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ANALYTIC MULTIFUNCTION, UNIFORM FRECHET ALGEBRAS AND THEIR EXTENSION

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Introduction

Analytic multivalued functions (shortly, analytic multifunctions) were studied first by Oka [13], and then by Nishino [12], Yamaguchi [21]. In recent years the theory of analytic multifunctions have been developed by several author, in particular by Aupetit [1,2,3], Slodkowski [16], Ransford [14,15] in the one-variable case, and by Slodkowski [17,18,19] in the several variable case.

In this note we study first the relation between analytic multifunctions and uniform Frechet algebras and give a generalization of Slodkowski's Theorem [17] for analytic multifunctions on an open subset G of $\mathbf{C}^n$ which can not be bounded on G. Then we investigate the extensibility of analytic multifunctions across thin sets which are removable for plurisubharmonic functions and we characterize the hyperbolicity of convex domains in terms of extensibility of analytic multifunctions.

1. Analytic multifunctions and uniform Frechet algebras

In [17] Slodkowski proved the following result on the relation between analytic multifunctions and uniform algebras.

SLODKOWSKI'S THEOREM. *Let G be a bounded planar domain and K : $G \to F_c(\mathbf{C}^k)$ an analytic multifunction such that* $\sup \max |K(\lambda)| < \infty$. *Then there exists a uniform algebra A and functions $f, g_1, ..., g_k \in A$ such that*

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(i) $\hat{f}(M_A) \setminus \hat{f}(\partial_A) = G$, *where* $\hat{f}$ *denotes the Gelfand transformation of* f, M_A *and* ∂_A *are the maximal ideal space and the Shilov boundary of* A.

(ii) $\hat{g}(\hat{f}^{-1}(x)) = K(x)$ *for every* $x \in G$, *where* $\hat{g} = (\hat{g}_1, ..., \hat{g}_k)$.

Following Slodkowski [16] we say that an upper semi-continuous multifunction $K : X \to F_c(Y)$ (where X and Y are complex spaces, $F_c(Y)$ denotes the hyperspace of non-empty compact subsets in Y) is analytic if for every open subset W in X and every plurisubharmonic function ψ on a neighbourhood of the graph $\Gamma K_{|W}$ of K on W, the function

$$\varphi(x) = \max\{\psi(x, y) : y \in K(x)\}$$

is plurisubharmonic on W. With this notion we can generalise Slodkowski's Theorem as follows.

THEOREM 1.1. *Let* G *be an open subset of* $\mathbf{C}^n$ *and* $K : G \to F_c(\mathbf{C}^k)$ *be an upper semi-continuous multifunction. Then* K *is analytic if and only if for every sequence* $\{G_j\}$ *of bounded open subsets of* G *increasing to* G, *there exists a uniform Frechet algebra* $A = \lim \text{proj } A_j$, *where* A_j *are uniform algebras,* $f_1, ..., f_n, g_1, ..., g_k \in A$ *such that*

(i) $\hat{g}(\hat{f}^{-1}(x) = K(x)$ *for* $x \in G$.

(ii) $\hat{f}(\partial_{A_{j\hat{f}^{-1}(L)}}) \cap G_j = \emptyset$ *for every* $j \geq 1$ *and for every complex line* L *in* $\mathbf{C}^n$, *where* $A_{j\hat{f}^{-1}(L)}$ *denotes the completion of* A_j *for the supnorm on* $\hat{f}^{-1}(L) \cap M_{A_j}$.

To prove Theorem 1.1 we first show that the function given by Slodkowski [17, (1.1)] can be chosen such that $L(x, \delta) \subset L(x, \delta')$ for every $x \in D_\delta = \{x \in \mathbf{C}^n : ||x|| > \delta\}$ and for $0 < \delta' \leq \delta$.

LEMMA 1.2. *Let* $\delta > 0$. *Then for every* $n \geq 1$ *there exists an analytic multifunction* $L(\, , \delta) : D_\delta \to F_c(\mathbf{C}^n)$ *such that*

(i) $(x, z) := x_1 z_1 + ... + x_n z_n = 1$ *for every* $z \in L(x, \delta)$.

(ii) $L(x,\delta) \subset L(x,\delta')$ *for all* $0 < \delta \leq \delta$.

PROOF: For simplicity we only consider the case $n = 3$. As in [16] we define the multifunction $L(\ ,\delta)$ by

$$L(x,\delta) = \{z = ||x||^{-2}(\bar{x}_1, \bar{x}_2, \bar{x}_3) + t(x_2 - x_3, x_3 - x_1, x_1 - x_2) : \\ |t| \leq ||x||^2 \exp \rho_\delta(\log ||x||)\}$$

where $\rho_\delta : (\log(\delta^2), \infty) \to \mathbb{R}$ is a smooth function which will chosen such that $L(\ ,\delta)$ is analytic and for which the condition (ii) holds.

