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PROBLEMS OF VECTOR OPTIMIZATION

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Abstract. The paper discusses results in main problems of vector optimization. Optimality notions and general existence theorems presented with an emphasis on proper efficiency. Norm scalarization in normed spaces ordered by general convex cone and other scalar representations are considered. For duality, we propose a scheme of constructing dual problems in an axiomatic approach, which includes Lagrangean duality as a special case. Furthermore, both necessary optimality conditions and sufficient conditions are obtained under relaxed assumptions and for general problems so that the Pontryagin maximum principle for cooperative differential games can be derived as consequences. Finally, we extend Ekeland's variational principle to vector optimization problems in a general setting.

Anyone attempts to make decisions in an optimal way. Traditionally, good decisions making have been based on optimizing a single criterion, i.e., a single objective. In various sciences such as sociology, economics, politics, technology, however, the concern has always been the satisfaction of aspirations, resulting in a theory of multicriteria optimization (or, what is the same, multiobjective optimization or vector optimization).

Beside specific problems as optimality notions and scalarization, in vector optimization there arise the same issues as in scalar optimization existence problems, optimality conditions, duality, stability, and so on.

In this paper we present some results that we have recently developed in the field of vector optimization.

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1. Optimization notions

In contrast to scalar optimization, where the objective space is R with the unique natural ordering, in vector optimization there is a variety of fashions to order the objective space, depending on the preference attitude of the decision maker. In most cases, the ordering is defined by a convex cone in a linear space. However, with an ordering already fixed, there are still various notations of optimality: weak (or Slater) optimality, (Pareto) optimality, proper optimality and strong (or ideal) optimality. In this context, the terminology should be explained. In the literature the terms "efficient", "nondominated", "noninferior" or "preferred" are sometimes aliases of "optimal". The terms "efficient" and "nondominated" are even more frequently used.

Let X be a set and Y a vector space ordered by a convex cone K. A point $x_0 \in S \subseteq X$ is called a (Pareto) minimizer of a mapping $F : X \to Y$ on S if $(F(x_0) - K) \cap F(S) \subseteq F(x_0) + K$. When the relative interior ri$K$ is nonempty, a point $x_0 \in S$ is said to be a weak (or Slater) minimizer of F on S if $(F(x_0) - riK) \cap F(S) = \emptyset$. A point $x_0 \in S$ is referred to as a strong minimizer of F on S if $F(S) \subseteq F(x_0) + K$. If X is a topological space and if in the first two definitions $F(S)$ is replaced by $F(S \cap N)$ for some neighborhood N of x_0, then we have a local Pareto (or weak, resp.) minimizer (of some kind), $F(x_o)$ is naturally called a minimum (of the corresponding kind).

Proper optimality is the most complicated specific solution notion of vector optimization. There are a number of different definitions. Each of them shows that improper minimizers are in a certain sence anomalous. However, in the simplest case, where $Y = R^n, K = R^n_+$ and $F(S)$ is convex, all known definitions of properness coincide. In this section we discuss three properness definitions of Geoffrion [9] Kuhn and Tucker [33] and Benson [1] in a general setting.

In the remainder of the section let X and Y be normed spaces and let K be an ordering cone of Y. A minimum $y_o \in A \subset Y$ is set to be a Geoffrion proper minimum (G.P.min.) of A if there exists a real $\beta > 0$ with the property that if $\langle \lambda_1, y_0 \rangle > \langle \lambda_1, y \rangle$ for some $\lambda_1 \in Y^*, \| \lambda_1 \| = 1$, and some $y \in A$, then one has $\lambda_2 \in Y^*$, $\| \lambda_2 \| = 1$, such that

$$\langle \lambda_1, y_0 - y \rangle \leq \beta \langle \lambda_2, y - y_0 \rangle.$$

A minimum y_0 is called a Benson proper minimum (B.p. min.) of A if

$$(-K) \cap \mathrm{clcone}(A - y_0 + K) \subset K.$$

For the Kuhn-Tucker properness we have the following extension to a problem with parameter. Beside X and Y, let normed spaces Z and W be also given, where Z is ordered by a convex cone M. Let U be an arbitrary set. Let the mapping $F : X \times U \to Y, G : X \times U \to Z$ and $P : X \times U \to W$ be given. Then consider the vector optimization problem

$$\min F(x, u), \tag{1.1}$$

$$\text{s.t. } (x, u) \in X_0 \times U_0 \tag{1.2}$$

where $X_0 \times U_0$ consists of all (x, u) satisfying

$$G(x, u) \leq 0, \tag{1.3}$$

$$P(x, u) = 0, \tag{1.4}$$

$$x \in X, u \in U. \tag{1.5}$$

The infinite dimension of the concerned spaces and the presence of parameter u, on which no differentiability conditions is imposed, allow us to consider dynamic control problems and reduce them to Problem (1.1)-(1.2) (see Section 5).

