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AFFINE POLAR QUOTIENTS OF ALGEBRAIC PLANE CURVE

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Abstract. The aim of this paper is to introduce the notion of "affine polar quotients" of algebraic plane curve and its applications for the study at infinity.

1. Introduction

Let $f : C^2, 0 \to C, 0$ be a germ of a holomorphic function, $\ell = \beta x - \alpha y$ a generic linear form and $\gamma \subset \{(x,y)/\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} = 0\}$ an irreductible component of the polar curve of f. Then $v^0_\gamma(f/\gamma) := \frac{(f^{-1}(0).\gamma)_0}{mult_0(\gamma)}$ is called a polar quotient of $f^{-1}(0)$ (cf. [LMW]). The polar quotients have been used by several authors to study the local singularities. In particular, Le-Michel-Weber [LMW] have recently proved that the polar quotients of a plane curve singularity are the topological quotients defidned by the link of the singularity (namely the quotients of the linking coefficients by the braid indexes of all of Seifert fibers defined by the link of the singularity).

In this paper we give the notion of "affine polar quotient" of algebraic plane curve and its recent applications to the global topology of affine curves. This notion is a generalisation of the one in the local case to the global case. Our main result generalizes Theorem C of [LMW]. As consequences, we recover again the theorem of Moh-Ephaim about the regularity of algebraic plane curve which has only one place at infinity (cf. [E]) and Vui's proof for a Neumann's conjecture about the link at infinity [H].

2. Affine polar quotients

Let $P \in C[x,y]$ be a polynomial. Let γ be a *"component at infinity"* of the affine polar curve of P. This means that there is a linear form $\ell = \beta x - \alpha y$

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such that $\ell^{-1}(0)$ is not tangential to $P^{-1}(0)$, $\gamma \subset \{\alpha\frac{\partial P}{\partial x} + \beta\frac{\partial P}{\partial y} = 0\} \setminus P^{-1}(0)$ and there is a representation

$$\gamma = \begin{cases} x & = x_\gamma(t) \in t^{-N}C\{t\} \\ y & = y_\gamma(t) \in t^{-N}C\{t\} \end{cases}$$

wiht $||(x_\gamma(t), y_\gamma(t))|| \to \infty$ as $t \to 0$. In other words, γ is a Puiseux expansion at infinity of the affine polar curve of P.

DEFINITION: We set

$$v_\gamma^\infty(P/\gamma) = \frac{v_t(P(x_\gamma(t), y_\gamma(t)))}{v_t(\ell(x_\gamma(t), y_\gamma(t)))}$$

and call it an affine polar quotient of $P^{-1}(0)$.

REMARK: 1). If $\beta \neq 0$ and $y = y_\gamma(x) = \sum_{\lambda \in Q, \lambda \leq 1} a_\lambda x^\lambda$ is a representation of γ, then it is easy to see that

$$v_\gamma^\infty(P/\gamma) = v_x^\infty(P(x, y_\gamma(x))),$$

where

$$v_x^\infty(\sum_{\mu \in Q} b_\mu x^\mu) = \max\{\mu \ / \ b_\mu \neq 0\}.$$

Thereby an affine polar quotient of $P^{-1}(0)$ is the natural valuation (at infinity) of the restriction of P on some "component at infinity" of the affine polar curve of P.

2) If $v_\gamma^\infty(P/\gamma) > 0$, then

$$v_\gamma^\infty((P-a)/\gamma) = v_\gamma^\infty(P/\gamma) \quad \forall a \in C.$$

But this is not true in general if $v_\gamma^\infty(P/\gamma) \leq 0$.

