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NOTE ON FILTERING FROM POINT PROCESS

TRAN HUNG THAO

1. Introduction

The aim of the note is to consider the problem of filtering and unnormalized filtering for a real semimartingale from point process observations, particularly for the case of Fellerian signal.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which all relevant processes are defined and adapted to a filtration $(\mathcal{F}_t)$. "Usual conditions" are supposed to be satisfied by $(\mathcal{F}_t)$.

The system process will be a semimartingale

$$X_t = X_0 + \int_0^t H_s ds + Z_t, \tag{1.1}$$

where Z_t is a $\mathcal{F}_t$-martingale, H_t is a bounded $\mathcal{F}_t$-progressive process and $E[\sup_{s\leq t}|X_s|] < \infty$.

The observation is given by a point process $\mathcal{F}_t$ - semimartingale of the form

$$Y_t = Y_0 + \int_0^t h_s ds + M_t, \tag{1.2}$$

where M_t is a $\mathcal{F}_t$-martingale with mean 0, $M_0 = 0$ and $h_t = h(X_t)$ is a positive bounded $\mathcal{F}_t$-progressive processes.

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Denote by $\mathcal{F}_t^Y$ the natural filtration of Y which provide observation datas concerning X_t.

Suppose that the processes $u_s = \frac{d}{ds} < Z, M >_s$ is $\mathcal{F}_s$-predictable $(s \leq t)$, where $<,>$ stands for the quadratic variation of Z_t and M_t. Denote also by $\hat{u}_s$ the $\mathcal{F}_t^Y$-predictable projection of u_s.

We shall find an equation for the filtering process :

$$\pi(X_t) = E(X_t / \mathcal{F}_t^Y) . \tag{1.3}$$

and we shall also consider the unnormalized filtering in the general case and in the case of Markov-Feller observation process.

Let $\pi(h_t)$ be the filtering process corresponding to the process in (1.2). The following facts are well known :

a) The process

$$m_t = Y_t - Y_0 - \int_0^t \pi(h_s) ds \tag{1.4}$$

is a $\mathcal{F}_t^Y$-martingale and therefore Y_t is also a $\mathcal{F}_t^Y$ semi-martingale. Note that m_t can be then expressed by

$$m_t = M_t - \int_0^t [h_s - \pi(h_s)] ds . \tag{1.5}$$

b) $\sigma(m_s \, ; s \leq t) \subset \mathcal{F}_t^Y$ and the process m_t is called the innovation of the point process Y_t.

c) If m_t is the innovation of Y_t and R_t is an $\mathcal{F}_t^Y$-martingale then

$$R_t = R_0 + \int_0^t K_s dm_s \tag{1.6}$$

where K_t is a bounded $\mathcal{F}_t^Y$-predictable process such that

$$\int_0^t |K_s| \pi(h_s) ds < \infty \qquad \text{a.s.}$$

(See, for example, [1]).

It follows from (a) and (c) that the observations Y_t can be expressed as

$$Y_t = Y_0 + \int_0^t U_s dm_s \qquad (\mathcal{F}_t^Y - \text{semimartingale}) \tag{1.7}$$

with some U_t of the same properties as K_t.

2. Filtering equation.

THEOREM 1. . *Under the assumptions and notations mentioned in the previous section, the optimal state estimation $\pi(X_t)$ is given by*

$$\begin{aligned} \pi(X_t) = \pi(X_0) &+ \int_0^t \pi(H_s)ds \\ &+ \int_0^t \pi^{-1}(h_s)[\pi(X_{s-}h_s) - \pi(X_{s-})\pi(h_s) + \hat{u}_s]dm_s. \end{aligned} \tag{2.1}$$

The equation is up to an indistinguishability.

