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ON THE STRUCTURE OF COCOMMUTATIVE COALGEBRAS

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The note proves the existence of a special basis for cocommutative coalgebras, called the γ_2 -basis. This can be considered as a dual version of a result of J.P.May [2; Thm.2]. The basis-free form of this result can be considered as the splitting theorem for ξ -modules of finite type. An application of this result will appear in [3], where the algebra structure of cohomology of the wreath product $\sum_\infty \int X$ is determined. Throughout the paper the underground field will be the finite field $\mathbb{F}_2$ of two elements.

Let V be a cocommutative coalgebra over $\mathbb{F}_2$ with the comultiplication Δ ;$\underline{\Delta}$ the composition

$$\underline{\Delta} : V \overset{\Delta}{\to} V \otimes V \overset{pr}{\to} (V \otimes V)_{\sum_2}$$

where $(V\otimes V)_{\sum_2} = V\otimes V/\cup_{g\in\sum_2} Im(1-g)$- the coinvariants of $\sum_2$ in $V\otimes V$ and pr is the projection. By the cocommutativity, we have $\underline{\Delta}(V) \subset \{v \otimes v; v \in V\}$. Denote by V_{od} the kernel of $\underline{\Delta}$. An element of V_{od} will be said to be odd. Recall that the map $\underline{\Delta}$ gives rise to the root map

$$\xi : V \xrightarrow{\underline{\Delta}} \underline{\Delta}(V) \xrightarrow{\pi} V,$$

where π is given by $\pi(v \otimes v) = v$. With these notations the main result of this note may be formulated as follows.

THEOREM 1. *For each connected, cocommutative coalgebra V there exists a basis $\mathcal{B}$ of the $\mathbb{F}_2$-module V and a certain inverse map of ξ, $\gamma_2 : V \longrightarrow V$ such that*

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i) $\gamma_2(\mathcal{B}) \subset \mathcal{B} \cup \{0\}$.
ii) If $x \in \mathcal{B}$ *and* $\gamma_2(x) \neq 0$, *then* $\Delta\gamma_2(x) = x \otimes x$.
iii) Denoting $\mathcal{B}_{od} = \mathcal{B} \cap V_{od}$ *and* $\gamma_{2^h} = (\gamma_2)^h$ *we have*

$$\mathcal{B} = \bigcup_{x \in \mathcal{B}_od} \{x, \gamma_2(x), \cdots, \gamma_{2^{h(x)-1}}(x)\},$$

where $\gamma_{2^{h(x)}}(x) = 0, 0 < h(x) \leq \infty$.

Such a basis will be called a γ_2-basis of V and the integer $h(x)$ the depth of the element x. The notation γ_2 is an imitation of the divided powers in an algebra with divided powers (see[2]).

PROOF: Let ξ be the root map as above. Since $\xi(V)$ is a vector subspace of V, we can write $V = \xi(V) \oplus X$. From this we obtain

$$\xi(V) = V_\infty + \sum_{i>0} \xi^i(X),$$

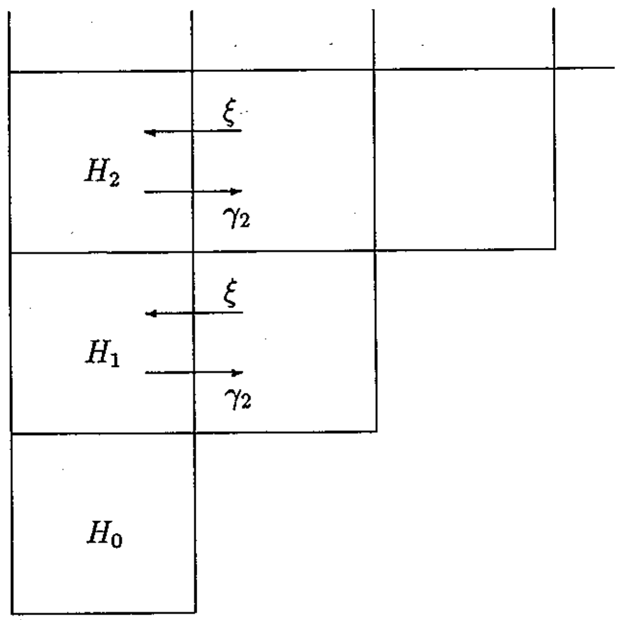
where $V_\infty = \bigcap_{n>0} \xi^n(V)$. Again, since V_∞ is a vector subspace of $\xi(V)$, we can write $\xi(V) = V_\infty \oplus W$ and we can choose W such that $W \subseteq \sum_{i>0} \xi^i(X)$. So we get

