ACTA MATHEMATICA VIETNAMICA
Volume 16, Number 2, 1991

AN INTERSECTION THEOREM AND RELATED PROBLEMS

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Introduction

Many studies on the existence of solutions of various problems could be reduced to the study of the non-emptiness of the intersection of a finite (or an infinite) family of sets. A well-known result equivalent to the Brouwer's fixed point theorem is the Knaster-Kuratowski-Mazurkiewicz's intersection theorem (see Ky Fan [6] for a generalized form) is particularly suitable for many applications [2], [3]. However, these results were valid only for topological vector spaces.

In this paper we shall give a new intersection theorem for Hausdorff topological spaces. Our arguments for proving this theorem are purely topological and very simple.

First we will introduce the notion of weakly connected multifunctions relative to a subset of a Hausdorff topological space and consider some classes of these multifunctions (Section 1). This notion is a generalization of the notion of α-connected function given in [9] and of the multifunctions which have been implicitly used in several papers [1], [3], [10]. Using this new notion we will obtain an intersection theorem for topological spaces (Section 2). Section 3 is devoted to some related problems including fixed point theorems in Hausdorff topological spaces, a generalization of well-known versions of the minimax theorems in game theory such as Wu-Wen-Tsun's [10, Theorem 1] and a result of

Received July 30, 1990

Hoang Tuy's minimax theorem [9]. We will also consider existence theorems in the theory of variational inequalities in Hausdorff topological spaces.

1. Weakly connected multifunctions relative to a set

Throughout this paper X, Y denote Hausdorff topological spaces and C, D nonempty subsets of X, Y, respectively. Let 2^C be the set of all subsets of C and $\mathcal{C}(2^C)$ the set of all nonempty compact subsets of C.

A multifunction $f : C \to 2^Y$ is upper-semicontinuous (u.s.c.) at $x \in C$ if for each neighborhood V of $f(x)$ in Y there exists a neighborhood U of x in C such that $f(U) \subset V$.

DEFINITION 1: A multifunction $f : C \to 2^Y$ is said to be *weakly connected relative to the set D* if for every pair of points $p, q \in C$ with the property

$$f(p) \cap D \neq \emptyset, \ f(q) \cap D \neq \emptyset \text{ but } f(p) \cap D \cap f(q) = \emptyset. \tag{1.1}$$

there exists a u.s.c. multifunction $u : [0,1] \to \mathcal{C}(2^C)$ such that $p \in u(0)$, $q \in u(1)$ and for each $a \in [0,1]$,

$$\text{either } f(u(a)) \cap D \cap f(p) = \emptyset \ \text{ or } f(u(a)) \cap D \cap f(q) = \emptyset. \tag{1.2}$$

The set of such multifunctions is large enough. We shall present below some classes of these multifunctions.

Recall that a subset D' of D is called *connected relative* to D if D' can not be represented as the union of two nonempty disjoint relative open (or relative closed) subsets of D.

Class $\mathcal{A}$. Assume that $f : C \to 2^Y$ is a multifunction such that for every $p, q \in C$ satisfying (1.1) there is a u.s.c. multifunction $u : [0,1] \to \mathcal{C}(2^C)$ such that $p \in u(0)$, $q \in u(1)$ and for every $a \in [0,1]$, $f(u(a)) \cap D$ is nonempty, open, connected relative to D and $f(u(a)) \cap D \cap (f(p) \cup f(q))$ is nonempty connected relative to D. Then f is weakly connected relative to D.

Class $\mathcal{B}$. Assume that $f : C \to 2^Y$ is a multifunction such that for every pair $p, q \in C$ satisfying condition (1.1), there is a continuous mapping $u : [0,1] \to C$ with $u(0) = p$, $u(1) = q$ and for each $a \in [0,1]$ the condition (1.2) holds. Then f is weakly connected relative to D.

Subclass $\mathcal{B}_1$. Let C be an arc connected set in X, and $C = C_1 \cup C_2$, $D = D_1 \cup D_2$, where C_1, C_2 are disjoint, D_1, D_2 are nonempty disjoint. Suppose that $f : C \to 2^Y$ satisfies the following conditions

(i) For any $x', x'' \in C_1$, $f(x') \cap D_1 \cap f(x'') \neq \emptyset$, and $f(x) \subset Y \setminus D_2$ for all $x \in C_1$.

