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NONCONVEX PERTURBATION OF DIFFERENTIAL INCLUSIONS WITH MEMORY

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1. Introduction

Nonconvex-valued differential inclusions have attracted much attention in recent years. See the monographs [1], [14] and the papers [4]-[8], [12], [13], [15], [16] for an overview on this area of research. However, there are a few results devoted to differential inclusions with memory [9]-[11], [13].

Let E be a Hilbert space, I an interval of R and τ a positive scalar. Denote by $\mathcal{C}_E(I)$ the Banach space of continuous functions from I into E. By $\mathcal{C}_0$ we mean the Banach space $\mathcal{C}_E[-\tau,0]$ with the norm $\|\varphi\|_0 = \max_{s\in[-\tau,0]}\|\varphi(s)\|$. For $t_0 \geq 0, a > 0$, $x \in \mathcal{C}_E[t_0-\tau, t_0+a]$, and for any $t \in [t_0, t_0+a]$ we define a map $T(t)$ from $\mathcal{C}_E[t_0-\tau, t_0+a]$ into $\mathcal{C}_0$ as follows

$$T(t)x(s) = x(t+s), \quad s \in [-\tau, 0].$$

For an arbitrary nonempty set $A \subset E$, denote by $m(A)$ the (unique) element of A with the smallest norm. In this paper we prove the following

THEOREM 1.1. *Let E be a separable Hilbert space , $\Omega \subset R \times \mathcal{E}_0$ an open subset containg (t_0, φ_0). Assume that*

1) φ_0 is a Lipschitz function;

2) F is an upper semicontinuous map from Ω into non-empty closed convex subsets of E and the map $(t,x) \longrightarrow m(F(t,x))$ is locally compact,

3) G is a uniformly continuous map from Ω into non-empty compact subsets of E whose image $G(\Omega)$ is relatively compact.

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Then there exist a positive scalar a and an absolutely continuous function $x(\cdot) \in \mathcal{E}_E[t_0 - \tau, t_0 + a]$ such that

$$T(t_0)x = \varphi_0, \tag{1.1}$$

$$\dot{x}(t) \in F(t, T(t)x) + G(t, T(t)x) \tag{1.2}$$

for almost all $t \in [t_0, t_0 + a]$.

Recall that a map Φ is locally compact if for each point in Dom φ there exists a neighborhood which is mapped into a compact subset. The map G is uniformly continuous on Ω if for any $\epsilon > 0$, there exists a positive scalar δ such that if $(t_1, x_1), (t_2, x_2) \in \Omega$ and $\|(t_1, x_1) - (t_2, x_2)\|_{R \times \mathcal{C}_0} \leq \delta$, then $h(G(t_1, x_1), G(t_2, x_2)) \leq \epsilon$, where $\|(t, x)\|_{R \times \mathcal{C}_0} = |t| + \|x\|_0$ and $h(A, B)$ is the Hausdorff distance between the nonempty subsets A, B of E.

Note that in [8] Gamal developed the discretization method initiated by Filippov, Moreau for studying evolution equations perturbed by non-convex-valued maps in separable Hilbert spaces. This method and some techniques of [1] will be used in the proof of Theorem 1.1. We shall also present an existence theorem for global solutions to differential inclusions with memory (1.1)-(1.2). The obtained results extend Theorems 2.1.3, 2.3.1, 2.1.4 of [1] and some results of [8], [13].

2. Proof of Theorem 2.1

Let us first recall some compactness criterions that will be used in the sequel.

PROPOSITION 2.1 ([3],[8]). *Given a Banach space E, let $\mathcal{H}$ be a family of ds-measurable functions from [0,1] into the unit ball of E satisfying*
i) For any compact set $A \subset [0,1], \mathcal{H}_A = \{\int_A f(s)ds | f \in \mathcal{H}\}$ is relatively compact,
ii) For any $\epsilon > 0$, there exists a number $\lambda_\epsilon \in (0,1)$ such that for all $\eta \in (0, \lambda_\epsilon)$ and all $f \in \mathcal{H}$

$$\int_0^{1-\eta} \|f(s+\eta) - f(s)\| ds \leq \epsilon.$$

Then $\mathcal{H}$ is relatively compact in $L^1_E[0,1]$.