EXAMPLE: $P = x(xy-1)$. It is evident that $\ell = x - y$ is generic for $P^{-1}(0)$ and the affine polar curve

$$\{2xy - 1 + x^2 = 0\} \quad \text{or} \quad \{y = \frac{1}{2x} - \frac{x}{2}\}$$

has only one irreductible component. But there are two "components at infinity" (namely two directions go to infinity): $\gamma_1 = \left\{ \begin{array}{l} x = t^{-1} \\ y = \frac{1}{2}t - \frac{1}{2}t^{-1} \end{array} \right\}$ and $\gamma_2 = \left\{ \begin{array}{l} x = t \\ y = \frac{1}{2}t^{-1} - \frac{1}{2}t \end{array} \right\}$. We have

$$v^{\infty}_{\gamma_1}(P/\gamma_1) = 3$$
$$v^{\infty}_{\gamma_2}(P/\gamma_2) = -1$$

and $v^{\infty}_{\gamma_1}((P-a)/\gamma_1) = 3, v^{\infty}_{\gamma_2}((P-a)/\gamma_2) = 0 \quad \forall a \in C^*$.

3. Link at infinity and topological quotients

Let $P \in C[x,y]$ be a reduced polynomial. Then the intersection of $P^{-1}(0)$ with any sufficiently large sphere S^3 around the origin in C^2 is transverse and give a well-defined link (S^3, L), called the *link at infity* of $P^{-1}(0)$. In [N] the link at infinity of $P^{-1}(0)$ can be described as follows.

Let $C = \overline{P^{-1}(0)}$ be the compactification of $P^{-1}(0)$ by an embedding $C^2 \subset \mathbb{P}^2$. We can assume that the line $\mathbb{P}^1$ at infinity is not a component of C. So C meets $\mathbb{P}^1$ in finitely many points $Y_1, ..., Y_n$, say. Choose an embedded disk D_0 in $\mathbb{P}^1$ which contains $C \cap \mathbb{P}^1$ and let D be a thin 4-disk regular neighborhood of D_0 in $\mathbb{P}^2$ whose boundary $S = \partial D$ meets C and $\mathbb{P}^1$ transversely.

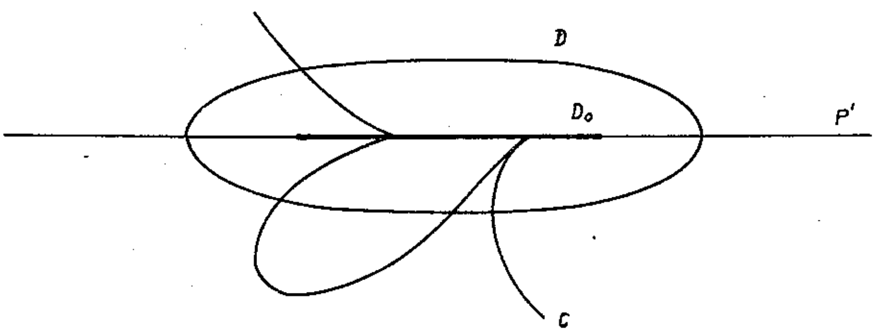

Figure 1.

Let L_i be the link of $\mathbb{P}^1 \cup C$ at the point Y_i (the link of a singularity) and

$L_0 = (S, (\mathsf{P}^1 \cup C) \cap S)$ denote the link of $\mathsf{P}^1 \cup C$ on S. Then L_0 is the splicing of L_i $(i = 1, ..., n)$ along the component $K_0 = \mathsf{P}^1 \cap S$.

Now let $N(\mathsf{P}^1)$ be a thin closed tubular neighborhood of P^1 in P^2. Then $S^3 \doteq \partial N(\mathsf{P}^1)$ is a "sphere at infinity" in C^2, and $L_0^{'} = (S^3, S^3 \cap C)$ is, but for orientation the link at infinity that interests us. Note that $N(\mathsf{P}^1)$ is obtained from D by adding a 2-handle along $K_0 \subset S = \partial D$. So $L_0^{'}$ is obtained from L_0 by $(+1)$-Dehn surgery on K_0.