(ii) For any $x', x'' \in C_2$, $f(x') \cap D_2 \cap f(x'') \neq \emptyset$ and $f(x) \subset Y \setminus D_1$ for all $x \in C_2$.

Then f is a multifunction of class $\mathcal{B}$.

Subclass $\mathcal{B}_2$. This subclass consists of multifunction f satisfying the conditions :

(i) For every $x \in C$, $f(x) \cap D$ is nonempty, relative closed, connected relative to D.

(ii) For $p, q \in C$ satisfying (1.1), there is a continuous $u : [0,1] \to C$ such that $u(0) = p$, $u(1) = 1$ and for all $a \in [0,1]$, $fu(a) \cap D \cap (f(p) \cup f(q))$ is nonempty, connected relative to D.

Class $\mathcal{C}$. This class consists of multifunctions $f : C \to 2^Y$ such that for every pair $p, q \in C$ satisfying (1.1) one can find a u.s.c. multifunction $u : [0,1] \to \mathcal{C}(2^C)$ with $u(0) = p$, $u(1) = q$ and for all $a \in [0,1]$ the following conditions hold :

(i) $fu(a) \cap D$ is nonempty, relative closed, connected relative to D,

(ii) $fu(a) \cap D \subset (f(p) \cup f(q)) \cap D$.

Let α be a real number. Following [9], a real function $F : C \times D \to R^1$ is said to be *α-connected* on $C \times D$ if the multifunction $f_\alpha : C \to 2^D$, defined by $f_\alpha(x) = \{y \in D \mid F(x,y) \geq \alpha\}$ for each $x \in C$, satisfies the following conditions:

(i) The set $\cap\{f_\alpha(p_i) \mid i = 1, ..., k\}$ is connected for any finite set $\{p_1, p_2, ..., p_k\} \subset C$.

(ii) For any pair $p, q \in C$, there exists a continuous mapping $u : [0,1] \to C$ such that $p = u(0)$, $q = u(1)$ and

$$f_\alpha(u(t')) \subset f_\alpha(u(t)) \cup f_\alpha(u(t'')) \tag{1.3}$$

for any t, t', t'' with $0 \le t \le t' \le t'' \le 1$.

Clearly, the multifunction f_α corresponding to each α-connected real mapping F is a multifunction of the class $\mathcal{C}$.

Subclass $\mathcal{C}_1$. Let $f : C \to 2^D$ be a multifunction satisfying the following conditions :

(i) For each $x \in C$, $f(x)$ is a nonempty, relative closed, connected subset of D.

(ii) For any pair $p, q \in C$, there exists a continuous mapping $u' : [0,1] \to C$ connecting p with q such that for any $y \in f(u'([0,1]))$,

$$\text{either} \quad \{p, q\} \setminus C_y \neq \emptyset \quad \text{or} \quad u'([0,1]) \subset C_y, \tag{1.4}$$

where $C_y = \{x \in C : y \notin f(x)\}$.

Then f is a multifunction of the class $\mathcal{C}$.

Indeed, if $\{p, q\} \setminus C_y \neq \emptyset$ for all $y \in f(u'([0,1]))$, then $y \in f(p) \cup f(q)$, hence the condition (ii) of class $\mathcal{C}$ is satisfies for all $a \in [0,1]$. For, if $\{p, q\} \subset C_{y'}$ for some $y' \in f(u'([0,1]))$, then there is a point $a' \in [0,1]$ such that $y' \in f(u'(a')) \setminus \{f(p) \cup f(q)\}$. Hence, by (1.4) we obtain $u'([0,1]) \subset C_{y'}$. It follows that $u'(a') \subset C_{y'}$ and $u' \notin f(u'(a'))$, a contradiction.

Subclass $\mathcal{C}_2$. Let C be a nonempty convex subset of a Hausdorff topological vector space. Assume that $f : C \to 2^D$ is a multifunction such that for each $x \in C$, $f(x)$ is a nonempty, relatively open, connected subset of D and for each $y \in D$, the set

$$C_y = \{x \in C : y \notin f(x)\} \tag{1.5}$$

is convex. Then f is weakly connected relative to D.

