.))$ which satisfies the system

$$p(z) = q, \ x(z) = y,$$
$$\dot p(t) = -p(t)[f_{1\xi}(t,x(t)) + f_{2\xi}(t,x(t))u(t)] + L_{1\xi}(t,x(t)) + L_{2\xi}(t,x(t))u(t),$$
$$\dot x(t) = f_1(t,x(t)) + f_2(t,x(t))u(t),$$

we can define a function $\phi(.)$ as in (3.2). Moreover, upon computation similar to the proof of Theorem 2 in [8], the function $\phi(.)$ corresponding to $(x(.), u(.), p(.))$ mentioned above also possesses the representation (3.4).

LEMMA 3.3. *Let* $(x^*(.), u^*(.), p^*(.)) \in S^*$, $[t_1, t_2] \subset [0, T]$. *Suppose that*

$$x^*(t) = \alpha_i(t),\ i = 1, \text{ or } 2, \text{ for all } t \in [t_1, t_2], \tag{3.5}$$

then

$$\phi^*(t) = 0 \text{ for all } t \in (t_1, t_2]$$

PROOF: Because of (3.5), it follows from (1.4) that

$$u^*(t) \in (\beta_1, \beta_2), \text{ a.e. in } [t_1, t_2].$$

Hence,

$$\phi^*(t) = H_v(t, x^*(t), u^*(t), p^*(t), 1) = 0, \text{ a.e. in } [t_1, t_2]. \tag{3.6}$$

The conclusion of the lemma follows from (3.6) and the fact that $\phi^*(.)$ is continuous on the left on $(t_1, t_2]$ ∎

PROOF OF THEOREM 3.1:

We consider only the case $G \subset G^{ex}+$. The proof for the case $G \subset G^{ex}-$ is similar. Without loss of generality (pay attention to the continuity of the function $h(.,.)$), we assume $G^{ex} = G^{ex}+$.

a) (H1) holds for Problem (1.1)–(1.4) provided that $G \subset G^{ex}+$.

Suppose that $(x_i(.), u_i(.), p_i(.)) \in S_{z,y,q_i}$, i = 1, 2. Suppose further that $(t, x_i(t)) \in G^{ex}$ for $t \in [z, z'] \subset [0, T]$, and $x_1(z') = x_2(z')$. We have to prove that

$$x_1(t) = x_2(t) \text{ for all } t \in [z, z']. \tag{3.7}$$

Assume (3.7) is false. Then, without loss of generality, there exist $t_1, t_2 \in [z_1, z_2]$ which satisfy

$$\begin{aligned} &x_1(t_1) = x_2(t_1), \quad x_1(t_2) = x_2(t_2), \\ &x_1(t) < x_2(t) \text{ for } t \in (t_1, t_2). \end{aligned} \tag{3.8}$$

By Lemma 3.2 there exist $\zeta_1, \zeta_2 \in [t_1, t_2]$ such that

$$\begin{aligned} &\phi_1(t) < 0 \text{ for } t \in (t_1, \zeta_1), \text{ and } \phi_1(t) > 0 \text{ for } t \in (\zeta_1, t_2), \\ &\phi_2(t) < 0 \text{ for } t \in (t_1, \zeta_2), \text{ and } \phi_2(t) > 0 \text{ for } t \in (\zeta_2, t_2), \end{aligned} \tag{3.9}$$

where $\phi_i(.)$ is the function defined by (3.2) corresponding to $(x_i(.), u_i(.), p_i(.))$, i = 1, 2, respectively. It is sufficient to consider the case $\zeta_1 > \zeta_2$ (the other cases can be treated similarly).

If $\zeta_2 = t_1$, then (1.2)-(1.3), (3.9), together with Lemma 3.1, and Theorem I.1 in [6] yield

$$x_1(t) < x_2(t) \text{ for all } t > t_1$$

(whenever the graphs of $x_1(.)$ and $x_2(.)$ still remain in G^{ex}) which contradicts (3.8).

If $\zeta_2 > t_1$, then by (1.2), (3.9), and Lemma 3.1, we get $x_1(t) = x_2(t)$ for all $t \in [t_1, z_2]$ which contradicts (3.9). Consequently, (H1) holds.

b) (H3) holds for Problem (1.1)–(1.4) provided $G \subset G^{ex+}$.