To prove the analyticity of $L(\ ,\delta)$ it suffices to show that $\lambda \to L(x(\lambda),\delta)$ is analytic on every complex line $x = x(\lambda)$ in $\mathbb{C}^3$. Let

$$x(\lambda) = (a_1\lambda + b_1, a_2\lambda + b_2, a_3\lambda + b_3).$$

We may assume that $||a||^2 = |a_1|^2 + |a_2|^2 + |a_3|^2 = 1$. Then

$$L(x(\lambda),\delta) = \{(1/A)(\overline{a_1\lambda} + \bar{b}_1, \overline{a_3\lambda} + \bar{b}_2, \overline{a_3\lambda} + \bar{b}_3) + (t(a_2 - a_3)\lambda - (b_2 - b_3), \\ t(a_3 - a_1)\lambda + (b_3 - b_1), t(a_1 - a_2)\lambda - (b_1 - b_2)) : |t| \leq (1/2) \exp \rho_\delta(\log A)\},$$

where $A = ||x(\lambda)||^2 = |\lambda|^2 + \ < \bar{a}, \bar{b} > \bar{\lambda} + \ < a, b > \lambda + ||b||^2$ with $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $<,>$ is the scalar product in $\mathbb{C}^3$. Put

$$R(\lambda) = (1/A) \exp \rho_\delta(\log A).$$

Let ψ be a plurisubharmonic function on $\{(\lambda, z_1, z_2, z_3) : \lambda \in G, (z_1, z_2, z_3) \in L(x(\lambda),\delta)\}$. We put

$$\varphi(\lambda) = \max\{\psi(\lambda, z_1, z_2, z_3) : (z_1, z_2, z_3) \in L(x(\lambda),\delta)\} \\ = \max\{\widetilde{\psi}(\lambda, t) : |t| \leq R(\lambda)\},$$

where

$$\widetilde{\psi}(\lambda, t) = \psi(\lambda, (\overline{a_1\lambda} + \bar{b}_1)/A + t(a_2 - a_3)\lambda + (b_2 - b_3), (\overline{a_2\lambda} + \bar{b}_2)/A + \\ t(a_3 - a_1)\lambda + (b_3 - b_1), (\overline{a_3\lambda} + \bar{b}_3)/A + t(a_1 - a_2)\lambda + (b_1 - b_2))$$

By [16] to prove the subharmonicity of $\varphi(\lambda)$ it suffices to show that the domain

$$\Omega = \{(\lambda, t) \in \mathbf{C}^2 : |t| > R(\lambda)\}$$

is pseudoconvex. Hence it remains to prove that the function

$$\log(\operatorname{dist}(t, \partial\Omega_\lambda)) = -\log |\, |t| - R(\lambda)|$$

is plurisubharmonic, where $\Omega_\lambda = \Omega \cap (\lambda \times \mathbf{C})$ [16].

Let $t = c\lambda + d$ be an arbitrary complex line in $\mathbf{C}^3$. We may assume that $|c| = 1$. Put

$$\theta(\lambda) = -\log(|t| - R(\lambda)) = \log(|c\lambda + d| - R(\lambda)).$$

Then

$$\begin{aligned}
\partial\theta(\lambda)/\partial\lambda &= -\partial(\log(|c\lambda + d| - R(\lambda))/\partial\lambda \\
&= (\partial(|c\lambda + d| - R(\lambda))/\partial\lambda)/(|c\lambda + d| - R(\lambda)) \\
\partial^2\theta(\lambda)/\partial\bar{\lambda}\partial\lambda &= (1/(|c\lambda + d| - R(\lambda))^2)(-\partial^2/\partial\bar{\lambda}\partial\lambda(|c\lambda + d|- \\
&\quad - R(\lambda))(|c\lambda + d| - R(\lambda)) \\
&\quad + \partial/\partial\lambda(|c\lambda + d| - R(\lambda))\partial/\partial\bar{\lambda}(|c\lambda + d| - R(\lambda))) \geq 0 \iff \\
-\partial^2/\partial\bar{\lambda}\partial\lambda(|c\lambda + d| &- R(\lambda))(|c\lambda + d| - R(\lambda)) \\
&\quad + \partial/\partial\lambda(|c\lambda + d| - R(\lambda))\partial/\partial\bar{\lambda}(|c\lambda + d| - R(\lambda)) \geq 0
\end{aligned}$$

Thus it suffices to show that

$$\begin{aligned}
&-\partial^2/\partial\bar{\lambda}\partial\lambda(|c\lambda + d| - R(\lambda))(|c\lambda + d| - R(\lambda)) \geq 0 \quad \text{or} \\
&-\partial^2/\partial\bar{\lambda}\partial\lambda(|c\lambda + d| - R(\lambda) \geq 0 \iff -\partial^2/\partial\bar{\lambda}\partial\lambda(|c\lambda + d|) \\
&+\partial^2 R(\lambda)/\partial\bar{\lambda}\partial\lambda \geq 0
\end{aligned}$$

The proof will be completed when we can find a smooth function ρ_δ : $(\log(\delta^2), \infty) \to \mathbf{R}$ independent of the complex line $x(\lambda)$ such that

$$-\partial^2(|c\lambda + d|)/\partial\bar{\lambda}\partial\lambda + \partial^2 R(\lambda)/\partial\bar{\lambda}\partial\lambda \geq 0.$$

For this we shall need the following claim whose proof will be given below.