Indeed, for any pair $p, q \in C$, since C is convex, we can choose the continuous arc by setting for all $a \in [0,1]$, $u'(a) = (1-a)p + aq$. Clearly, (1.4) follows from (1.5).

2. Intersection theorem

Let $f : C \to 2^Y$ be a multifunction such that $f(x)$ is nonempty for each $x \in C$. We denote by $\mathcal{B}(x)$ the set of all neighborhood of x in C. For any $B \subset C$ let $f(B) = \cup\{f(u) \mid u \in B\}$. By $\overline{f(B)}$ we denote the closure of the set $f(B)$.

Let $(Lsf)(x) : C \to 2^Y$ be the multifunction defined by $(Lsf)(x) = \cap\{\overline{f(B)} \mid B \in \mathcal{B}(x)\}$. Recall that the multifunction f is quasi upper semicontinuous (q.u.s.c.) at $x \in C$ if $(Lsf)(x) \subset f(x)$ (see [2]).

THEOREM 1. *Let C and D be nonempty subsets of Hausdorff topological spaces X and Y, respectively. Let D be a compact subset of Y. Assume that $f : C \to 2^Y$ is q.u.s.c. on C and weakly connected relative to the set D. Then $f(p) \cap D \cap f(q) \neq \emptyset$ for all pairs of elements $p, q \in C$ such that $f(p) \cap D \neq \emptyset$, $f(q) \cap D \neq \emptyset$.*

PROOF: Suppose that there exists a pair of elements $p, q \in C$ such that $f(p) \cap D \neq \emptyset$, $f(q) \cap D \neq \emptyset$ but $f(p) \cap D \cap f(q) = \emptyset$. Since f is weakly connected relative to D, there exists a u.s.c. multifunction $u : [0,1] \to \mathcal{C}(2^C)$ such that $p \in u(0)$, $q \in u(1)$ and for all $a \in [0,1]$, either $f(u(a)) \cap D \cap f(p) = \emptyset$ or $f(u(a)) \cap D \cap f(q) = \emptyset$. Set

$$A_1 = \{a \mid 0 \leq a < 1,\ f(u(a)) \cap D \cap f(q) = \emptyset\},$$
$$A_2 = \{a \mid 0 < a \leq 1,\ f(u(a)) \cap D \cap f(q) = \emptyset\}.$$

We have $0 \in A_1$, $1 \leq A_2$, $A_1 \cap A_2 = \emptyset$ and $[0,1] = A_1 \cup A_2$. Now we shall show that the sets A_1, A_2 are relative open.

Let us define a multifunction $g : C \to 2^D$ by $g(x) = f(x) \cap D$ for each $x \in C$. Then the multifunction $gu : [0,1] \to 2^D$ is a q.u.s.c. multifunction defined on $[0,1]$ and $gu(a') \cap g(q) = \emptyset$ for all $a' \in A_1$.

Let a be an arbitrary point of A_1 and $y \in g(q)$. Since gu is q.u.s.c. on $[0,1]$, there are two neighborhoods $U(y)$ of a in $[0,1]$ and $V(y)$ of y in Y such that $gu(a') \cap V(y) = \emptyset$ for all $a' \in U(y)$.

Thus, the set $\{V(y) \mid y \in g(q)\}$ is an open covering of the compact set $g(q)$. Hence there exists a finite subset $\{y_1, y_2, ..., y_m\}$ of $g(q)$ such that $\{V(y_i) \mid i = 1, 2, ..., m\}$ is a covering of $g(q)$.

Set $U(a) = \cap\{U(y_i) \mid i = 1, 2, ..., m\}$. Then for any $a' \in U(a)$, we have $gu(a') \cap g(q) = \emptyset$. This means $a' \in A_1$ and hence $U(a) \subset A_1$, i.e. A_1 is a relative open subset of $[0,1]$. By the same argument, A_2 is a nonempty relative open subset of $[0,1]$. But this is a contradiction because $A_1 \cup A_2 = [0,1]$.

Let $N_k(C)$ denote the set of all subsets U_k of C consisting of k different elements of C, $k = 1, 2, ...$

We define

$$S_k(f) = \{\cap\{f(x_i)\ i = 1, ..., k,\ x_i \in U_k\} \mid U_k \in N_k(C)\},$$
$$S(D) = \{D \cap B \mid B \in S_k(f),\ k = 1, 2, ...,\} \cup \{D\}.$$

A multifunction $f : C \to 2^Y \setminus \{\emptyset\}$ is called *connected relative to the set* D if f is weakly connected relative to each closed set of $S(D)$.