PROPOSITION 2.2 [2]. *Let $(\Omega, \mathcal{A}, \mu)$ be a measured space with finite μ and E a separable Banach space. Assume that $\mathcal{H}$ is a bounded uniformly integrable family of $L^1_E(\Omega, \mathcal{A}, \mu)$ satisfying the following condition : For any $\eta > 0$ there exist $A_\eta \in \mathcal{A}$ with $\mu(\Omega \backslash A_\eta) < \eta$ and a map G_η from A_η into E with nonempty compact values such that $f(\omega) \in G_\eta(\omega)$ for all $f \in \mathcal{H}$ and $\omega \in A_\eta$. Then the set $M = \{\int_\Omega f d\mu | f \in \mathcal{H}\}$ is relatively compact in E.*

PROOF OF THEOREM 1.1: We shall prove Theorem 1.1 by adapting the original technique used by Gamal in [8].

Observe first that since $(t,x) \longrightarrow m(F(t,x))$ is locally compact, there exist a compact convex subset $K_1 \subset E$ and positive scalars a, b such that

$$Q = \{(t,x) \in \Omega : |t - t_0| \leq a, \|x - \varphi_0\|_0 \leq b\} \subset \Omega$$

and $m(F(t,x)) \in K_1$ for all $(t,x) \in Q$.

Let $K_2 \subset E$ be a compact convex set containing $G(\Omega)$. Put

$$\mu_1 = \max\{\|u\| : u \in K_1\},$$

$$\mu_2 = \max\{\|u\| : u \in K_2\}.$$

Let ℓ be the Lipschitz constant of φ_0. Without loss of generality we may assume that the scalar a satisfies the following condition

$$a < \frac{b}{\max\{\ell, \mu_1 + \mu_2\}}. \tag{2.1}$$

Let $\epsilon_n = 2^{-n}, n \geq 1$. By Assumption 3) and Lemma 1 in [8], there exists a strictly decreasing sequence of positive scalars $(e_n)_{n=1}^{\infty}$ converging to 0 as $n \longrightarrow +\infty$ such that $\frac{a}{e_{n-1}}, \frac{e_{n-1}}{e_n}$ are integers and $\frac{e_{n-1}}{e_n} \geq 2$ for every $n \geq 2$. Moreover, for every $(t_1,x_1),(t_2,x_2) \in \Omega$ with

$$\|(t_1,x_1) - (t_2,x_2)\|_{R \times C_0} \leq e_n(\max\{\ell, \mu_1 + \mu_2\} + 1)$$

we have $h(G(t_1,x_1), G(t_2,x_2)) \leq \epsilon_n$.

For each $n \geq 1$ we consider the partition of $I = [t_0, t_0 + a]$ given by

$$P_n = \{t_i^n = t_0 + ie_n : i = 0, 1, \dots, \nu_n = \frac{a}{e_n}\}.$$

As shown in [8], the sequence $(P_n)_{n=1}^{\infty}$ satisfies the following two properties

(P1) $P_n \subset P_{n+1}$,

(P2) For every $n \geq 2$ and for every $t_i^n \in P_n \backslash P_1$ there exists a unique couple (r, j) of positive integers depending on t_i^n such that

$$\begin{cases} r < n, \\ t_i^n \notin P_u, \quad u = 1, \dots, r, \\ t_i^n \in P_u, \quad u \geq r + 1, \\ 0 \leq j \leq \nu_r - 1, \\ t_j^r < t_i^n < t_{j+1}^r. \end{cases}$$

To every partition P_n we associate an absolutely continuous function $x_n : [t_0 - \tau, t_0 + a] \longrightarrow E$ and step functions $y_n : [t_0, t_0 + a] \longrightarrow E$, $z_n : [t_0, t_0 + a] \longrightarrow E$ such that the following relations are satisfied for all $n \geq 1$