$$V = V_\infty \oplus W \oplus X.$$

We first consider the restriction of the root map on the first summand $\xi \mid_{V_\infty}: V_\infty \longrightarrow V_\infty$. Let $\{x_j, j \in J\}$ be a basis of $V_\infty \cap ker\xi$. Since $x_j \in V_\infty, j \in J$, we can choose $\gamma_2(x_j) \in V_\infty$ to be an inverse image of x_j under ξ. We have the system $\{\gamma_2(x_j); j \in J\} \subset ker\xi^2 \cap V_\infty$. By the definition of the element $\gamma_2(x_j)$, we can easily check that $\{[\gamma_2(\xi_j)]; j \in J\}$ is a basis of $V_\infty \cap ker\xi^2 / V_\infty \cap ker\xi$. Inductively, for every $n \in Z_+, j \in J$ we may define an element $\gamma_{2^n}(x_j) \in V_\infty$ to be an inverse image of $\gamma_{2^{n-1}}(x_j)$ under ξ respectively. So we get the system $\{\gamma_{2^n}(x_j); j \in J\} \subset V_\infty \cap ker\xi^{n+1}$ such that $\{[\gamma_{2^n}(x_j)]; j \in J\}$ is a basis of $V_\infty \cap ker\xi^{n+1} / V_\infty \cap ker\xi^n$.

By this way we obtain the linearly independent system $\{\gamma_{2^n}(x_j); j \in J, n \in Z_+\} \subset V_\infty$. Moreover, by the definition of ξ for every $v \in V_\infty$, there exists an $N \in Z_+$ such that $v \in ker\xi^N \cap V_\infty = V_\infty \cap ker\xi^N / (V_\infty \cap ker\xi^{N-1}) \oplus \cdots \oplus (V_\infty \cap ker\xi^2 / V_\infty \cap ker\xi) \oplus (V_\infty \cap ker\xi)$. Since $\{[\gamma_{2^n}(x_j)]; j \in J\}$ is a basis for $V_\infty \cap ker\xi^{n+1}/V_\infty \cap ker\xi^n, n \in Z_+$, the element v can be expressed as a linear combination of the elements $\gamma_{2^n}(x_j), j \in J, 0 \le n \le N-1$. So the system $\{\gamma_{2^n}(x_j); j \in J, n \in Z_+\}$ is a basis for V_∞.

Next we consider the summand W. For $n \in Z_+$, let $F_n = X + \xi(X) + \cdots + \xi^n(X)$.



Put

$$H_0 = W \cap ker\xi \cap F_0$$

$$H_0 \oplus H_1 = W \cap ker\xi \cap F_1$$

$$\cdots$$

$$H_o \oplus \cdots \oplus H_{n-1} \oplus H_n = W \cap ker\xi \cap F_n$$

Let $\{x_i : i \in I_s\}$ be a basis for $H_s, s > o$, and $I = \coprod_{s>0} I_s$. By an argument similar to that used above, for every $i \in I_s, n \in Z_+, 0 \le n \le s$, we may define elements $\gamma_2^n(x_i)$ as inverse images of $\gamma_{2^{n-1}}(x_i)$ under ξ. We can also check that $\{[\gamma_{2^n}(x_i)]; i \in I \setminus \coprod_{s=1}^{n-1} I_s\}$ is a basis for $W \cap ker\xi^{n+1}/W \cap ker\xi^n, n \in Z_+$. Therefore, we obtain the linearly independent system $\coprod_{s>0} \{\gamma_{2^n}(x_i); i \in I_s, 0 \leq n \leq s\}$ in W. Since $W \subset \sum_{i>0} \xi^i(X)$, for each $v \in W$ there exists an $N \in Z_+$ such that $v \in W \cap ker\xi^n = (W \cap ker\xi^N/W \cap ker\xi^{N-1}) \oplus \cdots \oplus (W \cap ker\xi^2/W ker\xi) \oplus (W \cap ker\xi)$. Then the element v can be expressed as a linear combination of the elements $\gamma_{2^n}(x^i), i \in i_s, 0 \leq n \leq s, 0 < s < N$. So the system $\coprod_{s>0} \{\gamma_{2^n}(x_i); i \in I_s, 0 \leq n \leq s\}$ is a basis for W.

