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LEFT SF-RINGS WHOSE COMPLEMENT LEFT IDEALS ARE IDEALS

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Abstract. A ring R is called a left (right) SF-ring if every simple left (right) R-module is flat. It is known that von Neumann regular rings are left and right SF-rings. In this note, we prove that if R is a left SF-ring whose complement left ideals are ideals, then R is strongly regular.

All rings considered in this paper are associative with identity, and all modules are unital. A ring R is (von Neumann) regular provided that for every $a \in R$ there exists $b \in R$ such that $a = aba$. R is called a strongly regular ring if for each $a \in R$, $a \in a^2R$. Following [1], call a ring R a left (right) SF-ring if every simple left (right) R-module is flat. It is known that every von Neumann regular ring is a left and right SF-ring. Ramamurthi [1] initiated the study of SF-rings and the question whether a SF-ring is necessarily regular. Since several years, SF-rings have been studied by many authors and the regularity of SF-rings satisfying certain additional conditions are obtained (cf. for example, [2] to [5]). In [2], M.B. Rege proved that a ring R is strongly regular if R is a left SF-ring whose maximal right ideals are ideals . R.Yue Chi Ming [5, Theorem 4] proved the strong regularity of left SF-ring whose maximal left ideals are ideals, which answers a question raised in [6, p.441]. Using complement one-sided ideals instead of maximal one -sided ideals, R.Yue Chi Ming [3, Prop.3] showed that if R is a right SF-ring whose complement left ideals are ideals, then R is strongly regular, and he proposed the following question: Is R strongly regular if R is a left SF-ring whose complement left ideals are ideals? In this note, we gives a positive answer to the question.

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We first prove some important lemmas.

A ring without nonzero nilpotent elements is called a reduced ring. We use Z to denote the left singular ideal of a ring R.

LEMMA 1. *Let R be a ring. If every complement left ideal of R is an ideal, then R/Z is reduced.*

PROOF: Suppose there exists $a \in R, a \notin Z$ such that $a^2 \in Z$, then $l(a)$ is not left essential in $l(a^2)$ and hence there exists a nonzero left ideal I such that $l(a) \oplus I$ is left essential in $l(a^2)$. Let C be a complement of $l(a)$ in R such that $I \subseteq C$. By hypothesis, C is an ideal of R. Since $Ia \subseteq Ca \subseteq C$ and $Ia \subseteq l(a)$, then $Ia \subseteq C \cap l(a) = 0$ which implies $I \subseteq l(a)$. Therefore $I = I \cap l(a) = 0$, a contradiction to $I \neq 0$. This proves that R/Z is reduced.

LEMMA 2. *Let R be a left SF-ring. If every complement left ideal of R is an ideal, then R/Z is a strongly regular ring.*

PROOF: By Lemma 1, R/Z is reduced. Since R is a left SF-ring, then R/Z is a left SF-ring by [2, Prop. 3.2], and hence R/Z is strongly regular by [2, Remark 3.13].

LEMMA 3. *Let R be a left SF-ring. If every complement left ideal of R is an ideal, then $Z = 0$.*

PROOF: Let $Z \neq 0$. For every $0 \neq a \in Z$, consider

$$T = Z + r(a).$$

If $T \neq R$, then there is a maximal right ideal K of R such that $T \subseteq K$. Because of the known fact that stronly regular rings are right and left duo, it follows from Lemma 2 that K/Z is an ideal of R/Z. Then K is an ideal of R. Thus there is a maximal left ideal L such that

$$T \subseteq K \subseteq L \subset R.$$

Since R is a left SF-ring and $a \in L$ we have $a = ab$ for some $b \in L$ which implies $1 - b \in r(a) \subseteq L$, whence $1 = (1 - b) + b \in L$, contradicting $L \neq R$. Thus $T = Z + r(a) = R$. This implies that there exist some $u \in Z$ and

$d \in r(a)$ such that $u + d = 1$ and hence $au = a$. Since $l(u)$ is left essential in $R, l(u) \cap Ra \neq 0$ and hence there is $x \in R$ such that $xa \neq 0$ and $xa \in l(u)$. This gives $xau = 0$, that is $xa = 0$ since $au = a$, this contradicts $xa \neq 0$. Therefore $Z = 0$.

Now we state our main result which gives a positive answer to the question raised in [3].

THEOREM. *If R is a left SF-ring whose complement left ideals are ideals, then R is strongly regular.*

PROOF: It follows from Lemmas 2 and 3 that R is strongly regular.

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