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A CLASS OF CALIBRATED FORMS ON f-MANIFOLDS

HOANG XUAN HUAN

Introduction

The calibration method was introduced first by Dao Trong Thi [4-6] and later by R. Havey, H.B. Lawson [12] in order to study globally minimal currents and surfaces on Riemannian manifolds. The principle of this method can be described as follows. Given a closed k-form with comass $||\Omega||^* \leq 1$ on a Riemannian manifold M, we define the cone of maximal directions of Ω at x

$$F(\Omega) = \{\xi \in \wedge_k M_x \mid \Omega(\xi) = ||\xi||\}.$$

If the tangent space $\vec{S}_x$ of a surface S belongs to $F_x(\Omega)$ almost everywhere, then S is a globally minimal surface. In this paper we study a class of calibrated forms Ω on $(2r+p)$-dimensional f-manifolds and we find the cone of maximal directions $F_x(\Omega)$. Thereby, calibrated forms can be presented as

$$\Omega = \eta^1 \wedge \cdots \eta^q \wedge \omega^k/k! \ , \ 0 \leq q \leq p \ , \ 0 \leq k \leq r \ ,$$

where ω is a closed 2-form and η^i $(0 < i \leq p)$ are the 1-forms of the f-structure.

These results allow us to determine a class of minimal surfaces on f-manifolds and in particular, on contact manifolds.

1. Forms and currents

In this section we collect some concepts and facts of the current theory (for details see [7]).

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Let R^n be the n-dimensional Euclidian space, $\wedge_{k,n}$ and $\wedge^{k,n}$ the dual spaces of the k-vectors and k-covectors on R^n, respectively. The direct sums $\wedge_{*,n} = \oplus\wedge_{k,n}, \wedge^{*,n} = \oplus\wedge^{k,n}$ form the contravariant and covariant Grassmann algebras with the exterior multiplication $\wedge$.

The scalar product $(.,.)$ in R^n induces the scalar product $(.,.)$ and the corresponding norm $|.|$ in $\wedge_{*,n}$, and the mass of a k-vector $\xi \in \wedge_{k,n}$ is defined by

(1.1)

$$||\xi|| = \inf_B\{\sum_{\beta\in B}\beta \mid B \text{ is a finite set of simple } k\text{-vectors such that } \xi = \sum_{\beta\in B}\beta\}$$

The comass of a k-covector $\omega \in \wedge^{k,n}$ is defined by

$$||\omega||^* = \sup\{\omega(\eta) \mid \eta \text{ is a simple } k\text{-vector and } |\eta| \leq 1\} \tag{1.2}$$

One can prove that the infimum in (1.1) and the supremum in (1.2) are attained for a finite set B and a simple k-vector ξ respectively. If ξ is a simple k-vector, then $||\xi|| = |\xi|$.

Let M be a Riemannian manifold. Each differential k-form can be regarded as a smooth section of the Grassmann bundle $\wedge^{k,n}$. We denote by E^k the vector space of all real differential k-forms on M equiped with the topology of compact convergence of all partial derivatives. A k-current (with compact support) on M is a real continuous linear functional on E^k. The mass $M(S)$ of a k-current S is defined by

$$M(S) = \sup\{S(\varphi) \mid \varphi \in E^k(M)\ ,\ ||\varphi_x||^* \leq 1\ ,\ \forall x \in M\}.$$

If $M(S) < \infty$, one can define a measure $||S||$ by the formula

$$||S||(f) = \sup\{S(\varphi) \mid \varphi \in E^k(M)\ ,\ ||\varphi_x||^* \leq f(x)\ ,\ \forall x \in M\}.$$

for any real nonnegative continuous function f on M. In this case, there exists $||S||$-measurable section $\vec{S}$ of the bundle $\wedge M$ such that $||S_x|| = 1$ almost everywhere in the sense of the measure $||S||$ and such that

$$S(\varphi) = \int \varphi(\vec{S}_x d||S||(x) \tag{1.3}$$

for an arbitrary k-form $\varphi \in E^k M$. $\vec{S}_x$ is called the tangent k-vector of S at x. The boundary of a k-current S is a $(k-1)$-current defined by the formula $\partial S(\varphi) = Sd\varphi$ for any $\varphi \in E^{k-1}M$. A k-current S on M (with or without boundary) is said to be absolutely (respectively, homologically) minimal if $M(S) \leq M(S')$ for any k-current S' such that $\partial S = \partial S'$ (respectively, $S - S'$ is the boundary of some $(k+1)$-current on M).