For a given $(x_0, u_0) \in X_0 \times U_0$, set

$$M_0^* = \{\mu \in M^* : \langle \mu, G(x_0, u_0) \rangle\} = 0.$$

Assume that $\mathrm{int} M_0^{**} \neq \emptyset$, that $F(., u_0)$ and $G(., u_0)$ have directional derivatives in all directions at x_0, and that $P(., u_0)$ is Gateaux differentiable at x_0. Then, a minimizer (x_0, u_0) is called a Kuhn-Tucker proper minimizer of the first type

(KT.p.min.I) of Problem (1.1)-(1.2) if there do not exist $\tilde{x} \in X, \alpha_1 > 0, \alpha_s > 0, u_1 \in U, \cdots, u_s \in U$ for some integer s such that

$$F'_x(x_0, u_0; \bar{x}) + \sum_{j=1}^{s} \alpha_j (F(x_0, u_j) - F(x_0, u_0)) \in -K \backslash \{0\},$$

$$G'_x(x_0, u_0; \bar{x} + \sum_{j=1}^{s} \alpha_j (G(x_0, u_j) - G(x_0, u_0)) \in -\text{ int } M_0^{**},$$

$$P'_x(x_0, u_0)\bar{x} + \sum_{j=1}^{s} \alpha_j P(x_0, u_j) = 0$$

If $-$int M_0^{**} in (1.6) is repaced by $-M_0^{**}$, then (x_0, u_0) is called a KT.p.min.II, which is stronger than a KT.p.min.I.

Now we compare three properness notions. It is known [3] that every G.p.min. is also a B.p.min. For finite dimensions the reverse implication is true. However, in infinite dimensions we have the following counterexample.

EXAMPLE1.1: Let $Y = l_2, K = l_{2+} := \{y = (y_1, y_2, \cdots) \in l_2 : y_i \geq 0 \quad \forall i\}, A = z^k \cup \{0\}$, where

$$z^{k_i} = \begin{cases} -k & \text{if } i = k, \\ 1 & \text{if } i = 1, k \neq 1, . \\ (ik)^{-1} & \text{if } i \neq k, i \neq 1 \end{cases}$$

Then, 0 is a B.p.min. of A but not a G.p.min.

Dealing with KT.p.min. we need the following definition. The constraints of Problem (1.1)-(1.2) are said to satisfy the Kuhn-Tucker constraint qualification of the first type (KT.CQ.I) at $(\bar{x}, \bar{u})$ if from

$$G'_x(\bar{x}, \bar{u} : x') + G(\bar{x}, u') - G(\bar{x}, \bar{u}) \in -\text{int } M_0^{**}, \tag{1.7}$$

$$P'_x(\bar{x}, \bar{u})x' + P(\bar{x}, u') = 0,$$

for some $x' \in X$ and $u' \in U$, it follows the existence of $\tilde{x} : [0, T] \to X$ for some $t_0 > 0$ such that $t^{-1}\tilde{x}(t) \to 0$ as $t \to 0$ and that $(x(t), \bar{u})$ and$(x(t), u')$ satisfy (1.3)-(1.5) for all $t \in [0, t_0]$, where $x(t) = \bar{x} + tx + \tilde{x}(t)$. If $-$int M_0^{**} in (1.7) is replaced by $-M_0^{**}$ we have KT.CQ.II, which is stronger than KT.CQ.I

THEOREM 1.2. *Assume that Problem (1.1)-(1.2) with* $\mathrm{int} M_0^{**} \neq \emptyset$ *satisfies the following conditions:*

(i) at $x_0, F(., u_0)$ *has continuous directional derivatives,* $G(., u_0)$ *has directional derivatives and* $P(., u_0)$ *is Gateaux differentiable:*

(ii) $F(x_0 + tx, u) \to F(x_0, u)$ *as* $t \to 0$ *for each* u *and* x;

(iii)KT.CQ.I (or KT.CQ.II) is satisfied;

(iv) for each x *in a neighborhood of* x_0, *for each* $u_1, u_2 \in U$ *and each* $\alpha \in [0,1]$, *there exists* $u \in U$ *such that*

$$F(x,u) \leq (1-\alpha)F(x,u_1) + \alpha F(x,u_2),$$

$$G(x,u) \leq (1-\alpha)G(x,u_1) + \alpha G(x,u_2),$$

$$P(x,u) = (1-\alpha)P(x,u_1) + \alpha P(x,u_2).$$

If (x_0, u_0) *is a B.p.min., then it is also a KT.p.min.I (or KT.p.min.II, resp.).*

Without the KT.CQ., using the Lusternik theorem we can prove a similar relation as follows.

THEOREM 1.3. *Assume for Problem (1.1)-(1.2) with* $\mathrm{int} M \neq \emptyset$ *that*

(i)$P(., u_0)$ *is continuously Fréchet differentiable at* x_0;

(ii) F *and* G *satisfy condition (ii) of Theorem 1.2 ;*

(iii) $F(., u_0)$ *and* $G(., u_0)$ *have continous directional derivatives at* x_0;

(iv) condition (iv) of Theorem 1.2 is satisfied;

(v) $P'_x(x_0, u_0)X = W$.

If (x_0, u_0) *is a B.p.min., then it is a KT.p.min.I.*

Since Geoffrion's properness is the strongest among the three types, it is worth deriving the following sufficient condition for this properness via a scalar optimization problem.

THEOREM 1.4. *Assume that K has a weakly compact base. If $\lambda \in K^*$ exists with $\langle \lambda, y\rangle > 0 \quad \forall y \in K\backslash\{0\}$ and $\langle \lambda, y_0\rangle = min\langle \lambda, y\rangle$, then y_0 is a G.p. min. of A*

This is an extension of a result in [10]. Moreover, one can easily prove that the theorem is valid for B.p.min. without the compactness assumption.

