ACTA MATHEMATICA VIETNAMICA
Volume 16, Number 2, 1991

ON NECESSARY OPTIMALITY CONDITIONS FOR DISCRETE MINIMAX PROBLEMS

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1. Introduction

Let $f_1, ..., f_N$ be functionals defined on a real Banach space X, C is a non-empty closed convex subset of X, F is a map from X into a real Banach space Y and K is a closed convex cone in Y with vertex at the origin. We shall be concerned with a discrete minimax problem of the following form:

$$(P) \quad \begin{cases} \text{minimize } \max\limits_{1\leq i\leq N} f_i(x), \\ \text{subject to } F(x) \in K, \ x \in C. \end{cases}$$

First and second-order optimality conditions for discrete minimax problems are discussed in [2] by Dem'yanov and Malozemov for the finite - dimensional case. Necessary conditions for a class of nonsmooth minimization problems involving unconstrained infinite-dimensional discrete minimax problems are studied in [1] by Bel-Tal and Zowe. In the case $N = 1$, optimality conditions for Problem (P) are given by Zowe and Kurcyusz [9], Maurer and Zowe [8], and the author [7]. For the general case $N \geq 1$, several suffidcient optimality conditions for (P) are established by the author in [6].

The purpose of this paper is to exploit characteristics of discrete minimax problems to derive various necessary optimality conditions for (P). We also show that under suitable hypotheses the necessary conditions obtained here are equivalent. In Section 2, using a result of Hiriart-Urruty [4] we establish a necessary optimality condition in terms of directional derivative and sequential tangent cone. It should be noted that the sequential tangent cone coincides with the contingent cone of the set of feasible points of (P). In Section 3, we give a generalization of a necessary optimality condition in [2] under a regularity

Received January 15, 1991, in revised form July 12, 1991

assumption. Section 4 studies the Kuhn-Tucker type necessary condition for (P) in terms of Gâteaux derivative and linearizing cone, which includes a result of Zowe and Kurcyusz in [9] as a special case.

2. Necessary optimality conditions

Let M be the set of feasible points of Problem (P). For $\bar{x} \in M$ we consider the following cone

$$T_M(\bar{x}) = \{x \in X \mid x = \lim_{n \to \infty} \frac{x_n - \bar{x}}{\lambda_n},\ \lambda_n \to 0_+,\ x_n \in M\},$$

which is called the sequential tangent cone of M at $\bar{x}$ (see e.g. [8]). We recall that the set of points $d \in X$ such that there exist sequences $d_n \to d$ and $\lambda_n \downarrow 0$ with $\bar{x} + \lambda_n d_n \in M$ for all n is said to be the contingent cone to M at $\bar{x}$ (see e.g. [10]).

In the proof of Theorem 2.1 we will see that the sequential tangent cone of M at $\bar{x}$ coincides with the contingent cone to M at $\bar{x}$.

THEOREM 2.1. *Let $\bar{x}$ be a local solution of (P). Suppose that f_i is Lipschitz in a neighbourhood of $\bar{x}$ and the directional derivative $f_i'(\bar{x}; d)$ exists for all $d \in X$, $i = 1, \ldots, N$. Then*

$$\max_{i \in R(\bar{x})} f_i'(\bar{x}; d) \geq 0,\ \forall d \in T_M(\bar{x}), \tag{2.1}$$

where

$$R(\bar{x}) = \{i \in [1:N] \mid f_i(\bar{x}) = \max_{1 \leq j \leq N} f_j(\bar{x})\}.$$

PROOF: We first evaluate the directional derivative of the function $\varphi(x) = \max_{1 \leq i \leq N} f_i(x)$.