Let Ω be a differential k-form on M. We define $||\Omega||^* = \sup ||\Omega_x||^*$ and the cone

$$F_x(\Omega) = \{\xi \in \wedge_k M_k \mid \Omega_x \xi = ||\xi||.||\Omega||^*\}.$$

DEFINITION: A closed k-form Ω is called a calibration if $||\Omega||^* = 1$. In this case, a k-vector $\xi \in F_x(\Omega)$ is called Ω-maximal.

The following theorem is the main tool for the determination of minimal currents in the following sections.

THEOREM 1.2 [6]. *Let Ω be a calibration, and S a current such that S_x are Ω-maximal almost everywhere in the sense of the measure $||S||$. Then S is homologically minimal (if Ω is exact, then S is absolutely minimal).*

Based on Theorem 1.2 we can prove the globally minimal properties of a class of currents and surfaces by finding suitable calibrations and determining its cones of maximal directions.

2. Calibrations on manifolds having a closed 2-form

Let ω be an exterior 2-form on R^n. We denote by φ the skew-symmetric transformation associated with ω such that $(\varphi(u), v) = \omega(u, v)$ for all $u, v \in R^n$. If K is a two-dimensional invariant subspace of R^n with respect to φ, then the restriction $\varphi\,|_k$ of φ to K is also a skew-symmetric transformation. With respect to an arbitrary orthonormal basic of K, the matrix of $\varphi\,|_k$ takes the form

$$\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}.$$

We can choose the orientation of the basis such that $\lambda \geq 0$. The number λ does not depend on the choice of the orthonomal basis of K, and λ is called the characteristic value of φ corresponding to the space K. It is well-known that R^n always admits an orthogonal decomposition (which is not unique) $R^n = K_0 \oplus \cdots \oplus K_r$, where $K_0 = ker\ \varphi$ and K_i ($i \leq r$ is a two-dimensional invariant subspace corresponding to the characteristic value λ_i $(0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_r)$. Let H be an arbitrary $2k$-imensional subspace and p the orthogonal projection from R^n to H. We define by φ_H the restriction of the transformation $\rho = p\varphi$ on H. φ_H is a skew-symmetric transformation on H, so H can be expressed by an orthogonal sum

$$H = H_1 \oplus \cdots \oplus H_k,$$

where H_i $(i \leq k)$ is a two-dimensional invariant subspace of H corresponding to the characteristic value β_i of φ_H $(0 \leq \beta_i \leq \cdots \leq \beta_k)$.

THEOREM 2.1. *Let φ and H be as above. Then $\beta_1 \cdots \beta_k \leq \lambda_{r-k+1} \cdots \lambda_r$. Moreover, equality holds if and only if $H = H_1 \oplus \cdots \oplus H_k$, where H_i $(i \leq k)$ is a two-dimensional invariant subspace of R^n corresponding to the characteristic value λ_{r-k+i} of φ.*

PROOF: If $\beta_1 = 0$, the statement is trivial. Therefore, we may assume that $\beta_1 > 0$. For any subset X of R^n, we denote by span (X) the subspace spanned by the vectors of X. For simplicity we shall consider only the case $n = 2r$. The case $n = 2r + 1$ can be proved analogously. We shoose an orthonormal basis $\{v_i\}_{i=1}^n$ of R^n satisfying the conditions $\varphi(v_{2i-1}) = \lambda_i v_{2i}, \varphi(v_{2i}) = \lambda_i v_{2i-1}$ for any $i \leq r$ and an orthonormal basis $\{e_i\}_{i=1}^n$ of H satisfying the conditions $\varphi_H(e_{2i-1}) = \beta_i e_{2i}$, $\varphi_H(e_{2i}) = -\beta_i e_{2i-1}$ $(i \leq k)$. Suppose that

$$e_i = \sum_{j=1}^{2r} c_{i,j} v_j \qquad (i \leq k). \tag{2.1}$$

We have

$$\varphi(e_i) = \sum_{j=1}^{r} \lambda_j(c_{i,2j-1}v_{2j} - c_{i,2j}v_{2j-1}) \quad \text{and}$$

$$(2.2) \qquad (e_i, \varphi(e_m)) = -(e_m, \varphi(e_i)) = \sum_{j=1}^{r} \lambda_j(c_{m,2j-1}c_{i,2j}c_{i,2j-1})$$

In particular,

$$(2.3) \qquad \beta_i = (e_i, \varphi(e_{2i-1})) = \sum_{j=1}^{r} \lambda_j(c_{2i-1,2j-1}c_{2i,2j} - c_{2i-1,2j}c_{2i,2j-1}).$$