Finally, by reversing the ambient orientation of $L_0^{'}$ we obtain the link at infinity of $P^{-1}(0)$. Hence we have described the link at infinity (global link L) in term of the links of the singularities L_i (the local links). This description is canonical but not unique in general.

Let ρ be a (typical or exceptional) fiber of the Seifert structure of the (global) link L induced by the Seifert structures of the (local) links L_i. By the meaning of homotopical invariance, it is called a *virtual component* of L. For any virtual component ρ, let $\ell(\rho, L)$ denote the linking number between ρ and L (called the *linking coefficient* of ρ) and $\ell(\rho, K_0)$ the linking number between ρ and K_0 (called the *braid index* of ρ with respect to K_0). Then the quotient $\frac{\ell(\rho,L)}{\ell(\rho,K_0)}$ is called the *topological quotient* of L given by ρ, it with respect to K_0 (or to the chosen projectification).

4. Topological character of affine polar quotients.

4.1. THEOREM. *Suppose that $P \in C[x,y]$ reduced. Let Q be the set of all of affine polar quotients of $P^{-1}(0)$ and Q_{top} be the set of all of topological quotients of the link at infinity of $P^{-1}(0)$ with respect to a chosen projectification. Then $Q = Q_{top} \cup \{q_{\max}\}$, where $q_{\max} = \max\{q \in Q\}$.*

PROOF: Theorem 4.1 is reduced from the following lemmas :

4.2. LEMMA. *For any affine polar component at infinity γ we have*

$$v_\gamma^\infty(P/\gamma) = d - \frac{(\overline{P^{-1}(0)}.\overline{\gamma})}{(z = 0.\overline{\gamma})}.$$

where d is the degree of P, $\overline{P^{-1}(0)}$ and $\overline{\gamma}$ are compactifications of $P^{-1}(0)$ and γ, respectively, and $z = 0$ is the line at infinity of $\mathbb{P}^2$.

PROOF: . We may choose the coordinates such that the polar curve is given as

$$G = \{(x, y) \in C^2 | \frac{\partial P(x, y)}{\partial y} = 0\}$$

and, in addition, the component at infinity $\gamma \subset G$ has Puiseux expansion at infinity:

$$\gamma = \{(x, y) | y = y_\gamma(x)\} \quad \text{with} \quad y_\gamma(x) = \sum_{\alpha \in 1 + Q^-} a_\alpha x^\alpha$$

where $Q^- = \{\alpha \in Q | \alpha < 0\}$ and $P_y(x, y_\gamma(x)) \equiv 0$. In particular, the point at infinity in question is in $x \neq 0$. One has

$$v_\gamma^\infty(P/\gamma) = v_x^\infty(P(x, y_\gamma(x))).$$

On the other hand, if $\overline{P}(z, y) = z^d P(\frac{1}{z}, \frac{y}{z})$ is the equation for $\overline{P^{-1}(0)}$ in local coordinates at infinity, then $\overline{P}_y(z, z y_\gamma(\frac{1}{z})) \equiv 0$, which means that

$$\overline{\gamma} := \{(z, y) : y = z y_\gamma(\frac{1}{z}) =: y_{\overline{\gamma}}(z)\}$$

is the compactification of γ. Therefore,

$$v_z(\overline{P}(z, y_{\overline{\gamma}}(z))) = d - v_x^\infty(P(x, y_\gamma(x))).$$

But in the local situation, one knows that

$$v_z(\overline{P}(z, y_{\overline{\gamma}}(z))) = \frac{\overline{\gamma}.\overline{P}^{-1}(0)}{\overline{\gamma}.\{z = 0\}}.$$

Thus, Lemma (4.2) is proved.