Observe that for $i \in R(\bar{x})$ one has

$$f_i(\bar{x}) > f_j(\bar{x}) \quad \forall j \notin R(\bar{x}). \tag{2.2}$$

Now we put

$$\epsilon = f_i(\bar{x}) - \max_{j \notin R(\bar{x})} f_j(\bar{x}).$$

By virtue of the continuing of f_i and f_j, it follows from (2.2) that for all sufficiently small positive t,

$$f_i(\bar{x}+td) \ > \ f_i(\bar{x}) - \frac{\epsilon}{3} \ > \ f_j(\bar{x}) + \frac{\epsilon}{3} \ > \ f_j(\bar{x}+td)$$
$$(\forall i \in R(\bar{x}),\ \forall j \notin R(\bar{x}),\ \forall d \in X),$$

which implies that

$$\max_{1\le i\le N} f_i(\bar{x}+td) = \max_{i\in R(\bar{x})} f_i(\bar{x}+td).$$

Hence

$$\varphi'(\bar{x};d) = \max_{i\in R(\bar{x})} f_i'(\bar{x};d). \tag{2.3}$$

Since $f_1, ..., f_N$ are Lipschitz in a neighbourhood of $\bar{x}$, all hypotheses of Theorem 6 in [4] are satisfied. Applying this theorem we obtain

$$\varphi'(\bar{x};d) \ge 0 \qquad \forall d \in K_M(\bar{x}), \tag{2.4}$$

where $K_M(\bar{x})$ is the contingent cone to M at $\bar{x}$.

We now show that

$$K_M(\bar{x}) = T_M(\bar{x}). \tag{2.5}$$

Indeed, by the definition of $K_M(\bar{x})$ there exists a sequence $\lambda_n \downarrow 0$ and a sequence $d_n \to d$ ($d \in K_M(\bar{x})$) such that $\bar{x} + \lambda_n d_n \in M$. Setting $y_n = \bar{x} + \lambda_n d_n$, one gets $d_n = \frac{y_n - \bar{x}}{\lambda_n} \to d$ with $y_n \in M,\ \lambda_n \downarrow 0$. Hence $d \in T_M(\bar{x})$. This means that $K_M(\bar{x}) \subset T_M(\bar{x})$.

Conversely, by the definition of $T_M(\bar{x})$, for $d \in T_M(\bar{x})$ there exists a sequence $\lambda_n \downarrow 0$ and a sequence $y_n \in M$ such that $d = \lim\limits_{n\to\infty} \frac{y_n - \bar{x}}{\lambda_n}$. Choosing $d_n = \frac{y_n - \bar{x}}{\lambda_n}$ we obtain $y_n = \bar{x} + \lambda_n d_n \in M$. Hence $d \in K_M(\bar{x})$. This shows that $T_M(\bar{x}) \subset K_M(\bar{x})$.

Now we only need to combine (2.3), (2.5) and (2.4) to get (2.1).

From now on let f_1, f_N, F be differentiable in the sense of Gâteaux at $\bar{x}$ with derivatives $f_1'(\bar{x}), ..., f_N'(\bar{x}), F'(\bar{x})$.

We recall that a feasible point $\bar{x}$ is said to be regular for Problem (P) (see e.g. [9]) if

$$F'(\bar{x})C(\bar{x}) - K(F(\bar{x})) = Y,$$

where

$$C(\bar{x}) = \{\lambda(x - \bar{x}) \mid x \in C,\ \lambda \geq 0\},$$
$$K(F(\bar{x})) = \{y - \lambda K(\bar{x}) \mid y \in K,\ \lambda \geq 0\}.$$

COROLLARY 2.2. *If $\bar{x}$ is a regular local solution for (P), then*

(2.6) $$\max_{i \in R(\bar{x})} < f_i'(\bar{x}), d > \geq 0 \qquad \forall d \in L_M(\bar{x}),$$

where

$$L_M(\bar{x}) = \{x \in C(\bar{x}) \mid F'(\bar{x})x \in K(F(\bar{x}))\}$$

is the linearizing cone of M at $\bar{x}$.

PROOF: Since $\bar{x}$ is regular, $L_M(\bar{x}) \subset T_M(\bar{x})$ (see e.g. [9]). Hence, the conclusion of Corollary 2.2 follows from Theorem 2.1.