In Subsection 3.1 the following definition of Borwein [2] will be considered. A point $y_0 \in A \subset Y$ is said to be a Borwein proper minimum of A if 0 is a minimum of the tangent cone $T(A + K, y_0)$ of $A + K$ at y_0.

2. Existence of efficient points

This section is devoted to general existence theorems which contain most known results based on compactness assumptions. For the sake of generality we consider a topological vector space Y. A convex cone K is called correct if $\mathrm{cl}K + K\backslash(-K) \subseteq K$. A set $A \subseteq Y$ is said to be K- complete (strongly K-complete, resp.) if it has no covers of the form $\{(y_\alpha - \mathrm{cl}K)^c : \alpha \in I\}(or\{(y_\alpha - K)^c : \alpha \in I\}$, resp.) with $\{y_\alpha\}$ being decreasing net in A, where $(.)^c$ stands for the complement $Y\backslash(.)$. For $y \in A$ we call $(y - K) \cap A$ a section of A.

THEOREM 2.1. *Let K be a correct cone and A be a nonempty set in Y. Then there exist efficient points of A if and only if A has a nonempty K-complete section.*

The correctness assumption may be weakened by strengthening the completeness as follows.

THEOREM 2.2. *Let K be a cone and A be a nonempty set in Y. Then, efficient points of A exist if and only if A has a nonempty strongly K complete section.*

Roughly speaking, having K-complete sections is a general characterization for the existence of efficient points. By verifying criteria for K-completeness in various circumstances we obtain as consequences many known results, e.g. of Corley [5], [6], Borwein [4], Henig [11] and Jahn [16].

3. Scalarization

Scalarization means the replacement of an original vector optimization problem by a suitable scalar problem, i.e., by a problem with a real-valued objective functional. If this functional is a norm, we have a norm scalarization. This type of scalarization will be considered in the first subsection. As we shall see there, norm scalarization gives close relations between vector optimization and approximation theory and tells many things about the geometric structure of the objective space . However, norm scalarization seems to be suitable only for finding solutions directly in the objective space. For more general problems of the form

$$\begin{cases} \min F(x), \\ \text{s.t. } x \in S \subseteq X, \end{cases} \tag{3.1}$$

where $F : X \to Y$ is a mapping between two topological vector spaces with an ordering cone K in Y, we should look for a more general scalarization, i.e., for a scalar problem

$$\begin{cases} \min s(x), \\ \text{s.t. } x \in S \subseteq X, \end{cases} \tag{3.2}$$

where s is a functional on X. A natural idea is that s may be found in the form $s = \xi \circ F$, where $\xi : Y \to R$ is a functional. When ξ is linear, we have linear scalarization. Theorem 1.4 and the results in Section 5 are in the sense of linear scalarization. In Subsection 3.2 we consider more general forms of ξ.

3.1 Norm scalarization:

We introduce an orthogonality concept in a normed space Y as follows. Assume that Y is a direct sum $Y = L \oplus L^+$ of two closed subspaces, with codim $L > 1$. Then, we say that L^+ is orthogonal to L if the following monotonicity of the canonical projection $p : Y \to L^+$ holds:

$$d(y_1, z_1 + L) < d(y_2, z_2 + L) \tag{3.3}$$

implies

$$\| p(y_1) - p(z_1) \| < \| p(y_2) - p(z_2) \|, \tag{3.4}$$

and if (3.3) becomes an equality, then so does (3.4).

If Y is a Hilbert space, then L is orthogonal to L if and only if L^+ is the orthogonal complement of L in the usual sense.

Throughout this subsection assume that $A \subseteq Y$ is a nonempty subset, that $\text{cl}K \neq Y$and that $Y = L \oplus L^+$, where L^+ is orthogonal to L if codim $L >$ 1. Sometimes we denote $p(A)$ by A^+.

THEOREM 3.1. *Assume that, for all $y_1, y_2 \in K^+$,*

$$\| \alpha y_1 + \beta y_2 \| \leq \| y_1 + y_2 \| \tag{3.5}$$

whenever $\alpha, \beta \in [0, 1]$.

(a) If $A \subseteq \hat{y} + K$ for some $\hat{y} \in Y$, then any point $y_0 \in Y$ with

$$d(y_0, \hat{y} + L) < d(y, \hat{y} + L), \quad \forall y \in A \setminus y_0,$$

is an efficient point of A.

(b) Every point $y_0 \in (\tilde{y} - K) \cap A$, for some $\tilde{y} \in A$, satisfying

$$d(y_0, \tilde{y} + L) > d(y, \tilde{y} + L), \quad \forall y \in ((\tilde{y} - K) \cap A) \setminus \{y_0\},$$

is an efficient point of A.

THEOREM 3.2. *Assume that ri $K \neq \emptyset$ and that*

$$K^+ \cap (u - \text{ri}K^+) \subseteq L^+ \cap B(0, \| u \|) \quad \forall u \in L^+,$$

where $B(y, r)$ is the ball of radius r and centered at y.

(a) If $\subseteq \hat{y} + K$ for some $\hat{y} \in Y$, then any point $y_0 \in Y$ with

$$d(y_0, \hat{y} + L) \leq d(y, \hat{y} + L), \quad \forall y \in A \tag{3.6}$$

is a weakly efficient point of A.