Let $\{x_k; k \in K\}$ be a basis of X. Then by the above construction we can easily check that $\{x_i, \gamma_2(x_i), \cdots, \gamma_{2^{h_i}-1}(x_i); i \in I \sqcup J \sqcup L\}$ is a γ_2 -basis of V, where

$$h_i = \begin{cases} \infty, & \text{if } i \in J \\ n+1, & \text{if } i \in I_n \\ 1, & \text{if } i \in K \end{cases}$$

The proof of Therem 1 is now complete.

In the rest of this note we shall try to translate the above result into a basis-free form. First we need some definitions.

DEFINITION 2. An abelian restricted Lie algebra over $\mathbb{F}_2$ of two elements is a graded $\mathbb{F}_2$-vector space L such that

i) $L_0 = 0$.

ii) There is a vector space homomorphism $\gamma : L_n \to L_{2n}$ for all $n \geq 0$.

If H is a commutative, connected Hopf algebra over F_2, then the augmentation ideal IH is an abelian restricted Lie algebra with $\gamma(x) = x^2$.

DEFINITION 3. A ξ-module over the field $\mathbb{F}_2$ is a graded $\mathbb{F}_2$- vector space V such that

i)$V_0 = 0$.

ii) There is a vector space homomorphism $\xi : V_{2n} \to V_n$ for all $n \geq 0$.

We routinely extend ξ on V by setting it be 0 on elements of odd degree.

If C is a cocommutative coalgebra, the root map defined above $\xi : C_{2n} \to C_n$ gives C the structure of a ξ-module. Notice that the dual of a ξ-module is an abelian restricted Lie algebra.

DEFINITION 4. Let n be an integer $n \geq 0$ or $n = \infty$. Let j be an integer $j \geq 1$. Define a ξ-module $V(j,n)$ by

$$V(j,n) = \begin{cases} F_2 & \text{if } i = 2^t j, \quad 0 \leq t \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

and $\xi : V(j,n)_{2i} \to V(l,n)_i$ an isomorphism for $i = 2^t j, t < n$, and $\xi = 0$ otherwise.

The ξ-module $V(j,n)$ may be called cyclic of depth $n+1$ on a generator of degree j.

The following is a basis-free formulation of Theorem 1.

THEOREM 5. *Let V be a ξ-module of finite type. Then there exists a set of pairs of integers (j_i, n_i) so that there is an isomorphism of ξ-modules*

$$\oplus V(j_i, n_i) \cong V.$$

In other words, Theorem 5 says that every ξ-modules is a direct sum of "cyclic" modules. In this form, the result can be considered as a dual version of [2, Theorem 2]. Indeed, let V^* be the vector space dual to V. Then V^* is an abelian restricted Lie algebra of finite type. According to J.P.May [2, Theorem 3] V^* can be decomposed into a direct product of abelian restricted Lie algebras on a single generator. By the assumption on the finite type we get

$$V^* \cong \oplus V(j_i, n_i)^*.$$

Now dualizing again we obtain Theorem 5.

REMARK: Using this special basis for coalgebra H_*X and a remark on the universal enveloping of an abelian restricted Lie algebra we have determined the cohomology algebra of the wreath product $\sum_\infty \int X$ in [3].

References

[1] H. Cartan, *Collected works*, Springer-Verlag **3** (1979).
[2] J.P. May, *Some remarks on the structure of Hopf algebras*, Bull. Amer. Math. Soc. **23** (1969), 708-713.
[3] Nguyen Viet Dung, *The modulo 2 cohomology algebra of the wreath product* $\sum_\infty \int X$, Proceedings of 1990 Barcelona Conference on Algebraic Topology, to appear.

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