Let us consider the minimax problem studied by Dem'yanov and Malozemov [2]

$$(P_1) \qquad \begin{cases} \text{minimize } \max\limits_{1 \leq i \leq N} f_i(x), \\ \text{subject to } x \in C, \end{cases}$$

where $f_1, ..., f_N$ are functions defined on $\mathbf{R}^n$, C is a nonempty closed convex subset of $\mathbf{R}^n$.

COROLLARY 2.3 ([2]). *Let $\bar{x}$ be an optimal solution for Problem (P_1). Assume that $f_1, ..., f_N$ are differentiable at $\bar{x}$. Then,*

(2.7) $$\min_{\substack{d \in C(\bar{x}) \\ ||d||=1}} \max_{i \in R(\bar{x})} < f_i'(\bar{x}), d > \geq 0.$$

PROOF: Applying Corollary 2.1 to Problem (P_1) we obtain the following necessary condition

(2.8) $$\max_{i \in R(\bar{x})} < f_i'(\bar{x}), d > \geq 0, \quad \forall d \in C(\bar{x}).$$

It is easy to check that (2.8) is equivalent to the condition

(2.9) $$\inf_{\substack{d \in C(\bar{x}) \\ ||d||=1}} \max_{i \in R(\bar{x})} < f_i'(\bar{x}), d > \geq 0.$$

Hence from the compactness of the set $\{d \in C(\bar{x}) \mid ||d|| = 1\}$ we get (2.7).

3. A geometrical necessary condition

In this section we try to establish a geometrical necessary condition for the infinite-dimensional case which under suitable hypotheses is equivalent to Condition (2.6).

Define

$$E := \overline{\{ \sum_{i \in R(\bar{x})} \alpha_i f_i'(\bar{x}) \mid \alpha_i \geq 0, \sum_{i \in R(\bar{x})} \alpha_i = 1\}},$$

where the bar indicates the weak* closure.

THEOREM 3.1. *Let the cone $K(F(\bar{x}))$ be closed and $\bar{x}$ be a regular local solution for (P). Then,*

(3.1) $$L_M(\bar{x})^* \cap E \neq \emptyset,$$

where $L_M(\bar{x})^*$ is the conjuage cone to $L_M(\bar{x})$.

PROOF: It follows from Corollary 2.2 that (2.6) is fulfilled for all $d \in L_M(\bar{x})$. Therefore, we only need to show that (2.6) implies (3.1).

Assume the contrary that (2.6) holds but

(3.2) $$L_M(\bar{x})^* \cap E = \emptyset.$$

Observe that the set E is convex, bounded and weakly* closed in X^*, where X^* is the topological dual of X. Then E is a weakly* compact convex subset of X^*. It is obvious that $L_M(\bar{x})^*$ is a weakly* closed convex cone. By the separation Theorem 3.4 in [3], there exist $w_0 \in X^{**}$ and a number γ such that

$$(3.3) \qquad < z, w_0 > \, > \gamma \, > \, < y, w_0 > \qquad (\forall z \in L_M(\bar{x})^*, \ \forall y \in E).$$

Since $0 \in L_M(\bar{x})^*$, it follows that

$$< y, w_0 > \, < 0 \qquad (\forall y \in E)$$

and hence

$$(3.4) \qquad \max_{y \in E} < y, w_0 > \, < 0.$$

Since $< y, w_0 > \, > \gamma \quad (\forall y \in L_M(\bar{x})^*)$ and $L_M(\bar{x})^*$ has the vertex at the origin, it follows from [3, Lemma 5.1] that

$$< y, w_0 > \, \geq 0 \qquad (\forall y \in L_M(\bar{x})^*),$$

which means $w_0 \in L_M(\bar{x})^{**}$. Since $K(F(\bar{x}))$ is closed convex, $L_M(\bar{x})$ is weakly closed. Consequently,

$$L_M(\bar{x})^{**} = L_M(\bar{x})$$

which implies that $w_0 \in L_M(\bar{x})$. It is clear that

$$\max_{i \in R(\bar{x})} < f_i'(\bar{x}), w_0 > \, \leq \max_{y \in E} < y, w_0 > .$$