(b) Every point $y_0 \in (\tilde{y} - K) \cap A$, for some $\tilde{y} \in A$, with

$$d(y_0, \tilde{y} + L) \geq d(y, \tilde{y} + L), \quad \forall y \in (\tilde{y} - K) \cap A),$$

is a weakly efficient point of A.

THEOREM 3.3. *Assume that*

$$K^+ \cap (y - K^+) \subseteq L^+ \cap B(0, \| x \|) \cup \{x\} \quad \forall y \in L^+. \tag{3.7}$$

(a) If the relative algebraic interior recor $K \neq \emptyset$ and if $A \subseteq \hat{y} + K$ for some $\hat{y} \in Y$, then any point $y_0 \in \hat{y} +$ recor K with (3.6) is a Borwein properly efficient point of A.

(b) If $\mathrm{int} K \neq \emptyset$, then every $y_0 \in (\tilde{y} - \mathrm{int} K) \cap A$ for some $\tilde{y} \in A$ with

$$d(y_0, \tilde{y} + L) \geq d(y, \tilde{y} + L), \quad \forall y \in (\tilde{y} - \text{ int } K) \cap A,$$

is a Borwein properly efficient point of A.

It is noted that (3.5)and (3.7) are equivalent.

Now we pass to necessary conditions.

THEOREM 3.4. *Assume that K is closed, $L = K \cap (-K)$ and cor $K^+ \neq \emptyset$.*

(a) Each efficient point y_0 of an arbitrary subset $A \subset Y$ satisfies the condition: for a given $\hat{y}$ such that $p(\hat{y}) \in p(y_0) -$ cor $K^+, d(y_0, \hat{y} + L) \leq d(y, \hat{y} + L)$ and

$$\| p(y_0) - p(\hat{y}) \| < \| p(y) - p(\hat{y}) \| \quad \text{whenever } p(y_0) \neq p(y), \tag{3.8}$$

for all $y \in A$ if and only if

$$L^+ \cap \text{ cl } B(0, \| p(y_0) - p(\hat{y}) \|) = (p(\hat{y}) - p(y_0) + K^+) \cap (p(y_0) - p(\hat{y}) - K^+). \tag{3.9}$$

(b) Each efficient point y_0 of an arbitrary subset $A \subset Y$ satisfies the condition: for a given $\tilde{y}$ such that $p(\tilde{y}) \in p(y_0) +$ cor $K^+, d(y_0, \tilde{y} + L) \geq d(y, \tilde{y} + L)$ and

$$\| p(y_0) - p(\tilde{y}) \| > \| p(y) - p(\tilde{y}) \| \quad \text{whenever } p(y) \neq p(\tilde{y}), \tag{3.10}$$

for all $y \in (\tilde{y} - K) \cap A$ if and only if (3.9) holds.

Similar statements are true for weakly efficient points without conditions (3.8) and (3.10) (see[27]). The above theorems contain results of [51], [50], [14], [15] and have applications in control approximation problems (see [27]).

3.2. Scalar representations:

In this subsection we assume that K is not a linear subspace of Y. Given Problem (3.1), it is desirable to have a problem of type(3.2) with the property that any optimal solution of (3.2) is also a minimizer of (3.1). Problem (3.2) is therefore considered as a scalar representation of (3.1). More precisely, (3.2) is said to be a scalar representation of (3.1) if for every $x_1, x_2 \in X$,

$$F(x_1) \in F(x_2) + K \text{ implies } s(x_1) \geq s(x_2) \text{ and}$$

$$F(x_1) \in F(x_2) + K\ (-K) \text{ implies } s(x_1) > s(x_2).$$

In the case $\mathrm{ri}K \neq \emptyset$, we say that (3.20) is a scalar weak representation of (3.1) if, for $x_1, x_2 \in X$,

$$F(x_1) \in F(x_2) + \mathrm{ri}K \text{ implies } s(x_1) > s(x_2).$$

It is clear that any representation is also a weak representation.One can easily see that any optimal solution of (3.2) is a Pareto minimizer of (3.1) whenever (3.2) is a scalar representation, and is a weak minimizer of (3.1) whenever (3.2) is a scalar weak representation.

THEOREM 3.5. *In order that Problem (3.2) be a scalar representation of Problem (3.1), it is necessary and sufficient that s be a composition of F and an increasing function on $F(X)$.*

By virtue of this theorem, in order to get a minimizer of (3.1), it suffices to take any increasing functional $\xi \circ F$. Of course, one tries to choose ξ as simple as possible. Other requirements on ξ are sometimes needed. For instance, we wish to have the scalar problem solvable whenever so is the vector problem; or if the vector problem possesses certain specific properties(linearity, convexity, quasiconvexity etc.), then so does its scalar representation. Recall that a mapping F is said to be quasiconvex if for any $y \in Y, x_1, x_2 \in X$ and $t \in [0,1]$,

$$F(x_1), F(x_2) \in y - K \text{ imply } F(tx_1 + (1-t)x_2) \in y - K.$$

THEOREM 3.6. *Assume that Y is finite dimensional, F is linear and S is a polyhedral set in X. Then for every minimizer (resp., weak minimizer)x of (3.1), there exists a $\xi \in riK^*$(resp., $\xi \in K^*\backslash\{0\}$) such that x is an optimal solution of (3.2) with $s = \xi \circ F$. (Problem (3.2) is then a scalar (resp., weak)representation.)*