We define the function

$$F(c) = F(c_{1,1}, \cdots c_{2k,2k}) = \ell n(\prod_{i=1}^{k} \beta_i) =$$

$$\sum_{i=1}^{k} \ell n \sum_{j=1}^{r} \sum_{j=1}^{r} \lambda_j(c_{2i-1,2j-1}c_{2i,2j} - v_{2i-1,2j}c_{2i,2j-1}),$$

and consider the problem of minimizing $F(c)$ under the following constraints

$$(2.4) \qquad \sum_{j=1}^{2r} c_{i,j}^2 = 1$$

$$(2.5) \qquad \sum_{t=1}^{2r} c_{p,t}c_{q,t} = 0$$

$$(2.6) \qquad 0 = \sum_{j=1}^{r} \lambda_j(c_{p,2j-1}c_{q,2j} - c_{p,2j}c_{q,2j-1}) \ , \ \forall(p,q) \neq (2i-1, 2j).$$

It is easy to verify that this problem always has solutions. Assume that c is a solution of this problem. We shall prove that each pair of vectors (e_{2i-1}, e_{2i}) (renumbered if necessary) form an orthonormal basis of some two-dimensional invariant subspace (with respect to φ) corresponding to the characteristic value

λ_{r-k+i} $(1 \leq i \leq k)$. In fact, since c is a conditionally extremal point of F, it must satisfy the Lagrange's equations

$$\frac{\lambda_j}{\beta_i} c_{2i,2j} = 2u_{2i-1} c_{2i-1,2j-1} + \sum_{t \neq 2i-1} u_{2i-1,t} c_{t,2j-1} + \sum_{t \neq 2i-1,2i} \lambda_j p_{2i-1,t} c_{t,2j},$$

$$\frac{-\lambda_j}{\beta_i} c_{2i-1,2j} = 2u_{2i-1} c_{2i-1,2j} + \sum_{t \neq 2i-1} u_{2i-1,t} c_{t,2j} - \sum_{t \neq 2i-1,2i} \lambda_j p_{2i-1,t} c_{t,2j-1},$$

$$\text{(2.7)} \quad \frac{-\lambda_j}{\beta_i} c_{2i,2j-1} = 2u_{2i} c_{2i-1,2j-1} + \sum_{t \neq 2i} u_{2i,t} c_{t,2j-1} + \sum_{t \neq 2i-1,2i} \lambda_j p_{2i,t} c_{t,2j-1},$$

$$\frac{\lambda_j}{\beta_i} c_{2i-1,2j-1} = 2u_{2i} c_{2i,2j} + \sum_{t \neq 2i} u_{2i,t} c_{t,2j} + \sum_{t \neq 2i-1,2i} \lambda_j p_{2i,t} c_{t,2j-1},$$

$(1 \leq i \leq k,\ 1 \leq j \leq r)$, where $u_i, u_{i,j}$ and $p_{i,j}$ $(i < j)$ are Lagrange's multipliers corresponding to the contrains (2.4), (2.5), (2.6) respectively, and

$$\text{(2.8)} \qquad u_{j,i} = u_{i,j}\ ,\ p_{j,i} = -p_{i,j} \text{ for } i > j.$$

Taking (2.2), (2.3), (2.4) into account, we obtain from (2.7) the following equalities

$$\text{(2.9)} \quad -\varphi(e_{2i}) = 2\beta_i u_{2i-1} e_{2i-1} + \sum_{t \neq 2i-1} \beta_i u_{2i-1,t} e_t - \sum_{t \neq 2i-1,2i} \beta_i p_{2i-1,t} \varphi(e_t),$$

$$\text{(2.10)} \qquad \varphi(e_{2i-1}) = 2\beta_i u_{2i} e_{2i} + \sum_{t \neq 2i} \beta_i u_{2i,t} e_t - \sum_{t \neq 2i-1,2i} \beta_i p_{2i,t} \varphi(e_t).$$

Multiplying both sides of (2.9) by e_{2i-1} and those of (2.10) by e_{2i}, and taking (2.4), (2.5), (2.6) into account, we obtain $\beta_i = 2\beta_i u_{2i-1}, \beta_i = 2\beta_i u_{2i}$ $(1 \leq i \leq k)$. Hence $u_{2i-1} = u_{2i} = 1/2$. Multiplying (2.9) by e_{2i} yields

$$\text{(2.11)} \qquad \beta_i u_{2i-1,2i} = 0 \quad \text{or} \quad u_{2i-1,2i} = 0.$$