On the other hand,

$$\max_{y \in E} < y, w_0 > \, = \max\{< \sum_{i \in R(\bar{x})} \alpha_i f_i'(\bar{x}), w_0 >, \mid \alpha_i \geq 0, \ \sum_{i \in R(\bar{x})} \alpha_i = 1\}$$

$$\leq \sum_{i \in R(\bar{x})} \alpha_i \max_{i \in R(\bar{x})} < f_i'(\bar{x}), w_0 > \, = \max_{i \in R(\bar{x})} < f_i'(\bar{x}), w_0 > .$$

Therefore,

$$\max_{i \in R(\bar{x})} < f_i'(\bar{x}), w_0 > = \max_{y \in E} < y, w_0 > . \tag{3.5}$$

In view of (3.4) and (3.5) we see that

$$\max_{i \in R(\bar{x})} < f_i'(\bar{x}), w_0 > < 0,$$

which contradicts (2.6). Thus we have proved that (2.6 implies (3.1), which completes the proof of theorem 3.1.

REMARK: When $K(F(\bar{x}))$ is closed, the condition (3.1) is equivalent to (2.6). Indeed, the proof of Theorem 3.1 has shown that (2.6) implies (3.1). To prove the converse, suppose that (3.1) holds. Since $L = L^{**}$ and $L^* \cap L \neq \emptyset$, for $x \in L$ we have

$$< y, x > \geq 0 \qquad (\forall y \in L^* \cap E),$$

whence

$$\max_{y \in E} < y, x > \geq 0.$$

Taking (3.5) into account we get

$$\max_{i \in R(\bar{x})} < f_i'(\bar{x}), x > \geq 0,$$

as required by (2.6).

Theorem 3.1 is a generalization of [2, Chapt.4, Theorem 3.1] to our infinite - dimensional discrete minimax problem.

4. The Kuhn-Tucker type necessary condition

THEOREM 4.1. *Let $\bar{x}$ be a regular local solution of (P). Then there exists a Lagrange multiplier $\wedge \in K^*$ such that*

$$\max_{i \in R(\bar{x})} < f_i'(\bar{x}), x > \geq < \wedge, F'(\bar{x})x > \tag{4.1}$$

for all $x \in C(\bar{x})$ and

$$< \wedge, F(\bar{x}) > = 0. \tag{4.2}$$

PROOF: Using Corollary 2.1 we see that (2.6) is fulfilled for all $d \in L_M(\bar{x})$. We only need to show that (2.6) implies (4.1) and (4.2).

Consider the following set in $\mathbb{R} \times Y$:

$$Q = \{ \max_{i \in R(\bar{x})} < f_i'(\bar{x}), x > \ +\alpha, \ F'(\bar{x})x - y) \mid \alpha \geq 0, \ x \in C(\bar{x}), \ y \in K(F(\bar{x}))\}.$$

It is clear that Q is a convex cone with vertex at the origin of $\mathbb{R} \times Y$. By the definition of Q we can see that for $x \in C(\bar{x})$, if $F'(\bar{x})x - y = 0$, then $y = F'(\bar{x})x \in K(F(\bar{x}))$. Hence, by (2.6) one gets $\max_{i \in R(\bar{x})} < f_i'(\bar{x}), x > \ +\alpha \geq 0 \quad (\forall \alpha \geq 0)$. Therefore, $(-\infty, 0) \times \{0\} \not\subset Q$. Consequently, $Q \neq \mathbb{R} \times Y$. So $(0,0)$ is a boundary point of Q.