Suppose that $(x^*(.), u^*(.), p^*(.)) \in S^*$ and $x^*(t) = \alpha_i(t)$ for all $t \in [t_1, t_2] \subset [0, T]$, i = 1, or 2. Suppose further that $z \in [t_1, t_2]$ and $q = p^*(z)$ or $q = p^*(z+0)$. We have to prove that there exist $(x(.), u(.), p(.)) \in S_{z,x^*(z),q}$ and t_1', t_2' with $t_1 \leq t_1' \leq z \leq t_2' \leq t_2$ such that

$$\begin{aligned} &(t, x(t)) \in G \text{ for all } t \in [t_1', t_2'], \\ &\text{and for k} = 1, 2,\ t_k' = t_k \text{ or } x(t_k') = \alpha_j(t_k'),\ j \neq i. \end{aligned} \tag{3.10}$$

The proof is trivial if $t_1 = t_2$. For the case $t_1 \neq t_2$, observe first that Lemma 3.3 gives $\phi^*(t) = 0$ for all $t \in (t_1, t_2]$. Then, together with the fact that $G \subset G^{ex+}$ and $x^*(t) = \alpha_i(t)$, $t \in [t_1, t_2]$, i = 1 or 2, Theorem 4 in [8] implies that i = 1, i.e., $x^*(t) = \alpha_1(t)$ for all $t \in [t_1, t_2]$.

If $z \in (t_1, t_2)$ and $q = p^*(z + 0)$ or $q = p^*(z)$, then

$$\phi(z) = qf_2(z, x(z)) - L_2(z, x(z)) = \phi^*(z) = \phi^*(z + 0) = 0 \tag{3.11}$$

for any and $(x(.), u(.), p(.)) \in S_{z,x^*(z),q}$. It follows from Lemma 3.1 and Lemma 3.2 that

$$\phi(t) < 0 \text{ for } t < z, \text{ and } \phi(t) > 0 \text{ for } t > z,$$

$$u(t) = \beta_1 \text{ for } t < z, \text{ and } u(t) = \beta_2 \text{ for } t > z$$

as long as gr$x$ remains in G^{ex}. Clearly,

$$u(t) = \beta_1 \leq u^*(t) \text{ for } t < z, \text{ and } u^*(t) \leq u(t) = \beta_2 \text{ for } t > z.$$

Then the existence of $t_1', t_2' \in [t_1, t_2]$ with (3.10) follows from (1.2)–(1.3), Theorem I.5 in [6], and the admisibility of $x^*(.)$.

The case $z = t_1$ or $z = t_2$ can be treated by the same way. Theorem 3.1 is completely proved. ∎

Therefore, from Theorem 3.1 we can conclude that the Method of Orienting Curves developed in [16] can be applied for constructing the optimal process of Problem (1.1)–(1.4) of the first type. It is easy to see that Problem (2.5)–(2.6) considered in the previous section belongs to class (1.1)–(1.4) of the first type with $G \subset G^{ex-}$.

REMARK 3.3. In practice, there are different problems of which the mathematical models belong to Class (1.1)–(1.4) of the first type, for example, inventory problems (see [15]) and problems of production planning (see [1]). In [3] the result just obtained was used to solve an optimal control problem of a hydroelectric power plant.

3.2. Problem (1.1)–(1.4) of the second type

A problem of class (1.1)–(1.4) is called of the *second type* if there is a continuously differentiable function $\eta(.)$ which satisfies

$$f(t, \eta(t), \beta_1) < \dot{\eta}(t) < f(t, \eta(t), \beta_2) \tag{3.12}$$

whenever $(t, \eta(t)) \in G^{ex}$, $t \in [0, T]$. Moreover,

$$\begin{aligned} G^{ex+} &= \{(t, \xi) \in G^{ex} \mid \xi > \eta(t)\}, \\ G^{ex-} &= \{(t, \xi) \in G^{ex} \mid \xi < \eta(t)\}, \\ G^{ex0} &= \{(t, \xi) \in G^{ex} \mid \xi = \eta(t)\}. \end{aligned} \tag{3.13}$$

REMARK 3.4. There are a lot of practical problems belonging to Class (1.1)–(1.4) of the second type (see [10], [14], [15]), and for some of them, the function $\eta(.)$ can be defined completely (see [14]).

THEOREM 3.2. *For Problem (1.1)–(1.4) of the second type, the hypotheses (H1) and (H3) are fulfilled.*

The following lemmas are useful for the proof of Theorem 3.2.

LEMMA 3.4. *Suppose that* $(z, y) \in G, q \in \mathbb{R}$, *and* $(x(.), u(.), p(.)) \in S_{z,y,q}$.

(i) *If* $(z, y) \in G^{ex+} \cup G^{ex_0}$ *and* $\phi(z) > 0$ *then,*

$$u(t) = \beta_2 \text{ and } (t, x(t)) \in G^{ex+} \text{ for } t > z,$$

whenever gr$x$ *still remains in* G^{ex}.