THEOREM 2. *Let D be a nonempty subset of Y. If a q.u.s.c. multifunction $f : C \to 2^Y$ is connected relative to D such that for each $x \in C$, $f(x) \cap D$ is nonempty, then the set $\cap\{f(x) \mid x \in C\}$ is nonempty.*

PROOF: We shall show that $D \cap \bigcap\{f(x) \mid x \in C\} \neq \emptyset$. Since f is q.u.s.c. on C, $f(x)$ is closed, it suffices to show that

$$D \cap \bigcap\{f(x_k) \mid k = 1, ..., n,\ x_k \in C\} \neq \emptyset.$$

We proceed by induction on the number n. First we consider the case $n = 2$. Since $f(x_1) \cap D$ and $f(x_2) \cap D$ are nonempty, using the weak connection relative to the set D, we have $f(x_1) \cap D \cap f(x_2) \neq \emptyset$ by Theorem 1. Assume that $D \cap \bigcap\{f(x_k) \mid k = 1, 2, ..., i,\ x_k \in C\} \neq \emptyset$ for all $i < n$ and consider n points of $x_1, x_2, ..., x_n$ of C. We fix $n-2$ points $x_2, x_3, ..., x_{n-1}$ of C. By the induction hypothesis, $D \cap [f(x_2) \cap ... \cap f(x_{n-1})] \cap f(x_1) \neq \emptyset$ and $D \cap [f(x_2) \cap ... \cap f(x_{n-1})] \cap f(x_n) \neq \emptyset$. Since $D' = D \cap \bigcap\{f(x_i) \mid i = 2, 3, ..., n-1\}$ is a closed subset of $S(D)$ and f is connected relative to D', we obtain $D \cap \bigcap\{f(x_i) \mid i = 1, 2, ..., n\} \neq \emptyset$.

REMARK: By a similar argument we obtain the following result.

A multifunction $f : C \to 2^Y$ is said to be *i-connected relatively* to the set D if f is weakly connected relative to every closed set of the form

$$S_i(D) = \{D \cap B \mid B \in S_k(f),\ k = 1, 2, ..., i\} \cup \{D\}.$$

THEOREM 2'. *Let D be a nonempty compact subset of a Hausdorff topological space Y and let $f : C \to 2^Y$ be a q.u.s.c. multifunction defined on C. Suppose that $D \cap f(x) \neq \emptyset$ for all $x \in C$ and that for some integer i, f is i-connected relative to D. Then*

$$D \cap \bigcap\{f(x_k) \mid k = 1, ..., i + 2\} \neq \emptyset$$

for any arbitrary subset of $i + 2$ points $x_1, x_2, ..., x_{i+2}$ of the set C.

3. Related problems

A first immediate consequence of our intersection theorem is the following fixed point theorems for Hausdorff topological spaces.

THEOREM 3. *Let C be a nonempty subset of a Hausdorff topological space X and let $f : C \to 2^X$ be a q.u.s.c. multifunction defined on C. If there exists a nonempty compact subset D of C such that f is connected relative to D and $f(x) \cap D \neq \emptyset$ for each $x \in C$, then f has a fixed point in C.*

PROOF: Let $Y = X$. Then the multifunction f satisfies all conditions of Theorem 2, hence the set $D' = D \cap \bigcap\{f(x) \mid x \in C\} \neq \emptyset$ and every point $x' \in D' \subset C$ is a fixed point of the multifunction f, i.e. $x' \in f(x')$.

A modification of this result is the following

THEOREM 3'. *Let C be a nonempty subset of a Hausdorff topological space X and let $f : C \to 2^X$. If there exists a nonempty compact subset D of C such that the restriction $f' = f \mid D$ of f on D is connected relative to D and is q.u.s.c. on D and $f(x) \cap D \neq \emptyset$ for each $x \in C$, then f has a fixed point in C.*

Now we give a generalized minimax theorem in Hausdorff topological spaces which includes well-known minimax theorems of Ky Fan [5], Wu-Wen-Tsun [10] and Hoang Tuy [9].