PROOF: . It is easy to see that

$$\pi(X_t) = \pi(X_0) + \int_0^t \pi(H_s)ds + \pi(Z_t), \tag{2.2}$$

where $\pi(Z_t)$ is a 0-mean $\mathcal{F}_t^Y$-martingale which can be represented in the form (1.6) with $E[R_t] = E[R_0] = 0$:

$$R_t \equiv \pi(Z_t) = R_0 + \int_0^t K_s dm_s \tag{2.3}$$

So, it is enough to show that the suitable process K_t for $\pi(Z_t)$ is determined by

$$K_t = \pi^{-1}(h_t)[\pi(Z_{t-}h_t) - \pi(Z_{t-})\pi(h_t) + \hat{u}_t]. \tag{2.4}$$

Note that

$$E(X_tY_t) = E[\pi(X_t).Y_t] \qquad (*).$$

We are going to calculate the two products X_tY_t and $V_t.Y_t$, where $V_t = \pi(X_t)$. The differential rule for products gives :

$$X_tY_t = X_0Y_0 + \int_0^t X_{s-}dY_s + \int_0^t Y_{s-}dXs + [X,Y]_t,$$

where the third term denotes the quadratic covariation of the two $\mathcal{F}_t$ - semimartingales X_t and Y_t. We have

(2.6)

$$\int_0^t X_{s-}dY_s = \int_0^t X_{s-}U_sdm_s = \int_0^t X_{s-}U_sdM_s + \int_0^t X_{s-}U_s[h_s - \pi(h_s)]ds,$$

(2.7)

$$\int_0^t Y_{s-}dX_s = \int_0^t X_{s-}H_sd + \int_0^t Y_{s-}dZ_s.$$

Calculations on semimartingale brackets yield :

$$[X;Y]_t = [Z,Y]_t = [Z, \int_0^t U_sdm_s]_t = \int_0^t U_sd[Z,m]_s$$

$$(2.8) \qquad = \int_0^t U_sd < Z, M >_s = \int_0^t U_su_sds.$$

Because the first term of the right hand side of (2.6) and the last one of (2.7) are 0-mean martingales and

$$E[\int_0^t U_su_sds] = E[\int_0^t U_su_sds],$$

we obtain at last :

$$(2.9) \qquad \begin{aligned} E(X_tY_t) &= E(X_0Y_0) + E(\int_0^t Y_{s-}H_sds) \\ &\quad + E[\int_0^t U_s(X_{s-}h_s - X_{s-}\pi(h_s) + u_s)ds]. \end{aligned}$$

An analogous calculation for the product of two $\mathcal{F}_t^Y$-semimartingales $V_t = \pi(X_t) = V_0 + \int_0^t \pi(H_s)ds + \int_0^t K_s dm_s$ and $Y_s = Y_0 + \int_0^t U_s dm_s$ gives

$$(2.10)\qquad \begin{aligned} V_tY_t = V_0Y_0 + \int_0^t V_{s-}U_s dm_s \\ + \int_0^t Y_{s-}\pi(H_s)ds + \int_0^t Y_{s-}K_s dm_s + [V,Y]_t, \end{aligned}$$

where

$$(2.11)\qquad \begin{aligned} [V,Y]_t &= [\int_0^t K_s dm_s, \int_0^t U_s dm_s]_t \\ &= \int_0^t U_s K_s d < m, m >_s = \int_0^t U_s K_s h_s ds. \end{aligned}$$

The expectations of the second and the fourth terms of (2.10) are equal to 0 and $E[V_0Y_0] = E[X_0Y_0]$. We have now

$$(2.12)\qquad \begin{aligned} E(V_tY_t) &= E(X_0V_0) + E(\int_0^t Y_{s-}\pi(H_s)ds) + E[\int_0^t U_s K_s h_s ds] \\ &= E(X_0Y_0) + E(\int_0^t Y_{s-}H_s ds) + E[\int_0^t U_s K_s \pi(h_s)ds]. \end{aligned}$$

It follows from (*), (2.9) and (2.12) that

$$\pi[X_{s-}h_s - X_{s-}\pi(h_s)] + \hat{u}_s = \pi(h_s).K_s \qquad \text{a.s. for all} \qquad s \geq 0,$$

hence the relation (2.4) and the assertion of Theorem.