(ii) *If* $(z, y) \in G^{ex-} \cup G^{ex_0}$ *and* $\phi(z) < 0$, *then*

$$u(t) = \beta_1 \text{ and } (t, x(t)) \in G^{ex-} \text{ for } t > z,$$

whenever gr$x$ *still remains in* G^{ex}.

PROOF: It suffices to prove (i). (ii) can be proved analogously.

We first deal with the case $(z, y) \in G^{ex+}$. Assume the contrary that (i) is not true, then there is $z' \in [z, T]$ such that

$$x(z') = \eta(z') \text{ and } (t, x(t)) \in G^{ex+} \text{ for all } t \in [z, z'). \tag{3.14}$$

Since $h(.,.)$ is continuous, we get $z' > z$.

On the other hand, if

$$u_\eta(t) = \frac{f_1(t, \eta(t)) - \dot{\eta}(t)}{f_2(t, \eta(t))}, \quad t \in [0, T], \tag{3.15}$$

then (3.12) imlies $u_\eta(t) < \beta_2 = u(t)$, for $t \in [z, z')$. Due to (1.2)–(1.3) and Theorem I.1 in [6], we get

$$\eta(t) < x(t), \quad t \in [z, z'].$$

Especially, $\eta(z') < x(z')$ which contradicts (3.14).

We now consider the case $(z, y) \in G^{ex_0}$. Since $\phi(.)$ is continuous, there exists $\epsilon > 0$ such that $\phi(t) > 0$ for $t \in [z, z + \epsilon]$. Then, (3.12) gives $u_\eta(t) < \beta_2 = u(t)$, for $t \in [z, z + \epsilon]$. The same argument as above leads to

$$\eta(t) < x(t), \quad t \in [z, z + \epsilon]. \tag{3.16}$$

Take $z'' \in (z, z + \epsilon)$, then $(z'', x(z'')) \in G^{ex+}$ and $\phi(z'') > 0$. By the previous proof, we get

$$u(t) = \beta_2 \text{ and } (t, x(t)) \in G^{ex+} \text{ for } t > z''$$

which together with (3.16) prove (i). The proof is complete. ∎

The next two Lemmas can be proved analogously.

LEMMA 3.5. *Let* $(z,y) \in G, q_i \in I\!R$, *and let* $(x_i(.), u_i(.), p_i(.)) \in S_{z,y,q_i}$, $i = 1$, *2. If* $\phi_2(z) < 0 < \phi_1(z)$, *then*

$$x_1(t) - x_2(t) \begin{cases} < 0 \text{ for } t < z, \\ > 0 \text{ for } t > z, \end{cases}$$

as long as $\text{gr}x_i$, $i = 1, 2$, *remain in* G^{ex}.

LEMMA 3.6. *Suppose that* $q \in I\!R$, *and* $(z,y) \in G^{exo}$. *Suppose further that* $(x(.), u(.), p(.)) \in S_{z,y,q}$ *and* $\phi(z) = 0$.

If there is $\epsilon > 0$ *with* $(t, x(t)) \in G^{ex+}$ ($(t, x(t)) \in G^{ex-}$), *for all* $t \in (z, z+\epsilon)$, *then* $u(t) = \beta_2$ *and* $(t, x(t)) \in G^{ex+}$ ($u(t) = \beta_1$ *and* $(t, x(t)) \in G^{ex-}$, *resp.*), *for* $t > z$, *as long as* $\text{gr}x$ *remains in* G^{ex}.

PROOF OF THEOREM 3.2:

a) (H1) holds for Problem (1.1)–(1.4) of the second type.

Suppose that $(x_i(.), u_i(.), p_i(.)) \in S_{z,y,q_i}$, $(z,y) \in G$, $q_i \in I\!R$, i = 1, 2, and $(t, x_i(t)) \in G^{ex}$ for $t \in [z, z'] \subset [0,T]$. Suppose further that $x_1(z') = x_2(z')$. We now verify that

$$x_1(t) = x_2(t) \text{ for all } t \in [z, z'].$$

Assume the contrary, i.e., this equality does not hold. Then, without loss of generality, assume that there exist $t_1, t_2 \in [z, z']$ such that

$$x_1(t_1) = x_2(t_1), \quad x_1(t_2) = x_2(t_2), \tag{3.17}$$

$$x_1(t) > x_2(t), \text{ for } t \in (t_1, t_2). \tag{3.18}$$

We first consider the case $(t_1, x_1(t_1)) \in G^+$.

