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LOCAL BIFURCATION FROM CHARACTERISTIC VALUES WITH FINITE MULTIPLICITY AND APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS*

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1. INTRODUCTION

Throughout this paper by X^*, Y^* we denote the duals of Banach spaces X and Y, respectively. We use the same symbols $\|.\|$ and $\langle , \rangle$ to denote the norms and the pairings between elements of X, X^* and Y, Y^*, respectively. We consider equations of the form

$$F(\lambda, v) = 0, \qquad (\lambda, v) \in \Lambda \times \overline{D}, \tag{1}$$

where $\overline{D}$ is the closure of a neighbourhood D of the origin in X, and $F : \Lambda \times \overline{D} \to Y$ is a mapping with $F(\lambda, 0) = 0$ for all $\lambda \in \Lambda$. A point $(\lambda, 0)$ is called a trivial solution of (1). A point $(\overline{\lambda}, 0)$ si called a bifurcation point of (1) if for any real number δ, $\varepsilon > 0$ there exists a solution $(\lambda, v) \in \Lambda \times \overline{D}$ with $|\lambda - \overline{\lambda}|_{\Lambda} < \delta$ and $0 < \|v\| < \varepsilon$, where $|.|_{\Lambda}$ denotes the restricted norm to Λ of a normed space containning Λ. The purpose of this paper is to study the existence of bifurcation points of equations of the form (1) with

$F(\lambda, v) = T(v) - L(\lambda, v) - M(\lambda, v)$, which can be rewritten as

$$T(v) = L(\lambda, v) + M(\lambda, v), \ (\lambda, v) \in \Lambda \times \overline{D}, \tag{2}$$

where T is a linear continuous mapping from X into Y, L is a continuous mapping from $\Lambda \times X$ into Y such that for any fixed $\lambda \in \Lambda$, $L(\lambda, .)$ is a linear mapping, and M is a nonlinear mapping from $\Lambda \times \overline{D}$ into Y, $M(\lambda, 0) = 0$ for all $\lambda \in \Lambda$. In what follows, R^n stands for the real n- dimensional Euclidean space.

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It is customary to simplify the notation for R^1 by dropping the superscript, $R^1 = R$. We use also the same symbol $|\,.\,|$ to denote the norms in R^n for $n = 1, 2, \ldots$. Next, let $\bar{\lambda}$ be a characteristic value of the pair (T, L), (i. e., $T(v) = L(\bar{\lambda}, v)$ for some $v \in X, v \neq 0$) such that $T - L(\bar{\lambda}, .)$ is a Fredholm mapping with nullity p and index s, $p > s \geqslant 0$, and $\| M(\bar{\lambda}, v) \| = O(\| v \|)$ as $\| v \| \to 0$. Further, let $\{v^1, \ldots, v^p\}$ and $\{\psi^1, \ldots, \psi^q\}$, $q = p - s$, be bases of the null spaces $\operatorname{Ker}(T - L(\bar{\lambda}, .))$ and $\operatorname{Ker}(T - L(\bar{\lambda}, .))^*$, respectively, with $(T - L(\bar{\lambda}, .))^*$ denoting the adjoint mapping of $T - L(\bar{\lambda}, .)$. Using the Hahn-Banach Theorem, we can find p functionals $\gamma^1, \ldots, \gamma^p$ on X and q elements $z^1, \ldots, z^q$ in Y such that $\langle v^i, \gamma^j \rangle = \delta_{ij}$, $i, j = 1, \ldots, p$, and $\langle z^m, \psi^n \rangle = \delta_{mn}$, $m, n = 1, \ldots, q$. Here δ_{ij}, δ_{mn} denote the Kronecker delta. Setting

$$X_o = [v^1, \ldots, v^p], \; Y_o = [z^1, \ldots, z^q],$$

$$X_1 = \{x \in X \,/\, \langle x, \gamma^i \rangle = 0, \; i = 1, \ldots, p\}$$

$$Y_1 = \{Y \in y \,/\, \langle y, \psi^j \rangle = 0; \; j = 1, \ldots, q\},$$

one can easily verify that $X = X_o \oplus X_1$ and $Y = Y_o \oplus Y_1$, where $[w^1, \ldots, w^m]$ denotes the space spanned by $\{w^1, \ldots, w^m\}$.