Let C, D be nonempty subsets of Hausdorff topological spaces X and Y, respectively. Let $F : C \times D \to R^1$ be a real-valued function. Put $t_0 = \inf_{x \in C} \sup_{y \in D} F(x, y)$. Assume that $t_0 < +\infty$.

Associated with F is the multifunction $f : C \to 2^D$ defined by

$$f(x) = \{y \in D \mid F(x, y) \geq t_0\}, \; x \in C.$$

The function F is called *connected* on $C \times D$ if its associated multifunction f is connected relatively to D.

Let Z be a nonempty subset of a Hausdorff topological space and $\alpha \in R^1$. A real-valued function $G : Z \to R^1$ is called upper semicontinuous in Z at α if the set $\{z \in Z \mid G(z) < \alpha\}$ is relative open in Z. The function G is upper semicontinuous (u.s.c.) in Z if G is u.s.c. in Z at every $\alpha \in R^1$.

THEOREM 4. *Assume that D is compact and F is a function connected on $C \times D$ such that $F(x, .)$ is u.s.c. in y for all $x \in C$ and the graph of f is a relative closed subset of $C \times D$. Then*

$$\sup_{y \in D} \inf_{x \in C} F(x,y) = \inf_{x \in C} \sup_{y \in D} F(x,y) \tag{3.1}$$

PROOF: Since the inequality

$$\sup_{y \in D} \inf_{x \in C} F(x,y) \leq \inf_{x \in C} \sup_{y \in D} F(x,y)$$

is trivial, we only need to show that the intersection $\cap\{f(x) \mid x \in C\}$ is nonempty. For a fixed $x \in C$, since D is compact and F is u.s.c. in y, the set $f(x)$ is nonempty. Moreover, since the graph of f is a relative closed subset of $C \times D$, the multifunction f is q.s.u.c. on C and it is connected relative to D. Hence by Theorem 2, $\cap\{f(x) \mid x \in C\} \neq \emptyset$.

COROLLARY 1. *If D is compact and F is a function connected on $C \times D$ which is u.s.c. in (x, y) at t_0 such that $F(x, .)$ is u.s.c. in y for all $x \in C$, then (3.1) holds.*

Now we present another class of multifunctions weakly connected to the set D.

Class $\mathcal{D}$. This class consists of multifunctions $f : C \to 2^Y$ such that for every pair of elements $p, q \in C$ satisfying (1.1) one can find a u.s.c. multifunction $u : [0,1] \to \mathcal{C}(2^C)$ such that the following conditions hold :

(i) $p \in u(0)$, $q \in u(1)$.

(ii) For each $a \in [0,1]$, (1.2) is satisfied.

(iii) For each $y \in \{p, q\}$, if $fu(a) \cap D \cap f(y) = \emptyset$ then there is a neighborhood U of a in $[0,1]$ such that $fu(a') \cap D \cap f(y) = \emptyset$ for all $a' \in U$.

We say that a real function $F : C \times D \to R^1$ is *connected on* $C \times D$ of type $\mathcal{D}$ if its associated multifunction f is weakly connected relative to each closed set of $S(D)$.

It is not difficult to prove the following

THEOREM 5. *If D is compact, F is connected on $C \times D$ of type $\mathcal{D}$ and $F(x,.)$ is u.s.c. in y for all $x \in C$, then (3.1) holds.*

Theorem 1 of [9] and Theorem 1 of [10] (hence Ky Fan's minimax theorem [5]) are consequences of Theorem 5. To see it we only need to use the following

LEMMA 1. *Assume that D is compact and F is t_0-connected. Suppose that $F(.,y)$ and $F(x,.)$ are u.s.c. in x and in y, separately for all $(x,y) \in C \times D$. Then F is connected on $C \times D$ of type $\mathcal{D}$. Hence (3.1) holds.*

PROOF: Since D is compact and $F(x,.)$ is u.s.c. in y, $f(x)$ is a nonempty closed subset of D, $f(x)$ is nonempty compact for all $x \in C$.