Now we fix an arbitrary pair of (i,j), $1 \le i \ne j \le k$. Multiplying (2.9) by e_{2j-1} and then by e_{2i}, we get

$$\begin{aligned} &\beta_i u_{2i-1,2j-1} + \beta_i^2 p_{2i-1,2j} = 0, \\ &\beta_i u_{2i-1,2j} - \beta_i^2 p_{2i-1,2j-1} = 0. \end{aligned} \tag{2.12}$$

Since $\beta_i \ne 0$, from (2.12) it follows that

$$\begin{aligned} &u_{2i-1,2j-1} + \beta_i p_{2i-1,2j} = 0, \\ &u_{2i-1,2j} - \beta_i p_{2i-1,2j-1} = 0. \end{aligned} \tag{2.13}$$

Permuting i and j in (2.13) we obtain

$$\begin{aligned} &u_{2j-1,2i-1} + \beta_j p_{2j-1,2i} = 0, \\ &u_{2j-1,2i} - \beta_j p_{2j-1,2i-1} = 0. \end{aligned} \tag{2.14}$$

Doing the same operation for (1.10) we have

$$\begin{aligned} u_{2i,2j-1} + \beta_i p_{2i-1,2j} &= 0, \\ u_{2i,2j} - \beta_i p_{2i-1,2j-1} &= 0, \\ u_{2j,2i-1} + \beta_j p_{2j,2i} &= 0, \\ u_{2j,2i} - \beta_j p_{2j,2i-1} &= 0. \end{aligned} \tag{2.15}$$

Taking (2.8) into account we transform (2.13), (2.14), (2.15) into the following system

$$\begin{aligned} u_{2i-1,2j-1} + \beta_i p_{2i-1,2j} &= 0, \\ u_{2i-1,2j} - \beta_i p_{2i-1,2j-1} &= 0, \\ u_{2i-1,2j-1} + \beta_j p_{2i,2j-1} &= 0, \\ u_{2i,2j-1} - \beta_j p_{2i-1,2j-1} &= 0, \\ u_{2i,2j-1} + \beta_i p_{2i,2j} &= 0, \\ u_{2i,2j} - \beta_j p_{2i,2j-1} &= 0, \\ u_{2i-1,2j} + \beta_j p_{2j,2i} &= 0, \\ u_{2j,2i} - \beta_j p_{2i-1,2j} &= 0. \end{aligned} \tag{2.16}$$

We regard (2.16) as a system of linear equations of $u_{2i-1,2j-1}, u_{2i-1,2j}, u_{2i,2j-1}, u_{2i,2j}, p_{2i-1,2j-1}, p_{2i-1,2j}, p_{2i,2j-1}, p_{2i,2j}$. The determinant of this system is equal to $-(\beta_j^2 - \beta_i^2)^2$. So, if $\beta_i \neq \beta_j$, then the system has only the trivial solution

$$(2.17)\qquad \begin{aligned} u_{2i-1,2j-1} &= u_{2i-1,2j} = u_{2i,2j-1} = u_{2i,2j} = \\ &= p_{2i-1,2j-1} = p_{2i-1,2j} = p_{2i,2j-1} = p_{2i,2j} = 0 \end{aligned}$$

But if $\beta_i = \beta_j$, then from (2.16) it follows that

$$(2.18)\qquad \begin{aligned} p_{2i,2j} &= p_{2i-1,2j-1}, \quad p_{2i,2j-1} = -p_{2i-1,2j}, \\ u_{2i-1,2j} &= u_{2i-1,2j-1}, \quad u_{2i-1,2j} = -u_{2i,2j-1}, \\ u_{2i-1,2j-1} &= -\beta_i p_{2i-1,2j}, \quad u_{2i-1,2j} = \beta_i p_{2i-1,2j-1}. \end{aligned}$$

Putting (2.11), (2.18), (2.17) into (2.9) and (2.10) we obtain

$$(2.19)\qquad -\varphi(e_{2i}) = \beta_i e_{2i-1} + \sum_{\substack{\beta_t=\beta_i \\ t\neq i}} \{u_{2i-1,2t-1}[\beta_i e_{2t-1} + \varphi(e_{2t})] + u_{2i-1,2t}[\beta_i e_{2t} - \varphi(e_{2t-1})]\}$$

$$(2.20)\qquad \varphi(e_{2i-1}) = \beta_i e_{2i} + \sum_{\substack{\beta_t=\beta_i \\ t\neq i}} \{u_{2i-1,2t-1}[\beta_i e_{2t} - \varphi(e_{2t-1})] - u_{2i-1,2t}[\beta_i e_{2t-1} + \varphi(e_{2t})]\}$$