THEOREM 3.7. *Assume that (3.1) is convex, i.e., F is a convex mapping and S is a convex set. Then, for every weak minimizer x of (3.1), there exists $\xi \in K^*\backslash\{0\}$ such that x is an optimal solution of (3.2) with $s = \xi \circ F$ (Problem (3.2)is then a weak representation.)*

It is worthwhile noticing that if ξ is linear and belongs to $K^*\backslash\{0\}$ and (3.1) is convex, then(3.2) is convex but, in general, not a representation. For a given minimizer of (3.1), it is not necessary for $\xi \in K^*\backslash\{0\}$ to exist such that (3.2) with $s = \xi \circ F$ is a scalar representation of (3.1) and has x as an optimal solution. The existence of such ξ can be guaranteed for instance if x is a proper minimizer and Y is finite dimensional.

THEOREM 3.8. *Assume that* $int\, K \neq \emptyset$ *and Problem (3.1) is quasiconvex in the sense that* F *is quasiconvex and* S *is convex . Let* $e \in intK$ *be given. Then for every weak minimizer* x *of (3.1) there exists* $a \in Y$ *such that* x *is an optimal solution of (3.2) with* $s = h_a \circ F$, *where* h_a *is defined on* Y *by*

$$h_a(y) = \ min\ \{t : y \in a + te - K, t \in R\}.$$

Observe that with this h_a, (3.2) is a quasiconvex problem and a weak representation of (3.1). One cannot expect s to have a simpler structure. For instance, instead of h_a one should take any vector $\xi \in K^*\backslash\{0\}$, but the composition $s = \xi \circ F$ is no longer a quasiconvex functional. Moreover, it might happen that (3.1) has weak minimizers while (3.2) with $s = \xi \circ F$ is not solvable.

4.Duality

In this section we shall present a scheme for constructing dual problems of a given vector problem and prove some duality results. Let $g : X \to Z, Z$ being a topological vector space ordered by a convex cone M, be given. Consider the problem

$$\begin{cases} \min F(x), \\ \text{s.t. } x \in X, g(x) \in -M. \end{cases} \tag{4.1}$$

As in scalar optimization, a dual problem of (4.1) is of the form

$$\begin{cases} \max D(u), \\ \text{s.t.} u \in U, \end{cases} \tag{4.2}$$

where U is a nonempty set and D is a mapping from U to Y. Problem (4.2) means that we look for a maximizer (resp., weak maximizer) $u_0 \in U$. The mapping D and the set U must be constructed in such a manner that by solving (4.2) one can obtain optimal values of (4.1), and, of course, other duality relations must hold. In general it is impossible to construct a dual problem with the objective mapping single valued. Therefore, we consider (4.2) with D

set-valued. A maximizer of (4.2) is then defined as a point $u_0 \in U$ with the property that there exists $y_0 \in D(u_0)$ such that

$$(y_0 + K) \cap D(U) \subseteq y_0 - K.$$

Weak maximizers are defined similarly. For the sake of simplicity, K and M are assumed to be convex, pointed with nonempty interior.

Now we are able to give a definition of dual problems. Problem (4.2) is said to be dual of (4.1) if

$$F(x) \notin D(u) - K\backslash\{0\}, \text{ for every } x \in X, g(x) \in -M, u \in U.$$

Being a dual it is said to be an exact dual if there are $x_0 \in X$ with $g(x_0) \in -M$ and $u_0 \in U$ such that $F(x_0) \in D(u_0)$. In order to obtain a dual of (4.1) let us proceed as follows. Choose a linear space E ordered by a pointed convex cone C. Let $\mathcal{U}$ be the set of mappings from $Y \times Z$ to E with the property

$$u(y,0) \in u(y',0) + C \quad \forall y, y' \in Y \text{ with } y \in y' + K\backslash\{0\}.$$

In other words, $\mathcal{U}$ consists of mappings from $Y \times Z$ to E which are increasing in the first variable when the second is zero. We take any nonempty subset $U \subseteq \mathcal{U}$ and define a set-valued mapping $D : U \to Y$:

$$D(u) = \{y \in Y : u(y,0) \text{ is an efficient point of the set}$$

$$\bigcup_{x\in X} u(F(x) + K, g(x) + M)\}. \tag{4.3}$$

Remmeber that $u(y,0) \in E$ and the efficiency is considered with respect to C. It is not hard to prove that with these D and U, Problem (4.2) is a dual of (4.1). Now the question is when (4.2) is exact. By specifying E and $U \in \mathcal{U}$ we shall see that the exactness of (4.2) can be reached. We take up the case $E = Y$ and $C = K$.

THEOREM 4.1. *Assume that (4.1) possesses a minimizer. Then (4.2) with D defined by (4.3) is an exact dual of (4.1) in the cases:*

(i) U is the entire set $\mathcal{U}$;

(ii) U consists of mappings which are increasing in the first variable and nondecreasing in the second one and the set $F(X)$ has a lower bound.

Now we show that Lagrangean duality can be obtained by taking U in a special form. Denote the set of continuous linear nondecreasing operators from Z to Y by L. Let U be the set consisting of mappings u from $Y \times Z$ to Y which can be expressed in the form $u(y, z) = y + l(z)$ for every $y \in Y, z \in Z$ and for some $l \in L$.