4.3. LEMMA. (cf. [L], 2.1). *Let ρ be a virtual component of L (the link at infinity of $P^{-1}(0)$) and $L_{i(\rho)}$ the local link of singularity of $\overline{P^{-1}(0)}$ at $Y_{i(\rho)}$ which has also ρ as a (local) virtual component. Let $q(\rho, L)$ (resp. $q(\rho, L_{i(\rho)})$) be the topological quotient of L (resp. $L_{i(\rho)}$) given by ρ with respect to K_0. Then $q(\rho, L) + q(\rho, L_{i(\rho)}) = d$, where d is the degree of P.*

PROOF: Note that $(1/k)$-Dehn surgery on a knot K_0 in S^3 replaces the linking number $\ell(C_1, C_2)$ of any two disjoint 1-cycles, which are disjoint from K_0, by $\ell(C_1, C_2) - k\ell(C_1, K_0)\ell(C_2, K_0)$. Applying this with $C_1 = \rho,\ C_2 = L_0$ we have

$$\ell(\rho, L) = -\ell(\rho, L_0^{'}) = -[\ell(\rho, L_0) - \ell(\rho, K_0)\ell(L_0, K_0)]$$

On the other hand, $\ell(\rho, L_0) = \ell(\rho, L_{i(\rho)}$ and $\ell(L_0, K_0) = d$. Hence

$$q(\rho, L) = \frac{\ell(\rho, L)}{\ell(\rho, K_0)} = -\frac{\ell(\rho, L_{i(\rho)})}{\ell(\rho, K_0)} + d.$$

4.4. LEMMA. *Let $f : C^2, 0 \to C, 0$ be a germ of a holomorphic function, $z = \beta x - \alpha y$ a linear form, and $\gamma \subset \{\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = 0\}$ an irreducible component of the polar curve of $f^{-1}(0)$ with respect to $z = 0$. Let Q^{loc} (resp. Q^{loc}_{top}) be the set of the local polar (resp. topological) quotients of $f^{-1}(0)$ with respect to $z = 0$. Then*

$$Q^{loc} = Q^{loc}_{top} \cup \{q^{loc}_{min}\}$$

where $q^{loc}_{min} = \min\{q \in Q^{loc}\}$

PROOF: This is a generalisation of [LMW] (Theorem C) to the case where $\{z = 0\}$ may be not transversal to $f^{-1}(0)$ (i.e., z may be not generic). The proof of Lemma (4.4) is the same as in [LMW], word by word, but only replacing the multiplicities by the corresponding intersection numbers. Note that the first local polar quotient (in the order of [LMW]) is a minimum one. In the relative situation ($z = \beta x - \alpha y$ may be not generic) this minimum quotient may be not the intersection number between $f^{-1}(0)$ and $\{z = 0\}$.

5. APPLICATIONS: In this section we give some applications of Theorem 4.1 for the study at infinity of algebraic plane curves. We recall that the curve $P^{-1}(0)$ is called *regular at infinity* if outside of a compact domain of C^2 the polynomial P define a trivial fibration in some neighborhood of $P^{-1}(0)$.

5.1. THEOREM. *Let $P \in C[x,y]$ be a reduced polynomial. Then the following are equivalent :*

(i) $P^{-1}(0)$ is regular at infinity.

(ii) The topological quotients of the link at infinity of $P^{-1}(0)$ with respect to any projectification of C^2 are not negative.

(iii) The affine polar quotients of $P^{-1}(0)$ are not negative.

PROOF: That (i) $\Rightarrow$ (ii) was already proved in [N] because the braid index is always not negative.

That (ii) $\Rightarrow$ (iii) follows from Theorem 4.1.