Set

$$(2.21)\qquad \begin{aligned} e'_{2i-1} &= e_{2i-1} + \sum_{\substack{\beta_t\neq\beta_i \\ t\neq i}} [u_{2i-1,2t-1} e_{2t-1} + u_{2i-1,2t} e_{2t}], \\ e'_{2i} &= e_{2i} + \sum_{\substack{\beta_t=\beta_i \\ t\neq i}} [u_{2i-1,2t} e_{2t} - u_{2i-1,2t} e_{2t-1}]. \end{aligned}$$

Since $\{e_i\}_{i=1}^k$ are linearly independent, $e'_{2i-1} \neq 0$ and $e'_{2i} \neq 0$. Hence, we may put

$$e''_{2i-1} = e'_{2i-1}/|e'_{2i-1}|,$$
$$e''_{2i} = e'_{2i}/|e'_{2i}|.$$

By virtue of (2.19) and (2.20) we have $\varphi(e''_{2i-1}) = \beta_i e_{2i}$ and $\varphi(e''_{2i}) = \beta_i e_{2i-1}$.

Consequently, β_i is a characteristic value of φ and span (e''_{2i-2}, e''_{2i}) is an invariant subspace of R^n corresponding to β_i. Thus, there exists $j \leq r$ such that $\beta_i = \lambda_j$. Moreover, if $m \neq i$, $\beta_m = \lambda_j$ and span (e_{2m-1}, e_{2m}) is a two-dimensional invariant subspace with respect to φ, then in (2.19), (2.20) the terms with index $t = m$ vanish. Continuing the process of orthonormalizing, we can choose the orthonormal basis $\{e_i\}$ of H such that $e_{2i-1} = e''_{2i-1}$ and $e_{2i} = e''_{2i}$. Hence $F(c)$ attains the maximal value if and only if $\{\beta_i\}_{i=1}^k$ are k largest characteristic value of φ and H_i is the two-dimensional invariant subspace corresponding to β_i.

THEOREM 2.2 (GENERALIZED WIRTINGER'S INEQUALITY). *Let ω be an exterior 2-form on R^n, $\{\omega_i\}_{i=1}^n$ an orthonormal basis of $\Lambda^{1,n}$ such that $\omega = \lambda_1\omega_1 \wedge \omega_2 + \cdots + \lambda_r\omega_{2r-1} \wedge \omega_{2r}$ $(r = [n/2], 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2r})$. Then*

$$\omega^k(\xi) \leq k! \prod_{i=1}^{k} \lambda_{r-k+i}||\xi|| \tag{2.22}$$

for any $k \leq r$ and any $2k$-vector $\xi \in \Lambda_{2k,n}$. Equality holds if and only if $\xi = \xi_1 + \cdots + \xi_p$, where ξ_i $(i \leq p)$ is a simple $2k$-vector which has the form $a_i e_1 \wedge \cdots \wedge e_{2k}$, $\{e_j\}_{j=1}^{2k}$ is an orthonormal system such that $\{e_{2i-1}, e_{2i}\}$ form an orthonormal basis of a two-dimensional invariant subspace (with respect to φ) corresponding to characteristic value λ_{r-k+1} $(j \leq k)$.

PROOF: First of all, we assume that ξ is a simple $2k$-vector in $\Lambda_{2k,n}$, $\xi = e_1 \wedge \cdots \wedge e_{2k}$. Put $K = \text{span}\,(e_1 \cdots e_{2k})$. Let $f : K \hookrightarrow R^n$ be the embedding of K into R^n. Then $f^*(\omega)$ is an exterior 2-form on K associated with the skew-symmetric transformation φ_k. Denote by $\beta_1, \cdots, \beta_k$ the characteristic values of φ_k and consider an orthonormal basis $\{\theta_1, \cdots, \theta_{2k}\}$ of $\wedge^{1,k}$ such that

$$f^* = k!\ \beta_1\theta_1 \wedge \theta_2 + \cdots + \beta_k\theta_{2k-1} \wedge \theta_{2k}.$$