Evidently, $U \in \mathcal{U}$. Hence (4.2) with this U and D defined by (4.3) is actually a dual of (4.1). It is also seen that D takes the form

$$D(u) = \{y \in Y : y \text{ is an efficient point of the set}$$

$$\bigcup_{x \in X}(F(x) + l(g(x))), \text{ where } l \in L \text{ determines } u\}.$$

THEOREM 4.2. *Assume that $g(X) \cap (-\operatorname{int} M) \neq \emptyset$ (Slater's condition) and the set $\bigcup_{x\in X}(F(x) + K, g(x) + M)$ is convex. Then, (4.2) with U as above is an exact dual of (4.1) whenever(4.1) possesses a B. p. min..*

The dual problem obtained in Theorem 4.2 is quite simply structured. Its objective mapping D is a generalization of usual Lagrangean functions. The convexity assumption in the theorem holds, for instance, if $F(.)$ and $g(.)$ are quasiconvex. This assumption, Slater's condition and properness assumption are all we have to pay in order to get exact duals with the constraint set relatively simple in comparison with the exact duals obtained in Theorem 4.1.

5. Optimality conditions

5.1. NECESSARY CONDITIONS:

Consider Problem (1.1)-(1.2). We define the Lagrangean

$$L(x, u, \lambda, \mu, \nu) = \langle \lambda, F(x, u)\rangle + \langle \mu, G(x, u)\rangle + \langle \nu, P(x, u)\rangle$$

and the simplex

$$\sum^{s} = \{a = (\alpha_1, \ldots, \alpha_s) : \alpha_j \geq 0, \sum_{j=1}^{s} \alpha_j \leq 1\}.$$

THEOREM 5.1. *Assume for Problem (1.1)-(1.2) that*

(i) int $K \neq \emptyset$ *and int* $M \neq \emptyset$;

(ii) $P(.,u_0)$ *is continuously Fréchet differentiable at* x;

(iii) $F(.,u)$ *and* $G(.,u)$ *are continuous in a neighborhood* V *of* x_o *for each* $u \in U$ *and regularly locally convex (see[30]) at* x_0 *for* $u = u_0$;

(iv) for every finite set of points $u_1, \cdots, u_s \in U$ *and every* $\delta > 0$ *there are a neighborhood* V' *of* x_0, $V' \subset V, \epsilon > 0$, *a mapping* $v : V' \times \epsilon \sum^s \to U$ *and points* $e \in K$ *and* $g \in M$ *such that for all* $x, x' \in V'$ *and* $a, a' \in \epsilon \sum^s$,

$$v(x,0) = u_0$$

$$\| P(x, v(x,a)) - P(x', v(x',a')) - P'_x(x_0,u_0)(x-x')$$

$$-\sum_{j=1}^{s}(\alpha_j - \alpha'_j)P(x_0,u_j) \| \leq \delta(\| x - x' \| + \sum_{j=1}^{s} | \alpha_j - \alpha'_j |)$$

$$F(x, v(x,a)) - F(x,u_0) - F(x,u_0) - \sum_{j=1}^{s} \alpha_j (F(x,u_j) - F(x,u_0))$$

$$\leq \delta(\| x - x_0 \| + \sum_{j-1}^{s} \alpha_j)e;$$

$$G(x, v(x,a)) - G(x,u_0) - \sum_{j=1}^{s} \alpha_j (G(x,u_j) - G(x,u_0))$$

$$\leq \delta(\| x - x_0 \| + \sum_{j-1}^{s} \alpha_j)g;$$

(v) $P'_x(x_0,u_0)X$ *has finite codimension.*

If (x_0,u_0) *is a local weak minimizer, then there exists* $(\lambda_0, \mu_0, \nu_0) \in (K^* \times M_0 \times W^*)\backslash\{0\}$ *such that*

$$0 \in \delta_x L(x_0, u_0, \lambda_0, \mu_0, \nu_0),$$

$$L(x_0, u_0, \lambda_0, \mu_0, \nu_0) = \min_{u \in U} L(x_0, u, \lambda_0, \mu_0, \nu_0).$$

Theorem 5.1 is an extension of a main result in [13] and contains many Lagrange multiplier rules. Its proof, although complicated (see [30], goes on the same way as most of the proofs of necessary optimality conditions using approximation theorems and separation theorems. For such general approximation theorems see [20], [22] and [23]. The assumption that int$K \neq \emptyset$ is commonly used in the literature, but it is rather restrictive. In [31] we obtain also a multiplier rule under the relaxed condition that ri$K \neq \emptyset$. Note that if K is not a subspace of Y, any minimizer is also a weak minimizer and hence necessary conditions for the latter also hold for the former.

5.2. SUFFICIENT CONDITIONS:

In this subsection we confirm a common fact that multiplier rules are also sufficient optimality conditions under appropriate convexity assumptions. We present general sufficient conditions for proper minimizers (and so for minimizers and weak minimizers as well) under relaxed assumptions on convexity and differentiability.

Consider Problem (1.1)-(1.2) with the additional constraint $x \in \Omega \subseteq X$ and a pointed ordering cone N for W.