3. Filtering of a Markov process from point process observation

In this section, the system process will be a Fellerian process X_t and observations will be provided by a point process Y_t of intensity h_t

$$Y_t = \int_0^t h_s ds + M_t,$$

where M_t is a $\mathcal{F}_t$-martingale and independent of X_t.

Suppose now, that the state space S is a subspace of R, and denote by $C(S)$ the space of all real-valued bounded continuous functions over S.

The filtering of X_t is defined now by the conditional distributions

$$\pi(f(X_t)) = E[f(X_t) \mid \mathcal{F}_t^Y], \quad f \in C(S). \tag{3.1}$$

A modification of a theorem of Kunita [2,3], for the case of point process observation will be made :

THEOREM 2. *If A is the infinitesimal generator of the semigroup P_t of the signal process, then $\pi(f)$ satisfies the following equations :*

a/

$$\begin{aligned} \pi(f(X_t)) = \pi(f(X_0)) + \int_0^t \pi(Af(X_s))ds+ \\ + \int_0^t \pi^{-1}(h_s)[\pi(f(X_{s-})h(X_s)) - \pi(f(X_{s-}))\pi(h(X_s))]dm_s, \end{aligned} \tag{3.2}$$

b/

(3.3)

$$\begin{aligned} \pi(f(X_t)) = \pi_0(P_t f)+ \\ + \int_0^t \pi^{-1}(h_s)[\pi(h(X_s).P_{t-s}f(X_{s-})) - \pi(P_{t-s}f(X_{s-}))\pi(h(X_s))]dm_s, \end{aligned}$$

where f belongs to the domain $\mathcal{D}(A)$ of the generator A and m_t is the innovation process of X_t by the point process observation Y_t.

PROOF: a) Recall the process $C_t^f \stackrel{\text{def}}{=} f(X_t) - f(X_0) - \int_0^t Af(X_s)ds$ is a $\mathcal{F}_t$-martingale. Then a direct application of the formula (2.1) for the semimartingale

$$f(X_t) = f(X_0) - \int_0^t Af(X_s)ds + C_t^f$$

yields (3.2) in noticing that the corresponding process u is 0, hence $\hat{u} = 0$ because of the independence of C_t^f and M_t.

b) It is also known that if $f \in C(S)$ and $t > 0$ the process

$$Q_t \stackrel{\text{def}}{\equiv} \begin{cases} f(X_t) & \text{if } s \geq t, \\ P_{t-s}f(X_s) & \text{if } s \leq t, \end{cases}$$

is an $\mathcal{F}_t$-martingale of the Fellerian process X_s [4].

Writing the equation (2.1) for the signal Q_t at a fixed instant t and using an argument on a monotone class, we get (3.3).

4. Zakai equation for unnormalized filtering

4.1. *General case.* Assumptions are the same as in Sections 1 and 2. Suppose now that the probability P is obtained from a probability Q by an absolutely continuous change of measure $Q \to P$ such that

$$\mu_t = Y_t - t$$

is a $(Q, \mathcal{F}_t^Y)$-martingale.

Let us denote $E[\frac{dP}{dQ} \mid \mathcal{F}_t^Y] = L_t$

A Bayes formula give us

$$E_P[X_t \mid \mathcal{F}_t^Y] = \frac{E_Q[X_t L_t \mid \mathcal{F}_t^Y]}{E_Q[L_t]}$$

Denote by $\sigma(X_t)$ the unnormalized filtering of X_t under Q :

$$\sigma(X_t) \stackrel{\text{def}}{=} E_Q[X_t L_t \mid \mathcal{F}_t^Y]$$

Then we have $\pi(X_t) = \frac{\sigma(X_t)}{\sigma(1)_t}$

We can get from (2.1) by some transformation :

$$\sigma(X_t) = \sigma(X_0) + \int_0^t \sigma_s(H_s)ds + \int_0^t [\sigma(X_{s-}h_s) - \sigma(X_{s-})]d\mu_s \,, \tag{4.17}$$

where $\mu_t = Y_t - t$.