If $\phi_1(t_1) = \phi_2(t_1)$, then, by Lemma 3.2 and the continuity of $\phi(.)$, and $h(.,.)$, one can choose $\epsilon > 0$ such that $x_1(t) = x_2(t)$ for $t \in [t_1, t_1+\epsilon)$ which contradicts (3.18).

If $\phi_2(t_1) > \phi_1(t_1)$, then by the continuity of $\phi_i(.)$, i=1, 2, we get $\phi_2(t) > \phi_1(t)$, and hence, $u_2(t) \geq u_1(t)$, for all $t > t_1$ and close to t_1. This yields $x_2(t) \geq x_1(t)$ for $t > t_1$, which contradicts (3.18). Therefore, $\phi_2(t_1) < \phi_1(t_1)$.

Using the same argument as above and paying attention to Lemma 3.2, and Lemma 3.5, we would obtain either $x_1(t) = x_2(t)$ for all $t > t_1$ and close to t_1, or $x_1(t) > x_2(t)$ for all $t > t_1$. But these are impossible (see (3.17) and (3.18)).

Let us turn to the case $(t_1, x_1(t_1)) \in G^{ex_0}$.

If further $\phi_1(t_1) = \phi_2(t_1) = 0$, then by (3.18), for all $\epsilon > 0$ with $t_1 + \epsilon < t_2$, there exist $i \in \{1,2\}$ and $t_\epsilon \in (t_1, t_1 + \epsilon)$ such that either (i) or (ii) below occurs

(i) $(t_\epsilon, x_i(t_\epsilon)) \in G^{ex+}$.

(ii) $(t_\epsilon, x_i(t_\epsilon)) \in G^{ex-}$.

It suffices to deal with (i). Part (ii) can be considered similarly.

Suppose that (i) occurs, then by the continuity of $x_i(.)$, i = 1,2, there is a subinterval $I \subset [t_1, t_2]$ with $t_\epsilon \in I$ and such that $(t, x_i(t)) \in G^{ex+}, t \in I$. Let us set

$$\bar{t} = \inf\{t \in [t_1, t_\epsilon] \mid (l, x_i(l)) \in G^{ex+}, \forall l \in (t, t_\epsilon)\}.$$

It is easy to prove that $x_i(\bar{t}) = \eta(\bar{t})$ and $\phi_i(\bar{t}) \geq 0$ (see Lemma 3.4). Hence, Lemma 3.4 and Lemma 3.6 imply $(t, x_i(t)) \in G^{ex+}$ and $u_i(t) = \beta_2,\ t > \bar{t}$.

If $x_j(\bar{t}) < x_i(\bar{t}), j \neq i$, then from $u_j(t) \leq u_i(t) = \beta_2$ for $t > \bar{t}$ it follows that $x_j(t) < x_i(t)$ for all $t > \bar{t}$. This contradicts (3.17).

If $x_j(\bar{t}) > x_i(\bar{t})$, then by the same argument as above we get $(t, x_j(t)) \in G^{ex+}$ and $u_j(t) = \beta_2$, for all $t > \bar{t}'$ where $\bar{t}' = \inf\{t \in [t_1, \bar{t}] \mid (l, x_j(l)) \in G^{ex+}, \forall l \in (t, \bar{t})\}$. Due to (1.2) we arrive at

$$x_j(t) > x_i(t) \text{ for all } t > \bar{t}$$

(as long as $\mathrm{gr}x_1$ and $\mathrm{gr}x_2$ remain in G^{ex}) which conflicts with (3.17).

The case in which $(t_1, x_1(t_1)) \in G^{ex_0}$ and there exists $j \in \{1,2\}$ such that $\phi_j(t_1) \neq 0$, can be treated by the same way.

The proof for the case $(t_1, x_1(t_1)) \in G^{ex-}$ is similar. Consequently, (H1) holds.

b) Verification (H3) for Problem (1.1)–(1.4) of the second type.

Suppose that $(x^*(.), u^*(.), p^*(.)) \in S^*$, $x^*(t) = \alpha_i(t)$ for $t \in [t_1, t_2] \subset [0, T]$, i = 1 or 2. Suppose further that $z \in [t_1, t_2]$ and $q = p^*(z)$ or $q = p^*(z+0)$.

We verify (H3) for i = 1. The case i = 2 can be proved analogously.