(i) $T(t_0)x_n = \varphi_0$,

(ii) For every $i = 0, 1, \dots, \nu_n - 1$ and for every $t \in (t_i, t_{i+1})$,

$$\begin{cases} y_n(t) = m(F(t_i^n, T(t_i^n)x_n)) \in K_1 \subset \mu_1 B, \\ z_n(t) \in G(t_i^n, T(t_i^n)x_n) \subset K_2 \subset \mu_2 B, \\ \dot{x}_n(t) = y_n(t) + z_n(t), \end{cases}$$

where $B = \{x \in E, \|x\| \leq 1\}$,

(iii) For all $t \in [t_0, t_0 + a]$,

$$x_n(t) \in x_0 + [0, a]\{K_1 + K_2\},$$

where $x_0 = \varphi_0(0)$,

(iv) For all $t, t' \in [t_0, t_0 + a]$,

$$\|T(t)x_n - T(t')x_n\|_0 \leq |t - t'| \max\{\ell, \mu_1 + \mu_2\},$$

(v) For every $i = 1, 2, \dots, \nu_n - 1$,

$$\int_0^{e_n} \|z_n(t_i^n + s) - z_n(t_{i-1}^n + s)\| ds \le e_n \epsilon_n \quad \text{if} \quad t_i^n \in P_1,$$

$$\int_0^{e_n} \|z_n(t_i^n + s) - z_n(t_j^r + s)\| ds \le e_n \epsilon_r \quad \text{if} \quad t_i^n \notin P_1,$$

where (r, j) is the unique couple of integers satisfying property $(P2)$ and depending on t_i^n.

Notice first that properties (i) and (iv) imply

$$\|T(t)x_n - \varphi_0\|_0 = \|T(t)x_n - T(t_0)x_n\|_0 \le |t - t_0| \max\{\ell, \mu_1 + \mu_2\}$$
$$\le a \max\{\ell, \mu_1 + \mu_2\},$$

which together with (2.1) yields

$$\|T(t)x_n - \varphi_0\|_0 \le b.$$

Then for all $t \in [t_0, t_0 + a]$ we have

$$m(F(t, T(t)x)) \in K_1 \subset \mu_1 B.$$

Further, from property (iv) it follows that

$$\|T(t_i^n)x_n - T(t_{i+1}^n)x_n\|_0 \le e_n \max\{\ell, \mu_1 + \mu_2\}.$$

Then for every $i = 0, 1, \dots, \nu_n - 1$ we have

$$h(G(t_i^n, T(t_i^n)x_n), G(t_{i+1}^n, T(t_{i+1}^n)x_n)) \le \epsilon_n.$$

Let us construct functions $x_n(\cdot), y_n(\cdot)$ and $z_n(\cdot)$ which satisfy properties (i)-(v). Let $n \ge 1$ be fixed. Firstly, for $t \in [t_0 - \tau, t_0]$ we set $x_n(t) = \varphi_0(t - t_0)$. Set $x_0^n = \varphi_0(t_0)$, $y_0^n = m(F(t_0^n, T(t_0^n)x_n))$ and let z_0^n be an arbitrary point of $G(t_0^n, T(t_0^n)x_n)$. The functions $x_n(\cdot), y_n(\cdot)$ and $z_n(\cdot)$ can be defined on $[t^0, t_1^n]$ as follows:

$$\begin{cases} x_n(t) = x_0^n + (t - t_0^n)(y_0^n + z_0^n), \\ y_n(t) \equiv y_0^n, \\ z_n(t) \equiv z_0^n. \end{cases} \tag{2.2}$$

It is obvious that the defined functions have properties (i) – (iii) on $[t_0 - \tau, t_1^n]$. We shall verify property (iv) for $t, t' \in [t_0, t_1^n]$. Let $s \in [-\tau, 0]$ be given. If $t + s \in [t_0, t_1^n]$ and $t' + s \in [t_0, t_1^n]$, then

$$\begin{aligned} \|T(t)x_n(s) - T(t')x_n(s)\| &= \|x_n(t+s) - x_n(t'+s)\| \\ &\leq |t - t'|(\mu_1 + \mu_2) \\ &\leq |t - t'| \max\{\ell, \mu_1 + \mu_2\}. \end{aligned}$$