Suppose that $p, q \in C$ such that $f(p) \cap f(q) = \emptyset$. Since F is t_0-connected, there exists a continuous mapping $u : [0,1] \to C$ such that (1.3) holds for $0 \leq t^1 \leq t \leq t^2 \leq 1$, i.e. $f(u(t)) \subset f(u(t^1)) \cup f(u(t^2))$ and $p \in u(0)$, $q \in u(1)$, and $fu(a)$ is connected for all $a \in [0,1]$. Thus, either $a \in A_1$ or $a \in A_2$, where

$$A_1 = \{a \mid 0 \leq a < 1,\ fu(a) \cap f(q) = \emptyset\},$$
$$A_2 = \{a \mid 0 < a \leq 1,\ fu(a) \cap f(p) = \emptyset\}.$$

Fix $a \in A_1$, $y \in f(q)$. We have $F(u(a),y) < t_0$. Since $F(.,y)$ is u.s.c. in x at t_0, the set $\{x \in C : F(x,y) < t_0\}$ is a relative open subset of C, which contains a neighborhood of $u(a)$ in C. Using the continuity of u, we have a neighborhood $U(y,a) \subset [0,1]$ of a (i.e. there are two points $a', a'' \in [0,1]$, $a' = a$ if $a = p$) such that $a \in U(y,a)$, $F(u(t),y) < t_0$ for all t with $a' \leq t < a''$. Since F is u.s.c. in y at t_0, there are two neighborhoods $V(a',y)$, $V(a'',y)$ of y in D such that $F(u(a'),v) < t_0$ for all $v \in V(a',y)$ and $F(u(a''),v) < t_0$ for all $v \in V(a'',y)$.

Further, since F is t_0-connected, using (1.3) we can see that $F(u(t), v) < t_0$ for all (t, v) such that $t \in U(y, a)$, $v \in V(y)$, where $V(y) = V(a', y) \cap V(a', y)$.

Since the set $f(q)$ is compact and $\{V(y) \mid y \in f(q)\}$ is an open covering of $f(q)$, there exists a finite covering $\{V(y_j) \mid j = 1, ..., m,\ y_j \in f(q)\}$ of $f(q)$.

Let $U(a) = \cap\{U(y_j, a) \mid j = 1, ..., m\}$. Then $F(u(t), v) < t_0$ for all $v \in f(q)$, $t \in U(a)$, i.e. $f(u(t)) \cap f(q) = \emptyset$ for all $t \in U(a)$. Hence the condition (iii) is verified for the point q. Similarly, this condition is also satisfied for the point p. So F is a function connected on $C \times D$ of type $\mathcal{D}$.

Finally we are concerned with existence theorems for variational inequalities.

Let $G : C \times D \to R^1$ be a real-valued function. We associate with G the multifunction $g : C \to 2^D$ defined by

$$g(x) = \{y \in D \mid G(x, y) \geq 0\}.$$

With a similar argument we can prove the following

THEOREM 6. *If G is u.s.c. in (x,y) at 0 and if there exists a nonempty compact subset D_1 of D such that its associated multifunction g is connected relative to D_1 and* $\inf_{x \in C} \sup_{y \in D_1} G(x, y) \geq 0$, *then there exists $y_0 \in D_1$ such that $G(x, y_0) \geq 0$ for all $x \in C$.*

COROLLARY 2. *If D is compact and G is u.s.c. in (x,y) at 0 and if moreover its associated multifunction g is connected relative to D and if* $\inf_{x \in C} \sup_{y \in D} G(x, y) \geq 0$, *then there exists $y_0 \in D$ such that $G(x, y_0) \geq 0$ for all $x \in C$.*

A modification of Theorem 6 is the following

THEOREM 7. *If G(x,.) is u.s.c. in y at 0 for all $x \in C$ and if there exists a nonempty compact subset D_1 of D such that*

(i) $\inf_{x \in C} \sup_{y \in D_1} G(x, y) \geq 0$,

(ii) G is connected in $C \times D_1$ of type $\mathcal{D}$.

Then there exists $y_0 \in D$ such that $G(x, y_0) \geq 0$ for all $x \in C$.

Using the generalized Knaster-Kuratowski-Mazurkiewicz principle (see [3], [6]), from the definition of the class $\mathcal{C}$ and Lemma 1 we can also see that Theorem 1 of [1] which deals with variational inequalities in Hausdorff topological vector spaces is a direct consequence of Corollary 2 or of Theorem 7.

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