Claim. $-\partial^2(|c\lambda + d|)/\partial\bar{\lambda}\partial\lambda + \partial^2 R(\lambda)/\partial\bar{\lambda}\partial\lambda = -(1/4(|c\lambda + d|))$
$+ (1/A^3)|\alpha|^2 \exp\rho_\delta(\log A)(\rho_\delta'' - 2\rho_\delta' + 2) + (1/A^2)\exp\rho_\delta(\log A)(\rho_\delta' - 1) +$
$$(1/A^3)|\alpha|^2 \exp\rho_\delta(\log A)\rho_\delta'(\rho_\delta' - 1),$$

where ρ_δ' and ρ_δ'' are derivatives of the first order and second order of ρ_δ, respectively, and $\alpha = \lambda + < \bar{a}, \bar{b} >$.

We shall choose $\rho_\delta : (q, \infty) \to \mathbf{R}$, where $q = \log(\delta^2)$, as follows:

$$\rho_\delta(x) = \exp 3(x - q) + (x - q) \quad \text{for} \quad q \leq 0,$$
$$\rho_\delta(x) = \exp 3x + x \quad \text{for} \quad q > 0.$$

We shall check that ρ_δ satisfies the inequality

$$-\partial^2(|c\lambda + d|)/\partial\bar{\lambda}\partial\lambda + \partial^2 R(\lambda)/\partial\bar{\lambda}\partial\lambda \geq 0$$

Observe first that

$$\rho_\delta' \geq 1$$
$$\rho_\delta'' \geq 2\rho_\delta' - 2.$$

We must prove that

$$(1/A^2)\exp\rho_\delta(\log A)(\rho_\delta' - 1) \geq (1/4(|c\lambda + d|))$$

or equivalently

$$(1/A^2)\exp\rho_\delta(\log A)(\rho_\delta' - 1)4(|c\lambda + d|) \geq 1$$

For $q \leq 0$ we have

$$(1/A^2)\exp\rho_\delta(\log A)(\rho_\delta' - 1)4(|c\lambda + d|) \geq$$
$$(1/A^2)\exp\rho_\delta(\log A)(\rho_\delta' - 1)(4/A)\exp\rho_\delta(\log A) =$$
$$(4/A^3)(\exp\rho_\delta(\log A))^2(\rho_\delta' - 1) = 4\exp 2(\exp 3(x - q) +$$
$$+ (x - q))3\exp 3(x - q)/\exp 3x \geq 1$$

where $x = \log A$. For $q > 0$ we have

$$(1/A^2)\exp \rho_\delta(\log A)(\rho_\delta - 1)4(|c\lambda + d|) =$$
$$(4\exp 2(\exp 3x + x)3\exp 3x)/\exp 3x \geq 1,$$

where $x = \log A > q > 0$.

Proof of the claim. We have

$$X = -\frac{\partial^2(|c\lambda+d|)}{\partial\bar{\lambda}\partial\lambda} + \frac{\partial^2 R(\lambda)}{\partial\bar{\lambda}\partial\lambda} = -\frac{|c|^2}{4|c\lambda+d|} + \frac{\partial^2(\frac{1}{A}\exp\rho_\delta(\log A))}{\partial\bar{\lambda}\partial\lambda} =$$
$$-\frac{1}{4|c\lambda+d|} + \frac{\partial^2(\frac{1}{A}\exp\rho_\delta(\log A))}{\partial\bar{\lambda}\partial\lambda} \quad (|c|^2 = 1),$$

$$\frac{\partial R(\lambda)}{\partial\lambda} = \frac{\partial}{\partial\lambda}(\frac{1}{A}\exp\rho_\delta(\log A)) = \frac{\partial}{\partial\lambda}(\frac{1}{A})\exp\rho_\delta(\log A) + \frac{1}{A}\frac{\partial}{\partial\lambda}(\exp\rho_\delta(\log A))$$
$$= \frac{1}{A^2}\frac{\partial A}{\partial\lambda}\exp\rho_\delta(\log A) + \frac{1}{A}\exp\rho_\delta(\log A)\rho'\frac{\partial}{\partial\lambda}(\log A)$$
$$= \frac{1}{A^2}(\bar{\lambda} + <a,b>)\exp\rho_\delta(\log A) + \frac{1}{A}\exp\rho_\delta(\log A)\rho'\frac{1}{A}(\bar{\lambda} + <a,b>)$$
$$= \frac{1}{A^2}((\bar{\lambda} + <a,b>)\exp\rho_\delta(\log A)(\rho'_\delta - 1)),$$

$$\frac{\partial^2 R(\lambda)}{\partial\bar{\lambda}\partial\lambda} = \frac{\partial}{\partial\bar{\lambda}}(\frac{1}{A^2}(\bar{\lambda} + <a,b>)\exp\rho_\delta(\log A)(\rho'_\delta - 1))$$
$$= \frac{\partial}{\partial\bar{\lambda}}(\frac{1}{A^2}(\bar{\lambda} + <a,b>)\exp\rho_\delta(\log A))(\rho'_\delta - 1) +$$
$$+ \frac{1}{A^2}(\bar{\lambda} + <a,b>)\exp\rho_\delta(\log A)\frac{\partial}{\partial\bar{\lambda}}(\rho'_\delta - 1).$$