THEOREM 5.2. *Assume that for a feasible point $(x_0, u_0) \in (\Omega \cap X_0) \times U_0)$ there exist nonempty sets G_1, G_2 and G_3 with $K \backslash (-K) \subseteq G_1 \subseteq Y, M_0^* * \subseteq G_2 \subseteq Z$ and $N \cup (-N) \subseteq G_3 \subseteq W$, such that F, G and P have partial directional variations (see [31]) on x at (x_0, u_0) with respect to g_1, G_2 and G_3, resp. . Assume further that the composite mapping (F, G, P, P) is partially differentiably (- C)-quasiconvex (see [31])at(x_0, u_0) with*

$$C = K \backslash (-K) \times M_0^{**} \times N \times (-N).$$

Assume, finally, that there exists $(\lambda, \mu, \nu) \in K^* \times M_0^* \times W^*$ satisfying

(i) $\langle \lambda, y \rangle > 0$ for all $y \in K \backslash (-K)$;

(ii) $\langle \lambda_0, F'_x(x_0, u_0)(x - x_0) \rangle + \langle \mu_0, G'_x(x_0, u_0)(x - x_0) \rangle$

$$+\langle \nu_0, P'_x(x_0, u_0)(x - x_0) \rangle \geq 0 \text{ for all } x \in \Omega;$$

(iii) $L(x_0, u_0, \lambda_0, \mu, \nu) = \min_{u \in U} L(x_0, u_0, \lambda_0, \mu, \nu)$

Then, (x_0, u_0) is a B. p. min. of Problem (1.1)-(1.2).

Theorem 5.2 contains some results of [17] and [31] as consequences.

5.3 Applications: Pontryagin maximum principle for cooperative differential games:

Let us fix a time interval $[t_0, t_1]$. The following many players game is considered:

$$\dot{x}(t) = \varphi(t, x(t), u_1(t), \cdots, u_m(t)), \quad x(.) \in C^n[t_0, t_1],$$

$$u_j(.) \in L_\infty^{r_j}[t_0, t_1], u_j(t) \in U_j \subseteq R^{r_j}, j = 1, \cdots, m$$

$$h_l(x(t_l)) = 0 \text{ with } h_l : R^n \to R^{sl} \text{ given }, l = 0, 1,$$

$$\max{}_{t \in [t_0, t_1]} g_i(t, x(t)) \leq 0, i = 1, \cdots, k,$$

$$F(x(.), u_1(.), \cdots, u_m(.)) = \xi(x(t_1)) + \int_{t_0}^{t_1} f(t, x(t), u_1(t), \cdots, u_m(t))dt \to \min .$$

The Hamilton is taken as (with $u = (u_1, \cdots, u_m)$)

$$H(t, x, u, p, \lambda, \mu) = \langle p, \varphi(t, x, u,) \rangle - \langle \lambda, f(t, x, u) \rangle - \langle \mu, g(t, x, u) \rangle.$$

Applying Theorem 5.1 we get the following maximum principle.

THEOREM 5.3. *Assume that $\varphi, h_0, h_1, g, \xi$ and f are jointly continuous and continuously differentiable with respect to x. Let $u_1^0(.), \cdots, u_m^0(.)$ be local weakly optimal controls with the resulting state $x_0(.)$. Then, there exist $\lambda \in K^*, l_0 \in R^{s_0}, l_1 \in R_1^s, p(.) : [t_0, t_1] \to R^n$ and nonegative regular measures $\mu_i, i = 1, \cdots, k$ on $[t_0, t_1]$, supported on the sets*

$$T_i = \{t \in [t_0, t_1] : g_i(t, x^0(t)) = 0\},$$

resp., not all zero and such that

(a) $p(.)$ is a solution of the integral equation

$$p(t) = -\xi(x^0(t))\lambda - h_1'(x^0(t_1))l_1 + \int_{t_1}^{t} H_x'(r, x^0(r), u^0(r), p(r), \lambda)dr$$

$$-\sum_{i=1}^{k} \int_{t}^{t_1} g_{ix}'(r, x^0(r))d\mu_i$$

with the initial condition $p(t_0) = h_0'(x^0(t_0))l_0$;

(b) for almost all $t \in [t_0, t_1]$,

$$H(t, x^0(t), u^0(t), p(t), \lambda) = \sup_{u \in U} H(t, x^0(t), u, p(t), \lambda).$$

Under appropriate convexity assumptions the maximum principle becomes a sufficient condition as follows. Consider the case $h_0(x(t_0)) = x(t_0) - x^0$ for a fixed $x^0 \in R^n$ (i. e., the left end-point is fixed). Then, we have

THEOREM 5.4. *Let* K *be pointed. Let* $u_1^0(.), \ldots, u_m^0(.)$ *be admissible with the resulting state* $x^0(.)$. *Let the following differentiability conditions be satisfied:* h_1 *and* ξ *are Fréchet differentiable at* $x^0(t_1)$; $f(t,.,.), \varphi(t,.,.)$ *and* $g_i(t,.)$ *have partial derivatives at* $(x^0(t), u^0(t))$ *and at* $x^0(t)$, *resp., for all* $t \in [t_0, t_1]$. *Moreover, assume that there exists* $\lambda \in K^*$ *with* $\langle \lambda, y \rangle > 0 \quad \forall y \in K \backslash \{0\}, l_1 \in R^{s_1}, p(.) : [t_0, t_1] \to R^n$, *and nonnegative regular measures* $\mu_i, i = 1, \cdots, k$ *on* $[t_0, t_1]$ *such that assertions (a) and (b) of Theorem 5.3 hold. Assume further that*