If $t + s \in [t_0 - \tau, t_0]$ and $t' + s \in [t_0, t_1^n]$, then

$$\begin{aligned} \|T(t)x_n(s) - T(t')x_n(s)\| &= \|x_n(t+s) - x_n(t'+s)\| \\ &= \|\varphi_0(t + s - t_0) - \varphi_0(0) + x_n(t_0) - x_n(t'+s)\| \\ &\leq \ell(t_0 - t - s) + (\mu_1 + \mu_2)(t' + s - t_0) \\ &\leq |t - t'| \max\{\ell, \mu_1 + \mu_2\}. \end{aligned}$$

Suppose that $t + s \in [t_0 - \tau, t_0], t' + s \in [t_0 - \tau, t_0]$. Then we have

$$\begin{aligned} \|T(t)x_n(s) - T(t')x_n(s)\| &= \|x_n(t+s) - x_n(t'+s)\| \\ &= \|\varphi_0(t + s - t_0) - \varphi_0(t' + s - t_0)\| \\ &\leq \ell|t - t'| \\ &\leq |t - t'| \max\{\ell, \mu_1 + \mu_2\}. \end{aligned}$$

Thus property (iv) is satisfied for $t, t' \in [t_0, t_1^n]$.

Next, put $x_1^n = x_n(t_1^n), y_1^n = m(F(t_1^n, T(t_1^n)x_n))$ and let $z_1^n \in G(t_1^n, T(t_1^n)x_n)$ be a point such that

$$\|z_1^n - z_0^n\| \leq h(G(t_1^n, T(t_1^n)x_n), G(t_0^n, T(t_0^n)x_n)).$$

Then for $t \in (t_1^n, t_2^n]$ we put

$$x_n(t) = x_1^n + (t - t_1^n)(y_1^n + z_1^n),$$
$$y_n(t) \equiv y_1^n, \tag{2.3}$$
$$z_n(t) \equiv z_1^n.$$

It is easy to see that $x_n(\cdot)$, $y_n(\cdot)$ and $z_n(\cdot)$ defined on $[t_0, t_2^n]$ by (2.2) and (2.3) satisfy conditions (i) – (iv). We now verify (v) for t_1^n. Since $\|T(t_1^n)x_n - T(t_0^n)x_n\| \le e_n \max\{\ell, \mu_1+\mu_2\}$, we have $h(G(t_1^n, T(t_1^n)x_n), G(t_0^n, T(t_0^n)x_n)) \le \epsilon_n$ and by our construction $\|z_1^n - z_0^n\| \le \epsilon_n$. Therefore,

$$\int_0^{e_n} \|z_n(t_1^n + s) - z_n(t_0^n + s)\| ds \le e_n \epsilon_n.$$

If $t_1^n \notin P_1$, we denote by (r, j) the unique couple satisfying property $(P2)$ and depending on t_1^n. Since

$$t_j^r = j e_r < t_1^n = e_n < t_{j+1}^r = (j+1)e_r$$

and $r < n$, it follows that $e_n < e_r$. Hence we obtain that $j = 0$, that is

$$t_0^n = t_0^r = t_j^r.$$

Therefore, using the above inequality, we derive that

$$\begin{aligned}\int_0^{e_n} \|z_n(t_1^n + s) - z_n(t_j^r + s)\| ds &= \int_0^{e_n} \|z_n(t_1^n + s) - z_n(t_0^n + s)\| ds \\ &\le e_n \epsilon_n \\ &\le e_n \epsilon_r.\end{aligned}$$