It remains to prove (iii) $\Rightarrow$ (i). Suppose that (iii) holds. That is, for all components at infinity γ of the affine polar curve

$$G = \{(x,y) \in C^2 : \alpha\frac{\partial P}{\partial x} + \beta\frac{\partial P}{\partial y} = 0\}$$

(α, β general), one has $v_\gamma^\infty(P/\gamma) \geq 0$. This means that $P(x,y)/_\gamma \not\to 0$ as $|(x,y)| \to \infty$. So there exists $\delta > 0$ such that for sufficiently large R one has

$$P^{-1}(\Delta_\delta) \cap (C^2 - B_R) \cap G = \emptyset,$$

where $\Delta_\delta = \{t \in C : |t| < \delta\}$ and $B_R = \{(x,y) \in C^2 : |(x,y)| < R\}$. In other words, we have a line field $\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}$ which is transverse to $P^{-1}(t) \cap (C^2 - B_R)$ for all $t \in \Delta_\delta$. Then the vector field

$$(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y})/(\alpha\frac{\partial P}{\partial x} + \beta\frac{\partial P}{\partial y})$$

trivializes the neighborhood at infinity $P^{-1}(\Delta_\delta) \cap (C^2 - B_R)$ of $P^{-1}(0)$. This completes the proof.

5.2. REMARK: Theorem (5.1) gives us a criterion for the regularity at infinity of algebraic plane curves. It was first proved in [L] (and then [NL]). Note that the implication (ii) $\Rightarrow$ (i) is a conjecture of [N] and was settled in [H] by using Lojasiewirz numbers at infinity.

5.3. COROLLARY. (Moh-Ephraim theorem [E]). *Any algebraic plane curve, which has only one place at infinity, is regular at infinity.*

PROOF: Let γ be a "component at infinity" of the affine polar curve of $P^{-1}(0)$. By Lemma 4.2, its corresponding polar quotient has the form

$$v_\gamma^\infty(P/\gamma) = d - \frac{(\overline{P^{-1}(0)}.\overline{\gamma})}{(z = 0.\overline{\gamma})}.$$

Let Γ be the irreducible component of the affine polar curve of $P^{-1}(0)$ which contains γ. Then Γ has not any other point at infinity because $P^{-1}(0)$ (and hence $P^{-1}(t)$ $\forall t$) has only one point at infinity. So we have $(\overline{P^{-1}(0)}.\gamma) = (\overline{P^{-1}(0)}.\overline{\Gamma})_\infty$ and $(z = 0.\gamma) = deg\ \Gamma$ (degree of Γ). Hence

$$v_\gamma^\infty(P/\gamma) = d - \frac{(\overline{P^{-1}(0)}.\overline{\Gamma})_\infty}{deg\ \Gamma}$$

$$v_\gamma^\infty(P/\gamma) deg\ \Gamma = d.deg\ \Gamma - (\overline{P^{-1}(0)}.\overline{\Gamma})_\infty$$

By Bezout theorem $d.deg\ \Gamma$ is the total intersection number in $\mathbb{P}^2$ of $\overline{P^{-1}(0)}$ and $\overline{\Gamma}$, so it is greater than $(\overline{P^{-1}(0)}.\overline{\Gamma})_\infty$ in general. Hence $v_\gamma^\infty(P/\gamma) \geq 0$ and the corollary is proved by Theorem 5.1 (iii).

REFERENCES

[E] R. Ephraim, *Special polars and curves with one place at infinity*, Proc. Symp. Pure Math. **40-1**, (1983), 353–360.

[H] Ha Huy Vui, *Sur l'irrégularité du diagramme splice pour l'entrelacement à l'infini des courbes planes*, C.R. Acad. Sci. Paris, t.313, Série I (1991), 277–280.

[L] Le Van Thanh, *Démonstration d'une conjecture de Neumann sur l'entrelacs à l'infini d'une courbe plane algebrique*, Max-Planck Inst. für Math. MPI/91-66.

[LMW] Le Dung Trang, F. Michel and C. Weber, *Courbes polaires et topologie des courbes planes*, Ann. Sc. Ec. Norm. Sup. 4^e Series, t.24 (1991), 141–169.

[N] W. Neumann, *Complex algebraic plane curves via their links at infinity*, Invent. Math. **98**(1989), 445–489.

[NL] W. Neumann and Le Van Thanh, *On irregular links at infinity of algebraic plane curves*, Math. Ann., to appear.

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