(c) the following convexity assumptions are satisfied: $U = U_1 \times \cdots \times U_m$ *is convex;* $\langle \lambda, \xi(.) \rangle$ *is convex at* $x^0(t_1)$; $\langle l_1, h_1(.) \rangle$ *is quasiconvex at* $x^0(t_1)$; $\langle \lambda, f(t,.,.) \rangle$ *and* $\langle p(t) + \sum_{i=1}^k \int_t^{t^1} g'_{ix} d\mu_i, \varphi(t,.,.) \rangle$ *are convex and concave, resp., at* $(x^0(t), u^0(t))$ *for almost all* $t \in [t_0, t_1]$;

(d) for almost all $t \in [t_0, t_1]$ *we have*

$$\varphi'_x(t, x^0(t), u^0(t)) \sum_{i=1}^k \int_t^{t_1} g'_{ix}(t, x(t)) d\mu_i = 0$$

and

$$\langle \sum_{i=1}^k \int_t^{t_1} g'_{ix}(t, x^0(t)) d\mu_i, \sum_{j=1}^m \varphi'_{u_j}(t, x^0(t), u^0(t))(u_j(t) - u_j^0(t)) \rangle \leq 0$$

for all admissible controls $u_j(.), j = 1, \cdots, m$.

Then, $u_1^0(.), \ldots, u_m^0(.)$ *are global optimal controls.*

Other maximum principles for slightly different games may be found in [48]

6. EKELAND'S VARIATIONAL PRINCIPLE FOR VECTOR OPTIMIZATION:

Ekeland's variational principle [8] is one of the most useful tools for nonlinear analysis. It has been used in various fields: optimization, global analysis, convex analysis, fixed point theory, generalized differential calculus, sensitivity. This section is devoted to an extension of this principle to vector optimization. To include known related results, we consider a general setting. Let N be the set of positive integers. A pair $(X, \rightarrow)$ of the set X and a subset of $X^N \times X$ is called an L-space if:

(i) $x_n = x \in X, \forall n \in N$, implies $(\{x_n\}, x) \in \rightarrow$; and

(ii) if $(\{x_n\}, x) \in \rightarrow$, then $(\{x_{n_i}\}, x) \in \rightarrow$ for every subsequence $\{x_{n_i}\}$.

In what follows we shall write $x_n \rightarrow x$ instead of $(\{x_n\}, x) \in \rightarrow$. Let Y be an ordered vector space with an ordering cone K and $d : X \times X \rightarrow K$ a mapping. The l-space X is called d-complete if $\sum_{n=1}^{m} d(x-n+1, x_n) \leq k$ for some $k \in K$ and for all $m \in N$ implies $x_n \rightarrow x$ for at leat one x in X. A mapping $d : X \times X \rightarrow K$ is said to be a vector halfmetric if the following two conditions hold:

(i) $d(x, z) = 0 \leftrightarrow x = z$;

(ii) $d(x, z) \leq d(x, y) + d(y, z)$ for $x, y, z \in X$.

A mapping $\varphi : X \rightarrow U$ between two L-spaces is closed if $\varphi(x) = y$ whenever $x_n \rightarrow x$ and $\varphi(x_n) \rightarrow y$. A mapping $J : X \rightarrow Y, X$ and Y being as above, is referred to as lower semicontinuous (l. s. c.) if $x_n \rightarrow a$ and $J(x_n) \leq y$ for some $y \in Y$ and for all $n \in N$ imply $J(a) \leq y$.

THEOREM 6.1. *Let X and U be nonempty L- spaces, Y be an ordered complete separable vector space with an ordering cone K and $B \subset X$. Let $\varphi : B \rightarrow U$ be closed. Let $d : B \times B \rightarrow K$ and $d_1 : \varphi B \times \varphi B \rightarrow K$ be vector halfmetrics such that B is d-complete and φB is d_1-complete. Let a mapping $J : \varphi B \rightarrow Y$ satisfy the following two conditions*

(i) $J(\varphi B)$ is minorized;

(ii) $\sup\{d(., x), d_1(\varphi(.), \varphi(x))\} + J(\varphi(.))$ is l. s. c. $\forall x \in B$.

Then

(a) for each $u \in B$, there exists a Pareto minimizer $v \in B$ of $J(\varphi(.)) + sup\{d(.,v), d_1(\varphi(.), \varphi(v))\}$ such that

$$J(\varphi(v)) \leq J(\varphi(u)) - sup\{d(v,u), d_1(\varphi(v), \varphi(u))\}$$

and, for all $x \in B \backslash \{v\}$,

$$J(\varphi(v)) \not\leq J(\varphi(x)) + sup\{d(x,v), d_1(\varphi(x), \varphi(v))\};$$

(b) if p is a Pareto minimum of $J(\varphi B)$ and is comparable with $J(\varphi(u))$, then for each $\epsilon \in K$, $J(\varphi(u)) \leq p + \epsilon$ implies

$$sup\{d(v,u), d_1(\varphi(v), \varphi(u))\} \leq \epsilon.$$

Theorem 6.1 has an equivalent form which is a generalization of the Caristi-Kirk fixed point theorem (see[24]). It includes most results in this direction, e.g. that of Downing and Kirk[7], Husain and Sehgal [12], Kasahara [18] and Park [49].

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