We now assume that $x_n(\cdot)$, $y_n(\cdot)$ and $z_n(\cdot)$ are defined on $[t_0^n, t_i^n]$ in such a way that

a) For every $k = 0, 1, \dots, i-1$ and for every $t \in (t_k^n, t_{k+1}^n)$,

$$\begin{cases} y_n(t) \equiv m(F(t_k^n, T(t_k^n)x_n)) \in K_1 \subset \mu_1 B, \\ z_n(t) \equiv z_k^n \in G(t_k^n, T(t_k^n)x_n) \subset K_2 \subset \mu_2 B, \\ \dot{x}_n(t) \equiv y_n(t) + z_n(t), \end{cases}$$

b) For all $t \in [t_0^n, t_i^n]$,

$$x_n(t) \in x_0 + [0, t_i^n]\{K_1 + K_2\},$$

c) For all $t, t' \in [t_0^n, t_i^n]$,

$$\|T(t)x_n - T(t')x_n\|_0 \leq |t - t'| \max\{\ell, \mu_1 + \mu_2\},$$

d) For every $k = 1, 2, \dots, i-1$,

$$\int_0^{e_n} \|z_n(t_k^n + s) - z_n(t_{k-1}^n + s)\| ds \leq e_n \epsilon_n \quad \text{if} \quad t_k^n \in P_1,$$

$$\int_0^{e_n} \|z_n(t_k^n + s) - z_n(t_j^r + s)\| ds \leq e_n \epsilon_r \quad \text{if} \quad t_k^n \notin P_1,$$

where (r, j) is the unique couple determined by t_k^n in the property (P2).

In order to define $x_n(\cdot)$, $y_n(\cdot)$ and $z_n(\cdot)$ on $(t_i^n, t_{i+1}^n]$ we put $x_i^n = x_n(t_i^n)$, $y_i^n = m(F(t_i^n, T(t_i^n)x_n))$ and choose the value z_i^n of $z_n(\cdot)$ on $(t_i^n, t_{i+1}^n]$ in the following way. If $t_i^n \in P_1$, then we take $z_i^n \in G(t_i^n, T(t_{i+1}^n)x_n)$ with

$$\|z_i^n - z_{i-1}^n\| \leq h(G(t_i^n, T(t_i^n)x_n), G(t_{i-1}^n, T(t_{i-1}^n)x_n)).$$

Assume that $t_i^n \notin P_1$. Because of property (P2), there exists a unique couple (r, j) depending on t_i^n such that

$$\begin{cases} r < n \\ t_i^n \notin P_u, \quad u = 1, \dots, r, \\ t_i^n \in P_u, \quad u \geq r + 1, \\ 0 \leq j \leq \nu_r - 1, \\ t_j^r < t_i^n < t_{j+1}^r. \end{cases}$$

Since $r < n$, then $t_j^r \in P_n$, that is $t_j^r = t_q^n$ for a unique integer q with $0 \leq q \leq i-1$. Hence we have

$$(i-q)e_n = t_i^n - t_q^n < e_r.$$

From property c) it follows that

$$\|T(t_k^n)x_n - T(t_{k+1}^n)x_n\| \leq e_n \max\{\ell, \mu_1 + \mu_2\}$$

for every $k = 0, 1, \ldots, i-1$. Therefore,

$$\begin{aligned}\|T(t_q^n)x_n - T(t_i^n)x_n\| &\leq \|T(t_q^n)x_n - T(t_{q+1}^n)x_n\| + \\ &\cdots + \|T(t_{i-1}^n)x_n - T(t_i^n)x_n\| \\ &\leq (i-q)e_n \max\{\ell, \mu_1 + \mu_2\} \\ &\leq e_r \max\{\ell, \mu_1 + \mu_2\}\end{aligned}$$

and we obtain

$$h(G(t_i^n, T(t_i^n)x_n), G(t_q^n, T(t_q^n)x_n) \leq \epsilon_r.$$

Let $z_i^n \in G(t_i^n, T(t_i^n)x_n)$ such that

$$\|z_i^n - z_q^n\| \leq h(G(t_i^n, T(t_i^n)x_n), G(t_q^n, T(t_q^n)x_n)).$$

Now, for $t \in (t_i^n, t_{i+1}^n]$ we put

$$\begin{aligned}x_n(t) &= x_i^n + (t - t_i^n)(y_i^n + z_i^n), \\ y_n(t) &\equiv y_i^n \\ z_n(t) &\equiv z_i^n.\end{aligned}$$