Put $T = \frac{\partial}{\partial\bar{\lambda}}(\frac{1}{A^2}(\bar{\lambda} + <a,b>)\exp\rho_\delta(\log A))$; $S = \frac{\partial}{\partial\bar{\lambda}}(\rho'_\delta - 1)$, when $\frac{\partial^2 R(\lambda)}{\partial\bar{\lambda}\partial\lambda} = T(\rho' - 1) + \frac{1}{A^2}((\bar{\lambda} + <a,b>)\exp\rho_\delta(\log A))S$. Then

$$T = \frac{\partial}{\partial\bar{\lambda}}(\frac{1}{A^2}(\bar{\lambda} + <a,b>)\frac{\partial}{\partial\bar{\lambda}}\exp\rho_\delta(\log A)$$
$$= (\frac{\partial}{\partial\bar{\lambda}}(\frac{1}{A^2}(\bar{\lambda} + <a,b>)) + \frac{1}{A^2}\frac{\partial}{\partial\bar{\lambda}}(\bar{\lambda} + <a,b>)\exp\rho_\delta(\log A) +$$
$$\frac{1}{A^2}(\bar{\lambda} + <a,b>))\exp\rho_\delta(\log A)\rho'_\delta\frac{\partial}{\partial\bar{\lambda}}(\log A)$$

$$= (-\frac{2}{a^3}(\lambda + <\bar{a},\bar{b}>)(\bar{\lambda} + <a,b>) + \frac{1}{A^2})\exp\rho_\delta(\log A) +$$
$$+\frac{1}{A^2}(\bar{\lambda} + <a,b>)\exp\rho_\delta(\log A)\rho'\frac{1}{A}(\lambda + <\bar{a},\bar{b}>)$$
$$= (-\frac{2}{a^3}(\lambda + <\bar{a},\bar{b}>)(\bar{\lambda} + <a,b>) + \frac{1}{A^2})\exp\rho_\delta(\log A) +$$
$$+\frac{1}{A^3}(\lambda + <\bar{a},\bar{b}>)(\bar{\lambda} + <a,b>)\exp\rho_\delta(\log A)\rho'_\delta.$$
$$S = \frac{\partial}{\partial\bar{\lambda}}(\rho'_\delta - 1) = \rho''_\delta\frac{1}{A}(\lambda + <\bar{a},\bar{b}>).$$

Put $\alpha = (\lambda + <\bar{a},\bar{b}>)$. Then

$$\frac{\partial^2 R(\lambda)}{\partial\bar{\lambda}\partial\lambda} = T(\rho'_\delta - 1) + \frac{1}{A^2}(\bar{\lambda} + <a,b>)\exp\rho_\delta(\log A))S$$
$$= ((-\frac{2}{A^3}|\alpha|^2 + \frac{1}{A^2})\exp\rho_\delta(\log A) + \frac{1}{A^3}|\alpha|^2\exp\rho_\delta(\log A)\rho'_\delta)(\rho'_\delta - 1)$$
$$\frac{1}{A^3}|\alpha|^2\exp\rho_\delta(\log A)\rho''_\delta)$$
$$= -\frac{2}{A^3}|\alpha|^2\exp\rho_\delta(\log A)(\rho'_\delta - 1) + \frac{1}{A^2}\exp\rho_\delta(\log A)(\rho'_\delta - 1) +$$
$$\frac{1}{A^3}|\alpha|^2\exp\rho_\delta(\log A)(\rho'_\delta - 1)\rho'_\delta + \frac{1}{A^3}|\alpha|^2\exp\rho_\delta(\log A)(\rho'_\delta - 1)\rho''_\delta$$
$$= \frac{1}{A^3}|\alpha|^2\exp\rho_\delta(\log A)(\rho''_\delta - 2\rho'_\delta + 2) +$$
$$\frac{1}{A^2}\exp\rho_\delta(\log A)(\rho'_\delta - 1) + \frac{1}{A^3}|\alpha|^2\exp\rho_\delta(\log A)\rho'_\delta(\rho'_\delta - 1).$$

Thus

$$X = \frac{\partial^2(c|\lambda + d|)}{\partial\bar{\lambda}\partial\lambda} + \frac{\partial^2 R(\lambda)}{\partial\bar{\lambda}\partial\lambda} = -\frac{1}{4|c\lambda + d|} + \frac{\partial^2 R(\lambda)}{\partial\bar{\lambda}\partial\lambda}$$
$$= -\frac{1}{4|c\lambda + d|} + \frac{1}{A^3}|\alpha|^2\exp\rho_\delta(\log A)(\rho''_\delta - 2\rho'_\delta + 2) +$$
$$\frac{1}{A^2}\exp\rho_\delta(\log A)(\rho'_\delta - 1) + \frac{1}{A^3}|\alpha|^2\exp\rho_\delta(\log A)\rho'_\delta(\rho'_\delta - 1)$$

The proof of Lemma 1.2 is now complete.