Then $(f^*\omega)^k = k!\ \beta_1 \cdots \beta_k\ \ \theta_1 \wedge \cdots \wedge \theta_{2k}$. Therefore $\omega^k(\xi)/k! = (f^*\omega)^k(\xi)/k! = \pm\beta_1 \cdots \beta_k||\xi||$. Using Theorem 2.1 we obtain $\omega^k(\xi) = k!\ \lambda_{r-k+1} \cdots \lambda_r||\xi||$. Moreover, equality holds iff ξ has the mentioned form. Consider now an arbitrary $2k$-vector $\xi \in \wedge_{2k,n}$. As it was mentioned above, one can choose simple $2k$-vectors $\eta_1, \cdots, \eta_p$ such that $\xi = \eta_1 + \cdots + \eta_p$ and $||\xi|| = ||\eta_1|| + \cdots + ||\eta_p||$. According to (2.4) we have

(2.23)

$$\begin{aligned}\omega^k(\xi) &= \omega^k(\eta_1) + \cdots + \omega^k(\eta_p)\\ &\leq k!\lambda_{r-k+1} \cdots + \lambda_r(||\eta_1|| + \cdots + ||\eta_1|| + \cdots + ||\eta_p||\\ &= k!\ \lambda_{r-k+1} \cdots \lambda_r||.\end{aligned}$$

Moreover, equality holds iff $\omega^k(\eta_i) = k!\ \lambda_{r-k+i} \cdots \lambda_r||\eta_i||$ for any $i \leq p$. Consequently, ξ has the desired form. The proof is complete.

Now let M be a Riemannian manifold, ω a closed differential 2-form on M. Then $\Omega = \omega^k/||\omega^k||^*$ ($k \leq [n/2]$) is also closed. We have the following consequence of Theorem 2.2.

COROLLARY 2.3. *Let ω be a closed differential 2-form on M and $\prod_{i=1}^{k} \lambda_{r-k+i} =$ constant $\neq 0$ for all $x \in M$, where $r = [n/2]$, $\lambda_i(x)$ $(i \geq r-k+1)$ are the k greatest characteristic values of the skew-symmetric transformation associated with the exterior 2-form ω_x. Then $\Omega = \omega^k/||\omega^k||^*$ are calibrations and the cones Ω-maximal directions were determined by Theorem 2.2.*

3. Calibrations on contact manifolds

In this section, M will be a $(2n+1)$-dimensional Riemannian manifold. M is said to be a contact manifold if there exists a pfaff-form having maximal rank η. If M is a contact manifold, there exists a vector field ξ such that

$$\eta(\xi)_x = 1, i\, d\eta_x = 0 \qquad \text{for all} \qquad x \in M. \tag{3.1}$$

(for details, see [9]). Moreover, for each $x \in M$ there exists a local coordinate of M such that

$$\eta = dx_1 + \sum_{i=1}^{n} x_{2i} dx_{2i+1} \qquad \text{and} \qquad \xi = \frac{\partial}{\partial x_1}. \tag{3.2}$$

Several authors [13, 14, 15] have studied the metric contact manifolds (α, ξ, η, g), where α is an affinor field and g is a Riemannian metric satisfying the following conditions

$$\text{rank } \alpha^i_j = 2n,\ \xi^i \eta_i = 1,\ \alpha^i_q \xi^q = 0, \tag{3.3}$$

$$\alpha^q_j \alpha^i_q = \delta^i_j + \eta_j \xi^i, \tag{3.4}$$

$$\eta_i = g_{i,q}\xi^q,\ g_{p,q}\alpha^p_i \alpha^q_j = g_{i,j} - \eta_i \eta_j \tag{3.5}$$

$$\alpha_{i,j} = \frac{\partial \eta_j}{\partial x_i} - \frac{\partial \eta_i}{\partial x_j} \qquad \text{where} \qquad \alpha_{i,j} = \alpha^q_i g_{q,j} \tag{3.6}$$

For the contact manifolds, the existence of metrics mentioned above has been proved in [11, 15]. In this case, ξ and α induce two distributions $t =$ span $(\{\alpha_q\}^n_{q=1})$ and $m = \xi$.

THEOREM 3.1. *Let M be a contact manifold as above and $\Omega = (d\eta)^k/\|((d\eta)^k\|^*$, $k \leq n$. Then Ω is a calibration, and the cones of Ω-maximal vectors belong to $\wedge_{2k}(t)$ and are determined by Theorem 2.2.*

PROOF: Because of (3.6), we have $g(\xi, \alpha_q) = \xi^i g_{i,k}\alpha^k_q = \xi^i \alpha_{i,q}$ for each q. On other hand, $i_\xi d\eta = 0$ by (3.1). Hence $\xi^i \alpha_{i,q} = 0$ for any q. This implies that two distributions t and ξ are orthonormal. If φ is the skew-symmetric transformation corresponding to $d\eta$, then ξ belongs to ker φ. Hence the assertion follows by applying Theorem 2.2.