Observe first that by Lemma 3.3, we obtain $\phi^*(t) = 0$ for all $t \in (t_1, t_2]$. Then Theorem 4 in [8] gives $\{(t, x^*(t)) \mid t \in (t_1, t_2)\} \subset G^{ex+} \cup G^{exo}$. Hence, by the continuity of $h(.,.)$, we obtain

$$(t, x^*(t)) \in G^{ex+} \cup G^{exo} \text{ for all } t \in [t_1, t_2].$$

Only two following cases occur

(i) $(z, \alpha_1(z)) \in G^{ex+}$,

(ii) $(z, \alpha_1(z)) \in G^{exo}$.

The case (i) will be treated as in b) of the proof of Theorem 3.1. For (ii), let us set

$$u(t) = \begin{cases} \beta_1, & \text{for } t < z, \\ \beta_2, & \text{for } t > z. \end{cases}$$

Denote by $(p(.), (x(.))$ the unique solution of the system

$$\begin{aligned} & p(z) = q, \quad x(z) = \alpha_1(z), \\ & \dot{p}(t) = -p(t) f_1(t, x(t)) + L_{1\xi}(t, x(t)) + L_{2\xi}(t, x(t)) u(t), \\ & \dot{x}(t) = f_1(t, x(t)) + f_2(t, x(t)) u(t). \end{aligned}$$

Since $u^*(t) \geq u(t) = \beta_1,\ t \in [t_1, z]$ and $u^*(t) \leq u(t) = \beta_2,\ \ t \in [z, t_2]$, it follows that

$$x(t) \geq \alpha_1(t) = x^*(t), t \in [t_1, t_2].$$

To verify (H3) it is sufficient to show that $(x(.), u(.), p(.)) \in S_{z, \alpha_1(z), q}$. For this aim, let us set

$$\begin{aligned} & t_1' := \max\{t_1, \hat{t}\} \text{ where } \hat{t} := \inf\{t \in [0, z) \mid (t, x(t)) \in G^{ex}\}, \\ & t_2' := \min\{t_2, \tilde{t}\} \text{ where } \tilde{t} := \sup\{t \in (z, T] \mid (t, x(t)) \in G^{ex}\}, \end{aligned}$$

and verify that (see Remark 3.2)

$$\phi(t) < 0 \text{ for } t \in (t_1', z), \tag{3.19}$$

$$\phi(t) > 0 \text{ for } t \in (z, t_2'). \tag{3.20}$$

Recall that $\phi(t)$ can be represented in the form

$$\phi(t) = f_2(t,x)\exp(-\int_z^t f_\xi(\tau,x,u)d\tau)[c_z + \int_z^t h(\tau,x)\exp(\int_z^\tau f_\xi(l,x,u)dl)d\tau] \tag{3.21}$$

where $c_z = [qf_2(z,x(z)) - L_2(z,x(z))]/f_2(z,x(z)) = \phi(z)/f_2(z,x(z))$.

If $z \in (z_1, z_2)$, then because of $\phi^*(z) = \phi^*(z+0) = 0$ and $q = p^*(z)$ or $q = p^*(z+0)$ we get $c_z = 0$. Define $u_\eta(.)$ as in (3.15), it follows from (3.12) that

$$u_\eta(t) > u(t) = \beta_1, \quad t \in (t_1, z_1) \text{ and } u_\eta(t) < u(t) = \beta_2, \quad t \in (z, t_2).$$

By virtue of Lemma 3.4 we obtain

$$x(t) > \eta(t) \text{ for } t \in (t_1', t_2') \setminus \{z\},$$

i.e., $h(t, x(t)) > 0$ for $t \in (t_1', t_2') \setminus \{z\}$.

Hence, (3.21) yields $\phi(t) > 0$ for $t \in (z, t_2')$, which is (3.20).

On the other hand, note that we also have

$$\phi(t) = f_2(t,x)\exp(-\int_{t_1}^t f_\xi(\tau,x,u)d\tau)[c_{t_1} + \int_{t_1}^t h(\tau,x)\exp(\int_{t_1}^\tau f_\xi(l,x,u)dl)d\tau] \tag{3.23}$$

where $c_{t_1} = \phi(t_1)/f_2(t_1, x(t_1))$. Then (3.22) and the fact that $\phi(z) = 0 = c_z$ imply $c_{t_1} < 0$. The inequality (3.19) follows from (3.22)–(3.23).

The proof for the case $z = t_1$ or $z = t_2$ is similar. The proof of Theorem 3.2 is completed. ∎

We have just proved that for Problem (1.1)–(1.4) of either the first or the second type, the hypotheses (H1), (H3)–(H5) are satisfied. Hence, the Method of Orienting Curves can be applied to construct its optimal process.

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