PROOF OF THEOREM 1.1: Let A and f, g be as in the theorem. To prove the analyticity of K it suffices to show that $K|_{G_j \cap L}$ is analytic for every $j \geq 1$ and for every complex line L in $\mathbb{C}^n$.

Given $j \geq 1$ and a complex line L in $\mathbb{C}^n$. From (i) and (ii) it follows that $F_j : M_{A_{j\hat{f}^{-1}(L)}} \setminus \partial A_{j\hat{f}^{-1}(L)} \to \Gamma(K|_{G_j \cap L}))$ is proper and surjective, where $F_j = (\hat{f}, \hat{g})|_{M_{A_{j\hat{f}^{-1}(L)}}}$ and $\Gamma(K|_{G_j \cap L})$ denotes the graph of $K|_{G_j \cap L}$. By a result of Slodkowski [18], $\Gamma(K|_{G_j \cap L})$ is an 0-maximum set in $L \times C^2$ in the sense of Slodkowski because of the 0-maximality of $M_{A_{j\hat{f}^{-1}(L)}} \setminus \partial A_{j\hat{f}^{-1}(L)}$ for $A_{j\hat{f}^{-1}(L)}$. Hence $K\,|_{G_j \cap L}$ is analytic [13].

Conversely, let $K : G \to F_c(\mathbb{C}^k)$ be an analytic multifunction and $\{G_j\}$ be a sequence of open subsets of G increasing to G such that

$$\sup\{\|z\| : z \in K(x),\ x \in G_j\} < \infty$$

for every $j \geq 1$. For simplicity we assume that $n = k = 2$. Let $L(\ ,\delta)$ be the analytic multifunction on D_δ defined by Lemma 1.2 for $n = 5$. For each $j \geq 1$ put

$$\begin{aligned} \widetilde{X}_j &= c\ell\{(\tilde{x}, z) \in \mathbb{C}^5 : \tilde{x} \in \widetilde{G}_j,\ z \in \widetilde{K}_j(\tilde{x})\} \\ U_j &= \mathbb{C}^5 \setminus X_j \\ T_j &= \{1,2,3,4,5\} \cup \{U_j \times \{1,2,3,4,5\}\}, \end{aligned}$$

where $\widetilde{G}_j = G_j \times \Delta$ and $\widetilde{K}_j(\tilde{x}) = K(x)$ for $\tilde{x} = (x, x_3) \in \widetilde{G}_j$. By the property (ii) of $L(\ ,\delta)$ and since $\widetilde{X}_j \subset \widetilde{X}_{j+1}$ we have $Y_j \subset Y_{j+1}$, where

$$\begin{aligned} Y_j = \{(y_\iota)_{\iota \in T_j} \in \mathbb{C}^{T_j} &: (y_1, \ldots, y_5) \\ &= (u, v) \in \widetilde{X}_j \quad \text{and for} \quad (\tilde{x}, z) \in U_j,\ (y_{\tilde{x},z,1}, \ldots, y_{\tilde{x},z,5}) \\ &\quad L(u - x, v - z,\ 1/2\operatorname{dist}((\tilde{x}, z), \widetilde{X}_j))\} \end{aligned}$$

Since $L(\ ,\delta)$ is upper semi-continuous, it follows that Y_j is compact. By A_j we denote the uniform algebra on Y_j generated by the functions

$$\begin{aligned} &e(y) = 1,\ f_1(y) = y_1,\ f_2(y) = y_2, \\ &f_3(y) = y_3,\ g_1(y) = y_4,\ g_2(y) = y_5, \\ &a_{\tilde{x},z,1}(y) = y_{\tilde{x},z,1}, \ldots, a_{\tilde{x},z,5} = y_{\tilde{x},z,5} \end{aligned}$$

with $(\tilde{x}, z) \in U_j$.

Put $A = \varinjlim(A_j R_{j+1})$, where $R_{j+1} : A_{j+1} \to A_j$ denotes the restriction map. Then as in [16] we can prove that $A, f = (f_1, f_2)$ and $g = (g_1, g_2)$ satisfy the conditions (i) and (ii) of Theorem 1.1.

The theorem is proved.