Now let M be a contact manifold without metric. We will construct a Riemannian metric and find suitable calibrations.

THEOREM 3.2. *Let M be a contact manifold. Then there exist an exact 1-form ω on M such that $\omega_x(\xi) = 1$ for all $x \in M$.*

PROOF: Let $\{u_\alpha\}_{\alpha\in N}$ be a locally finite open cover of M and $\{x^\alpha\}_{\alpha\in M}$ the correspondent coordinate system satisfying (3.2). Then

$$\eta = dx_1^\alpha + x_2^\alpha dx_3^\alpha + \cdots + x_{2n}^\alpha dx_{2n+1}^\alpha \tag{3.7}$$

on u_α for any $\alpha \in N$.

We choose a unit decomposition $\{\theta_\alpha\}_{\alpha\in N}$ refining $\{u_\alpha\}_{\alpha\in N}$ such that support $\theta_\alpha = v_\alpha \subset \{x \in M/|x_1^\alpha| < 1\}$. For each $p \in M$, define

$$f(p) = \sum_\alpha \int_0^{x_1^\alpha(p)} \theta_\alpha(t, x_2^\alpha(p) \cdots x_{2n+1}^\alpha(p))dt.$$

It is clear that $\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x_1^\alpha} = 1$ and $\omega = df$ is the required 1-form.

COROLLARY 3.3. *Let M be a contact manifold satisfying the assumption of Theorem 3.1. There exists a Riemannian metric g' on M such that g' induces the metric g on the distribution t, and the two distributions t, m are orthogonal. For the calibration $\Omega = (d\eta)^k \wedge \omega/||(d\eta)^k||^*$, $k \le n$, a $(2k+1)$-vector ρ is Ω-maximal if $\rho = \beta \wedge \xi$, where β is $(d\eta)^k/||(d\eta)^k||^*$-maximal*

PROOF: We denote by p the ξ-direction projection from TM to m and I the identical transformation on TM. Let $q = I - p$. We put

$$g'(X,Y) = g(q(X), q(Y)) + \omega(X)\omega(Y) \quad \forall X, Y \in TM. \tag{3.8}$$

Then g' is the desired metric.

COROLLARY 3.4:

1) Contact manifolds are cosymplectic manifold with the 1-form ω and 2-form $d\eta$ mentioned above.

2) For each $a \in R$, let $H^\alpha = \{x \in M \mid f(x) = a\}$. If $H^\alpha \neq \emptyset$, then H^α is symplectic with the 2-form $h_a^* d\eta$, where h_a is the embedding of H^α into M.

Consider the differential system $\gamma = \{X \in J(M) \mid \omega(X) = 0\}$ and the two distributions $t = \text{span}\ \{\gamma\}, m = \text{span}\ \{\xi\}$. Then $\dim t_x = 2n, t_x H_x^a$ for any $x \in M$, where $a = f(x)$.

THEOREM 3.5:

1) There exists a Riemannian metric on M such that m and t are orthogonal and $||\omega||^* = 1, \xi_x$ is ω-maximal for each $x \in M$.

2) Put $\Omega_1 = (d\eta)^k / ||(d\eta)^k||^*$. A $2k$-vector $\beta \in TM_x$ is ω_1-maximal iff β is $h_a^* \Omega_1$-maximal $(a = f(x))$. Moreover, $\Omega = \Omega_1 \wedge \omega$ is a calibration, and a $(2k+1)$-vector σ is Ω-maximal iff $\sigma = \beta \wedge \xi$, where β is Ω-maximal.

PROOF:

1) Let $\{v_\alpha\}$, $\{\theta_\alpha\}$, $\{x^\alpha\}$ be the families determined in the proof of Theorem 3.2. Because of (3.7) we have

$$d\eta = dx_2^\alpha \wedge dx_3^\alpha + \cdots + dx_{2n}^\alpha \wedge dx_{2n+1}^\alpha.$$

Consider H_α^a as a $2n$-dimensional symplectic manifold with the corresponding complex operation J_α. Denote by $P_\alpha : TM \to TH_\alpha^a$ the ξ-direction projection. Put

(3.9) $$g_\alpha(X, Y) = d\eta(J_\alpha P_\alpha(X), P_\alpha(Y)) + \omega(X)\omega(Y)$$

for any $X, Y \in TM_x$.