To apply the separation Theorem 1 of [5] we shall show that int $Q \neq \emptyset$. Using the open mapping Theorem 2.1 of [9] one finds a number $\rho > 0$ such that

$$(4.3) \qquad B_Y(0,\rho) \subset F'(\bar{x})((C - \bar{x}) \cap \overline{B_X(0,1)}) - (K - F(\bar{x})) \cap \overline{B_Y(0,1)},$$

where $B_Y(0,\rho)$ stands for the open ball around zero with radius $\rho > 0$ in Y. Consider the following subset of $\mathbb{R} \times Y$

$$Q_0 = \{(\alpha, y) \mid y \in B_Y(0,\rho), \ \alpha > \max_{1 \leq i \leq N} ||f_i'(\bar{x})||\}.$$

Observe that $f_i'(\bar{x})$ $(i = 1, ..., N)$ are bounded, as they are linear continuous mappings. Then for every $x \in (C - \bar{x}) \cap \overline{B_X(0,1)}$,

$$(4.4) \qquad \{ \max_{i \in R(\bar{x})} < f_i'(\bar{x}), x > \leq \max_{1 \leq i \leq N} ||f_i'(\bar{x})|| < +\infty.$$

Consequently, $Q_0 \neq \emptyset$. It follows from (4.3) and (4.4) that $Q_0 \subset Q$. Therefore, int $Q \neq \emptyset$.

By the separation Theorem 1 of [5] there exist $\Lambda \in Y^*$ and a number, not all zero, such that

$$(4.5) \qquad \beta \, (\max_{i \in R(\bar{x})} < f_i'(\bar{x}), x > \ + \alpha) - \ < \Lambda, F'(\bar{x})x - y > \ \geq 0$$

$$(\forall x \in C(\bar{x}), \ \forall y \in K(F(\bar{x})), \ \forall \alpha \geq 0)$$

For $x = 0$ and $\alpha = 0$ this implies $\Lambda \in (K(F(\bar{x})))^*$. Therefore $\Lambda \in K^*$ and $< \Lambda, F(\bar{x}) > = 0$.

Since the regularity condition is equivalent to (see e.e. [9])

$$0 \in \text{int}\{F'(\bar{x})C(\bar{x}) - K(F(\bar{x}))\},$$

one gets $\beta > 0$ and consequently one may assume that $\beta = 1$. Now, for $y = 0$ and $\alpha = 0$. (4.5) yields

$$\max_{i \in R(\bar{x})} < f_i'(\bar{x}), x > - < \Lambda, F'(\bar{x})x > \geq 0, \ \forall x \in C(\bar{x}),$$

which completes the proof of Theorem 4.1.

Let us consider the following problem whose necessary optimality conditions are given by Zowe and Kurcyusz [9] :

$$(P_2) \qquad \begin{cases} \text{minimize } f_1(x), \\ \text{subject to } F(x) \in K, \ x \in C. \end{cases}$$

Here f_1, F, C, K are as in Problem (P).

For $N = 1$ Theorem 4.1 has the following immediate consequence.

COROLLARY 4.2 ([9]). *Assume that $\bar{x}$ is a regular local solution of (P_2). Then there exists $\Lambda \in K^*$ such that*

$$f_1'(\bar{x}) - \Lambda \circ F'(\bar{x}) \in (C(\bar{x}))^*,$$
$$< \Lambda, F(\bar{x}) > = 0.$$

REMARK: It can be seen that Condition 2.6) is equivalent to the conditions (4.1), (4.2). Indeed, the proof of Theorem 4.1 has shown that (2.6) implies (4.1) and (4.2). Conversely, if (4.1), (4.2) hold, then $< \Lambda, F'(\bar{x})x > \geq 0$ for $x \in L_M(\bar{x})$.

In short, when $\bar{x}$ is a regular local solution of (P) and $K(F(\bar{x}))$ is closed, the necessary optimality conditions stated in Theorems 3.1, 4.1 and Corollary 2.2 are equivalent.

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