2. Extending analytic multifunctions and the hyperbolicity of convex domains.

The problem of extending analytic multifunction has been investigated first by Ransford. In [5] he proved that every analytic multifunction $K : D^* \to F_c(V)$, where $D = \{z \in \mathbb{C} : |z| < 1\}$, $D^* = D \setminus \{0\}$ and V is either D or $D_{rs} = \{z \in \mathbb{C} : r < |z| < s\}$, can be extended to an analytic multifunction $\widehat{K} : D \to F_c(V)$.

First we prove a theorem on extension by thin sets.

We shall need the following result which is a generalization of an important result of Werner [20].

LEMMA 2.1. *Let A be a uniform algebra with Shilov boundary ∂_A and U an open subset of $\mathbb{C}$. Let $h : U \to A$ be a holomorphic map. Then for every $f \in A$ such that $\sigma(f) \setminus \hat{f}(\partial_A) \subset U$, where $\sigma(f)$ is the spectra of f, the form*

$$\lambda \to K(\lambda) = \{\hat{h}(\lambda, w) = \hat{h}(\lambda)(w) : w \in \hat{f}^{-1}(\lambda)\}$$

defines an analytic multifunction on $\sigma(f) \setminus \hat{f}(\partial_A)$.

PROOF: We basically follow Slodkowski's argument in [16, Theorem 3]. It is enough to show that $K(\lambda)$ satisfies condition (ii) of [16, Theorem 3], i.e. for every polynomial $p(\lambda)$ and for every $a, b \in \mathbb{C}$, the function $\lambda \to \max|f_\lambda(K(\lambda))|$, where $f_\lambda(z) = (z - \lambda a - b)^{-1}\exp(p(\lambda))$, has local maximum property in G : $\{\lambda \in \sigma(f) \setminus \hat{f}(\partial_A) : a\lambda + b \notin K(\lambda)\}$. Let D be a disc such that $c\ell(D) \subset G$. Put $N = \hat{f}^{-1}(D) \subset M_A$, and let B denote the uniform closure of $A|_{c\ell(N)}$ on $c\ell(N)$. Then B is a uniform algebra with maximum ideal space $M_B = c\ell(N)$ and the form $k = (h(y) - af - b)^{-1} \exp p(f)$, where $a, b \in \mathbb{C}$ and p is a polynomial, defines an element of B. For $\lambda \in D$ we have

$$\begin{aligned} \max f_{\lambda *}(K(\lambda *)) &= \max |\hat{k} \hat{f}^{-1}(\lambda *)| \\ &= \max ||k|| \, |_{\partial N} \quad \text{(by Ross's local maximal principle)} \\ &= \max\{\max |\hat{k}(\hat{f}^{-1}(\lambda))| : \lambda \in \partial D\} \\ &= \max\{|f_\lambda(K(\lambda))| : \lambda \in D\} \end{aligned}$$

Thus, the function $\lambda \to \max |f_\lambda(K(\lambda))|$ has the local maximum property.

LEMMA 2.2. *Let $K : G \to F_c(Y)$, where G is an open subset of $\mathbb{C}^n$ and Y an analytic set in $\mathbb{C}^k$, be an upper semi-continuous multifunction. If $K : G \to F_c(\mathbb{C}^k)$ is analytic, then $K : G \to F_c(Y)$ is also analytic.*

PROOF: We may assume that $n = 1$. Let φ be a plurisubharmonic function on a neighbourhood W of $\Gamma K_{|U}$ in $G \times Y$, where U is an open subset of G. Consider the plurisubharmonic function $\tilde{\varphi}(z, w) = \varphi(z, \hat{g}(w)$ on $(id \times \hat{g})^{-1}(W)$, where f, g, A are constructed as in Theorem 1.1. By [7] we have

$$\tilde{\varphi}(z, w) = \lim_{n \to \infty} \max\{c_j^n \log |\hat{h}_j^n(z, w)|\}$$

for all $(z, w) \in (id \times \hat{g})^{-1}(W)$, $h_j^n \in CalO(U, A)$, where $CalO(U, A)$ is the set of all holomorphic maps from U into A.

Since for every sequence of upper semi-continuous functions ψ_n, $\psi = \lim \psi_n$ point-wise, $\lim \max(\psi_{n|F}) = \max(\psi_{|F})$ on every compact subset F, and since $(id \times \hat{g})^{-1}(\partial U) \supset \partial(id \times \hat{g})^{-1}(U)$, it suffices to prove the following formula

$$\max |\hat{h}_j^n(z, w)| \, |_{\partial(id \times \hat{g})^{-1}(U)} = \max |\hat{h}_j^n(z, w)| \, |_{(id \times \hat{g})^{-1}(\partial U)} .$$

But this formula follows from the analyticity of the multifunction $z \to \{\hat{h}_j^n(z, w) : w \in \hat{f}^{-1}(z)\}$ and from the relation $\hat{f}(\partial G) \supset \partial_A$.

Let $PSH(X)$ denote the set of all plurisubharmonic function on X.