4.2. *Fellerian signal.* Assumptions are the same as in Section1, where $\sigma(f(X_t))$ is the unnormalized filtering, $f \in C(S)$.

Then σ satisfies two following equations :

$$\sigma(f(X_t)) = \sigma(f(X_0)) + \int_0^t \sigma(Af(X_s))ds + \\ + \int_0^t [\sigma(h_s f(X_{s-}) - \sigma(f(X_{s-})]d\mu_s, \tag{4.2}$$

$$\sigma(f(X_t)) = \sigma(P_t f(X_0)) + \\ + \int_0^t [\sigma(h_s P_{t-s} f(X_{s-})) - \sigma(P_{t-s} f(X_{s-})]d\mu_s, \tag{4.3}$$

If X_t is of continuous sample paths, $X_{s-} = X_s$ then the two above equations can be briefly rewritten as follows

$$\sigma_t(f) = \sigma_0(f) + \int_0^t \sigma_s(Af)ds + \\ + \int_0^t [\sigma_s(hf) - \sigma_s(f)]d\mu_s, \tag{4.4}$$

$$\sigma_t(f) = \sigma_0(P_t f) + \\ + \int_0^t [\sigma_s(hP_{t-s}f) - \sigma_s(P_{t-s}f)].d\mu_s. \tag{4.5}$$

4.3. *A stochastic differential equation*

Suppose that X_t is a homogeneous and continuous Feller Markov process taking values in a compact separable Hausdorff space S. The semigroup P_t, $t \geq 0$ associated with the transition probabilities $P_t(x, E)$ is a Feller semigroup. Denote by $\mathcal{M}(S)$ the set of all probability measures over S. Then $\mathcal{M}(S)$ is also a compact Hausdorff space with the induced topology. Assume that the observation Y_t, $t \geq 0$ is a real valued point process of P-intensity

$h_t = h(X_t) \in C(S)$ and of Q-intensity 1. Denote again $\mu = Y_t - t$ which is an $(\mathcal{F}_t^Y, Q)$-martingale. Let σ_0 be an $M(S)$-valued random variable independent of (μ_t).

An $M(S)$-valued stochastic process σ_t is called a solution of the following stochastic differential equation

$$\sigma_t(f) = \sigma_0(P_t f) + \int_0^t [\sigma_s(hP_{t-s}f) - \sigma_s(P_{t-s}f)]d\mu_s, \tag{4.6}$$

where $\sigma_t(f) = \int f(X_t)d\sigma_t$ for $f \in C(S)$ and $\sigma_t \in \mathcal{M}(S)$, if σ_s is independent of σ-field $\sigma(\mu_v - \mu_u;\ s \leq u \leq v)$ for all $s \geq 0$ and satisfying this equation. One can prove that (refer to [5], where some corrections must be made) ;

THEOREM 3. *There exists a unique solution σ_t of (4.6) for arbitrary initial condition σ_0. Furthermore, this solution is measurable with respect to $\sigma(\mu_s - \mu_0;\ 0 \leq s \leq t) \vee \sigma(\sigma_0)$ where $\sigma(\sigma_0)$ is the σ-field generated by the $\mathcal{M}(S)$-valued random variable σ_0.*

REMARKS: (i) We can prove the existence in noticing that in this context, the unnormalized filtering is a solution of (4.6). The uniqueness can be proved by the method of Picard approximation.

(ii) We can verify that the solution of (4.6) is a Markov process.

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INSTITUTE OF MATHEMATICS, P.O. BOX 631 BO HO, HANOI, VIETNAM