It is easy to show that the functions $x_n(\cdot), y_n(\cdot)$ and $z_n(\cdot)$ satisfy conditions (i)-(iv) on $[t_0, t_{i+1}^n]$. We now verify (v) for $k = i$. If $t_i^n \in P_1$, we have

$$\int_0^{e_n} \|z_n(t_i^n + s) - z_n(t_{i-1}^n + s)\| ds = \int_0^{e_n} \|z_i^n - z_{i-1}^n\| ds \leq e_n \epsilon_n.$$

If $t_i^n \notin P_1$, we obtain

$$\int_0^{e_n} \|z_n(t_i^n + s) - z_n(t_j^r + s)\|ds = \int_0^{e_n} \|z_n(t_i^n + s) - z_n(t_q^n + s)\|ds$$
$$= \int_0^{e_n} \|z_i^n - z_q^n\|ds \leq e_n \epsilon_r.$$

Thus, the functions $x_n(\cdot)$, $y_n(\cdot)$ and $z_n(\cdot)$ with the desired properties can be defined on the whole interval $[t_0, t_0 + a]$.

In view of [1, Theorem 1.3.4], there is a function $g(\cdot) \in L_E^1[t_0, t_0 + a]$ such that $x_n(\cdot)$ converges uniformly to $x(\cdot)$ on compact subsets of $[t_0, t_0 + a]$ and $\dot{x}_n(\cdot) = y_n(\cdot) + z_n(\cdot)$ converges weakly to $g(\cdot)$ in $L_E^1[t_0, t_0 + a]$, where $x(t) = x_0 + \int_{t_0}^{t} g(s)ds$. We claim that there is a function $z(\cdot) \in L_E^1[t_0, t_0 + a]$ such that $z_n(\cdot)$ converges strongly to $z(\cdot)$ in $L_E^1[t_0, t_0 + a]$. Indeed, by an argument analogous to that of the proof for [8, Theorem 4] one can verify that for the sequence $\{z_n(\cdot)\}_{n=1}^{\infty}$ the following condition holds: For any positive scalar ϵ, there are a positive integer n_0 and a scalar $\alpha_\epsilon \in (0, a)$ such that for all $n > n_0$ and $\eta \in (0, \alpha_\epsilon)$,

$$\int_{t_0}^{t_0+a-\eta} \|z_n(t + \eta) - z_n(t)\|dt < \epsilon.$$

Using the definition of $z_n(\cdot)$ one can now show that for any positive scalar ϵ, there is a scalar $\alpha_\epsilon \in (0, a)$ such that for all $\eta \in (0, \alpha_\epsilon)$ and $n \geq 1$

$$\int_{t_0}^{t_0+a-\eta} \|z_n(t + \eta) - z_n(t)\|dt < \epsilon.$$

Consequently, by virtue of Proposition 2.1, for proving the relative compactness of $\{z_n(\cdot)\}_{n=1}^{\infty}$ we only need to verify the fact that for any compact measurable subset A of $[t_0, t_0 + a]$, the subset $\{\int_A z_n(t)dt \min_n \geq 1\}$ is relatively compact in E. Setting $\delta_n(t) = t_i^n$ for $t \in (t_i^n, t_{i+1}^n]$ and $\delta_n(0) = 0$ we define

$$\Phi_n(t) = G(\delta_n(t), T(\delta_n(t))x_n),$$

It is clear that $\alpha_n(\cdot), \alpha(\cdot)$ and $\beta_n(\cdot), \beta(\cdot)$ are measurable functions from $[t_0, t_0+a]$ into $[t_0, t_0 + a] \times C_0$ and E, respectively . Furthermore, by the results obtained above we get

(i) $\alpha_n(\cdot) \longrightarrow \alpha(\cdot)$ converges for all $t \in [t_0, t_0 + a]$,

(ii) $\beta_n(\cdot) \longrightarrow \beta(\cdot)$ weakly converges in $L^1_E[t_0, t_0 + a]$,

(iii) For all $[t_0, t_0 + a], (\alpha_n(t), \beta_n(t)) \in \text{graph } F$.