THEOREM 2.3. *Let g be an open set in $\mathbf{C}^n$ and S a closed subset of G such that $PSH(G \setminus S) = PSH(G)$. Let $K : G \setminus S \to F_c(Y)$ be an analytic multifunction, where Y is a Stein space. Then K can be extended analytically on G.*

PROOF: Without loss of generality we may assume that Y is an analytic set in $\mathbf{C}^k$. Then the function

$$\theta(x) = \{\sup \|y\| : y \in K(x)\}$$

is plurisubharmonic on $G_0 = G \setminus S$. Hence θ can be extended to a plurisubharmonic function on G. This implies that for every $x_0 \in S$, there exists a neighbourhood U of x_0 such that $K(U \cap G_0)$ is relatively compact. Define an upper semi-continuous extension of K by

$$\widehat{K}(x) = \begin{cases} K(x) & \text{for} \quad x \in G_0 \\ \{y \in Y : \exists\{(x_n, y_n)\} \subset \Gamma K,\ (x_n, y_n) \to (x, y)\}, & \text{for } x \in S. \end{cases}$$

We will prove that $\widehat{K}$ is analytic at every point $x_0 \in S$. Let G' be an open ball around x_0, $G' \subseteq G$. It suffices to show that $\widehat{K}|_{L \cap G'}$ is analytic for every complex line L in $\mathbf{C}^n$. Using Slodkowski's Theorem we can find a uniform algebra A and $f, g_1, \ldots, g_k \in A$ such that

(i) $\hat{g}\hat{f}^{-1}(z) = K(x)$ for all $z \in L \cap (G' \setminus S)$

(ii) $\hat{f}(\partial_A) = \partial(L \cap (G' \setminus S))$.

We have to prove that $\hat{f}(\partial_A) \cap (L \cap G') = \emptyset$.

For the contrary, suppose that there exist a complex line L in $\mathbf{C}^n$ such that $\hat{f}(\partial_A) \cap (L \cap G') = \emptyset$. Since K is analytic on $G' \setminus S$, it follows that $\hat{f}(\partial_A) \cap (L \cap (G' \setminus S)) = \emptyset$. Hence there exists $w_0 \in \partial_A$ such that $\hat{f}(w_0) = x_0$. Since G' is open and the set of pick points of A is dense in ∂_A [6], we may assume that w_0 is a pick point ($w_0 \in M_A$ is called a pick point of A if there exists $k \in A$ such that $|\hat{k}(w_0)| = 1$ and $|k(w)| < 1$ for $w \in M_A \setminus \{w_0\}$). Hence there exists $h \in A$ such that $|\hat{h}(w_0)| = 1$ and $|\hat{h}(w)| < 1$ for $w \in M_A \setminus \{w_0\}$.

Consider the plurisubharmonic function

$$\varphi(x) = \log\max|\hat{h}\hat{f}^{-1}(x)| \quad \text{on} \quad G' \setminus S.$$

Then φ is plurisubharmonic on G'. Since $\log\max|\hat{h}\hat{f}^{-1}(x)| \leq 0 = \log\max|\hat{h}\hat{f}^{-1}(x)|$ for every $x \in G'$, it follows that $\varphi = \text{const}$, a contradiction. Thus $\hat{f}(\partial_A) \cap (G' \cap L) = \emptyset$.

By [8], if $S = H \cap (G \setminus G')$, where H is an analytic set in G, G' is an open subset of G such that G' meets every irreducible component of H, or S is a set of zero $(2n-2)$-Hausdorff measure in G, then $PSH(G \setminus S) = PSH(G)$.

Let X be a complex space. Given $p, q \in X$, we choose points $p = p_1, ..., p_k = q$ of X, $a_1, ..., a_k$, $b_1, ..., b_k$ of the unit disc D, and holomorphic mappings $f_1, ..., f_k$ of D into X such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$ for $i = 1, ..., k$. Let

$$d_X(p, q) = \inf \sum_{i=1}^{k} \rho(a_i, b_i),$$

where ρ is the Poincare-Bergnan metric, the infimum is taken over all possible choice of points and mappings described above.

It is easy to see that d_X is a pseudo-distance on X.

A complex space X is called hyperbolic if its pseudo-distance d_X is a distance.

The following theorem discusses the relation between the hyperbolicity of convex domains and the extensibility of analitic multifunctions.

THEOREM 2.4. *Let D be a convex domain in $\mathbb{C}^k$. Then the following conditions are equivalent.*

(i) D is hyperbolic

(ii) For every analytic multifunction $K : \mathbb{C} \to F_c(D)$, the multifunction $\widehat{K} : \mathbb{C} \to F_c(D)$ given by $\widehat{K}(\lambda) = \widehat{K(\lambda}$, where $\widehat{K(\lambda}$ denotes the holomorphic convex hull of $K(\lambda)$, is constant.

(iii) Every analytic multifunction $K : \Delta^* \to F_c(D)$ can be extended analytically on Δ, where Δ is the unit disc, $\Delta^* = \Delta \setminus \{0\}$.