It is clear that g_α is symmetric and positive. Moreover, the two distributions t and m are orthogonal with respect to g_α. Put

$$g(X,Y) = \Sigma\theta_\alpha g_\alpha(X,Y). \tag{3.11}$$

It is straightforward that g is the required metric.

2) Because $\omega(Y) = 0$ for each $Y \in t$, $i_\xi d\eta = 0$, the assertion is only a simple consequence of the first assertion and Theorem 2.2.

4. A class of calibrations on f-manifolds

In this section, M denotes a $(2n+p)$-dimensional manifold. If M is equiped with an affinor field f of rank $2n$ such that

$$f^3 + f = 0, \tag{4.1}$$

then M is called a f-manifold. In this case, there exist the 1-forms $\{\eta^a\}$ ($a = 1, 2, \cdots, p$) and the vector fields $\{E_b\}$ ($b = 1, 2, \cdots, p$) on M such that

$$\eta^a E_b = \delta^a_b, f^2 = -I + \Sigma\eta^a \otimes E_a \tag{4.2}$$

Several authors (see [9]) have studied Riemannian metric g satisfying the conditions

(a) $g(X,Y) = g(f(X), f(Y)) + \Sigma\eta^a(X)\eta^a(Y)$,

(b) $F(X,Y) = g(X, f(Y))$ is a closed 2-form

THEOREM 4.1. *Let g be a metric and F defined as above. Then*

1) $f(E_a) = 0$ for any $a \le p$

2) Rank $F = 2n$ and for any $k \le n$, $\Omega = F^k/k!$ is a calibration and the Ω-maximal cones are determined by the Theorem 2.2.

PROOF: 1) By (4.1) we have $f(E_a) = -f^3(E_a) = -f(f^2(E_a))$ for any $a \le p$ By (4.2) this implies $f^2(E_a) = E_a - \eta^a(E_a)E_a = 0$. Hence $f(E_a) = 0$.

2) Because rank $f = 2n$, from (b) we get rank $F = 2n$. Using (a) and (b), we have

$$i_{E_a} F = 0. \tag{4.3}$$

Denote by φ_x the skew-symmetric transformation of F at x. Then E_a belongs to $\ker \varphi_x$ for any a and any $x \in M$. Let ω_i $(1 \leq i \leq 2n)$ be an orthonormal system such that

$$F = \lambda_1 \omega_1 \wedge \omega_2 + \cdots + \lambda_n \omega_{2n-1} \wedge \omega_{2n} \text{ at } x.$$

By (b) f is the skew-symmetric transformation of F. Let $\{e_i\}$ be the orthonormal system which satisfy the conditions

$$f(e_{2i-1}) = \lambda_i e_{2i},\ f(e_{2i}) = -\lambda_i e_{2i-1},\ \omega_j(e_i) = \delta^i_j. \tag{4.4}$$

From (a), (b) and (4.4) it follows that $\lambda_i = -F(e_{2i}, f(e_{2i})) = g((e_{2i}, f^2(e_{2i}))$. Taking (4.2) into account we have $\lambda_i = g(e_{2i}, e_{2i}) + \Sigma_a \eta^a(e_{2i}) g(e_{2i}, E_a)$. Because of (b) and (4.3) $g(e_{2i}, E_a) = 0$. Thus, $\lambda_i = 1$ for all $i \leq n$.

Now applying Theorem 2.2 we obtain the statement.

COROLLARY 4.2. *If $\eta = \eta^1 \wedge \cdots \wedge \eta^q$ is a closed differential form, then η is a calibration and $F_x(\eta) = \wedge_{2q}$ (span $\{E_1, \cdots, E_q\}$). Moreover, for any $k \leq n$, $\Omega = \eta \wedge F^k/k!$ is a calibration, and if γ is Ω-maximal, then $\gamma = \xi \wedge \beta$, where ξ is η-maximal and β is F^k-maximal.*

PROOF: By (4.2), (a) and Theorem 4.1, $\{\eta^a\}$ and $\{E_a\}$ are dual orthonormal systems. Moreover, E_a belongs to $\ker\varphi_s$ for any x on M, where φ_x is the skew-symmetric transformation with respect to F_x. Hence the assertion is a simple consequence of Theorem 4.1 and Theorem 2.2.

REMARK: The case η^a being closed for all $a \leq p$ has been considered in [12], [13] (see also [9]).

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DEPARTMENT OF MATHEMATICS
HANOI UNIVERSITY