So all assumptions of [1,Theorem 1.4.1] hold. Hence for almost all $t \in [t_0, t_0 + a]$,

$$(\alpha(t), \beta(t)) \in \text{graph } F,$$

or

$$g(t) - z(t) \in F(t, T(t)x).$$

Taking into account (2.4) we obtain

$$\dot{x}(t) \in F(t, T(t)x) + G(t, T(t)x)$$

for almost all $t \in [t_0, t_0 + a]$. The proof is now complete.

3. Existence of global solutions

Observe that the interval on which the solution is defined depends upon the size of Ω and upon the neighborhood which is mapped in a compact set. In the case where $\Omega = [t_0, \infty] \times C_0$ and when $m(F(t, x))$ remains in a compact set, we can take $a = \infty$ and $b = \infty$. Therefore we can take a arbitrarily in the proof of Theorem 1.1, and, consequently, obtain global results.

THEOREM 3.1. *Let E be a separable Hilbert space, $\Omega = [t_0, \infty) \times C_0$ and $\varphi_0 \in C_0$ a Lipschitz function. Assume that*

1) F is an upper semicontinuous map from Ω into non-empty closed convex subsets of E and $m(F(t, x))$ remains in a compact subset of E;

2) G is a uniformly continuous map from Ω into non-empty compact subsets of E whose image is relatively compact.

Then there exists an absolutely continuous function $x(\cdot) : [t_0 - \tau, \infty) \longrightarrow E$ *such that*

$$T(t_0)x = \varphi_0,$$

$$\dot{x}(t) \in F(t, T(t)x) + G(t, T(t)x)$$

for almost all $t \in [t_0, \infty)$.

REMARKS: 1. The main difficulty we meet here is the nonconvexity of the right-hand side of the differential inclusion (1.2). Therefore, the fixed point approach, a widely used tool in the theory of convex-valued differential inclusions, cannot be employed.

2. When $G \equiv \{0\}$ or $F \equiv \{0\}$ we obtain an extension of Theorems 2.1.3, 2.3.1, 2.1.4 of [1] and of some results of [8], [13].

3. In [16] Valadier proposed a new approach for solving the following non-convex evolution problem in R^d. Let $C(t) = R^d \setminus \mathrm{int} K(t)$, $K : [0,1] \longrightarrow R^d$ be an 1-Lipschitz closed convex valued map with $\mathrm{int} K(t) \neq \emptyset$ for all $t \in [0,1]$. Define

$$F_0(t,\xi) = \begin{cases} -N_{C(t)}\xi & \text{if } \xi \in C(t), \\ \emptyset & \text{otherwise,} \end{cases}$$

where $N_A x$ is Clarke's normal cone to the set A at $x \in A$, and let $F : [0,1] \times R^d \longrightarrow R^d$ be the smallest closed convex valued map which has a closed graph and contains the map $F_0 \cap B$, where $(F_0 \cap B)(t,\xi) = F_0(t,\xi) \cap B(0,1)$. He proved that $F = F_0 \cap B$, and that F is an upper semicontinuous map with compact convex values and obtained existence theorems for the following differential inclusion

$$\dot{x}(t) \in F(t, x(t)),$$

$$x(0) = x_0,$$

$$x(t) \in C(t).$$

Adapting Valadier's technique and following the proof of Theorem 1.1 one can establish the existence of solutions of the differential inclusion

$$\dot{x}(t) \in F(t, x(t)) + G(t, T(t)x),$$

$$T(0) = \varphi_0,$$

$$x(t) \in C(t)$$

with nonconvex-valued maps C and G .

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