PROOF: We first write $D = \bigcap_{\alpha \in I} \{Re x^*_\alpha < \epsilon_\alpha\}$, where $\{x^*_\alpha\}$ are linearly forms on $\mathbb{C}^k$. Without loss of generality we may assume that $0 \in D$. Then $\epsilon_\alpha > 0$ for all α.

Let $\{x^*_{\alpha_1}, ..., x^*_{\alpha_p}\}$ be a maximal linearly independent system of $\{x^*_\alpha\}$. Take $\theta_\alpha : H_\alpha \to \Delta$, where $H_\alpha = \{z \in \mathbb{C} : Re z < \epsilon_\alpha\}$, is a biholomorphism. Define a holomorphic map $\gamma : D_1 \to \Delta^p$, where $D_1 = \bigcap_{j=1}^{p} \{Re x^*_{\alpha_j}\}$ by $\gamma(x) = (\theta_{\alpha_1}(x^*_{\alpha_1}(x)), ..., \theta_{\alpha_p}(x^*_{\alpha_p}(x)))$. Obviously, γ is a biholomorphism if and only if $\bigcap_{j=1}^{p} \ker x^*_{\alpha_j} = \{0\}$ or, equivalently, D does not contain $\mathbb{C}$.

(i) $\Rightarrow$ (ii) Let $K : \mathbb{C} \to F_c(D)$ be an analytic multifunction. Suppose that $\widehat{K}(z_1) \neq \widehat{K}(z_2)$ for some two points $z_1, z_2 \in \mathbb{C}$. Take a plurisubharmonic function φ on Δ^p such that

$$\sup\{\varphi(y) : y \in \gamma\widehat{K}(z_1)\} \neq \sup\{\varphi(y) : y \in \gamma\widehat{K}(z_2)\}.$$

Since K is analytic, the function

$$\begin{aligned}\tilde{\varphi}(z) &= \sup\{\varphi(y) : y \in \gamma K(z)\} = \sup\{\varphi(y) : y \in \widehat{\gamma K(z)}\} \\ &= \sup\{\varphi(y) : y \in \gamma\widehat{K}(z)\}\end{aligned}$$

is subharmonic on $\mathbb{C}$. On the other hand, since $\gamma\widehat{K}(z) \subset \Delta^p$ for all $z \in \mathbb{C}$, we have $\tilde{\varphi}$ is bounded on $\mathbb{C}$. This is impossible because of the subharmonicity of $\tilde{\varphi}$ and of the relation $\tilde{\varphi}(z_1) \neq \tilde{\varphi}(z_2)$.

(ii) $\Rightarrow$ (i) From (ii) we have $\bigcap_{\alpha \in I} \{\ker x^*_\alpha\} = 0$, which implies that $\gamma : D_1 \to \Delta^p$ is a biholomorphism. Thus, D_1 and hence D is hyperbolic.

(i) $\Rightarrow$ (iii) By Theorem 2.3, γK and hence K can be extended to an analytic multifunction $\widetilde{K} : \Delta \to F_c(D_1)$. It remains to show that $\widetilde{K}(0) \subset D$.

Let $\alpha \in I$ and $\widetilde{x_\alpha^* K}$ be an extension of $x_\alpha^* K$ with values in $F_c(H_\alpha)$. Assume that $\widehat{x_\alpha^* K}(0) \neq x_\alpha^* \widetilde{K}(0)$. Take a plurisubharmonic function φ on $\mathbf{C}$ such that $\varphi_1(0) \neq \varphi_2(0)$, where

$$\varphi_1(z) = \sup\{\varphi(y) : y \in \widehat{x_\alpha^* K}(z)\} = \sup\{\varphi(y) : y \in \widetilde{x_\alpha^* K}(z)\}$$

and

$$\varphi_2(z) = \sup\{\varphi(y) : y \in x_\alpha^* \widehat{\widetilde{K}}(z)\} = \sup\{\varphi(y) : y \in x_\alpha^* \widetilde{K}(z)\}$$

for $z \in \mathbf{C}$.

Since φ_1 and φ_2 are plurisubharmonic on Δ and $\varphi_1 = \varphi_2$ on Δ^*, we have $\varphi_1(0) = \varphi_2(0)$. This is impossible because of the choice of φ. Thus, $\mathrm{Re} x_\alpha^*(z) < \epsilon_\alpha$ for all $z \in \widetilde{K}(0)$ and for all $\alpha \in I$. Hence $K(0) \subset D$.

(iii) $\Rightarrow$ (i) By [4], it suffices to show that every holomorphic map $\beta : \mathbf{C} \to D$ is constant. By the hypothesis, β can be extended to an analytic multifunction $\hat{\beta}$ on $\mathbf{C}P^1$. This implies that β is locally bounded on $\mathbf{C}P^1$. Thus $\hat{\beta}$ is holomorphic at ∞. Hence $\hat{\beta}$ and therefore β is constant.

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