Further, we assume $M = H + K$ with $H, K \in C^1(\Lambda \times D, Y)$ and make the following hypothesis on H:

HYPOTHESIS 1. There is a natural number $a \geqslant 2$ such that

$$P_Y H(\lambda, tv) = t^a P_Y H(\lambda, v) \text{ holds for all } t \in R, (\lambda, v) \in \Lambda \times \bar{D},$$

where P_Y is the projector from Y into Y_o.

Next, we define the mapping $\mathcal{A} : R^p \to R^q$, $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_q)$ by

$$\mathcal{A}_1(x) = \langle T(\sum_{j=1}^{p} x_j v^j) - H(\bar{\lambda}, \sum_{j=1}^{p} x_j v^j), \psi^i \rangle, \; i = 1, \ldots, q, \quad x = (x_1, \ldots, x_p) \in R^p. \tag{3}$$

In the case when L is differentiable, we define the mapping $\mathcal{B}: \Lambda \times R^p \to R^q$, $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_q)$, by

$$\mathcal{B}_1(\beta, x) = \langle D_\lambda L(\bar{\lambda}, \sum_{j=1}^{p} x_j v^j)(\beta) - H(\bar{\lambda} \sum_{j=1}^{p} x_j v^j), \psi^i \rangle, \; i = 1, \ldots, q, \quad \lambda \in \Lambda, x = (x_1, \ldots, x_p) \in R^p, \tag{4}$$

here D_λ denotes the partial Fréchet derivative with respect to $\lambda \in \Lambda$. Under suitable conditions on L and K, we have proved that $(\overline{\lambda}, 0)$ is a bifurcation point of Equation (2) provided that either

i/

$$\mathcal{A}(\overline{x}) = 0 \tag{5}$$

and

$$\text{Rank}\left(\frac{\partial \mathcal{A}_i}{\partial x_k}(\overline{x})\right)_{i,\, k=1,\dots,q} = q, \tag{6}$$

for some $\overline{x} \in R^p$, $\overline{x} \neq 0$, or ii/

$$\mathcal{B}(\overline{\beta}, \overline{x}) = 0 \tag{7}$$

and

$$\text{Rank}\left(\frac{\partial \mathcal{B}_i}{\partial x_k}(\overline{\beta}, \overline{x})\right)_{i,\, k=1,\dots,q} = q, \tag{8}$$

for some $(\overline{\beta}, \overline{x}) \in \Lambda \times R^p$, $\overline{x} \neq 0$. Moreover, we can describe the parameter families of nontrivial solutions in a neighbourhood of $(\overline{\lambda}, 0)$ in a an alytical form (see [5, Theorem 6 and Theorem 17, respectively]).

In Section 2 we shall apply these results to consider bifurcation points of Equation (2) assuming that $M = H + K$ and the mapping $A: R^q \to R^q$, $A = (A_1, \dots, A_q)$, defined by

$$A_i(x) = \langle H(\overline{\lambda}, \sum_{j=1}^{q} x_j v^j), \psi^i \rangle, \; i = 1, \dots, q,$$

$$x = (x_1, \dots, x_q) \in R^q,$$

is a potential operator with potential h. Let z^1 be a local relative extremum of h on the unit sphere S^q in R^q. Then under additional conditions on T, L, H and K we shall prove that $(\overline{\lambda}, 0)$ is a bifurcation point of Equation (2). Moreover, we can describe the parameter families of nontrivial solutions in a neighbourhood of $(\overline{\lambda}, 0)$ in an analytical form through z^1, $h(z^1)$ and $v^1, \dots, v^q$ (see Remark 14 below).

In Section 3 we shall apply the results obtained in Section 2 to consider bifurcation points of the problem of small amplitude free vibrations of a thin rectangular plate and the boundary problem of nonlinear elliptic differential equations (see in [3]). For the first problem we s' all show that if $\overline{\lambda}$ is a characteristic value of the linearized problem with multiplicity p, then there exist at least 2p distinct parameter famil es of nontrivial solutions in a neighbourhood of $(\overline{\lambda}, 0)$. For the second problem under additional conditions we shall show that there exist at least two distinct parameter families of nontrivial solutions. In addition, we can aloso describe these families in an analytical form.

2. THE MAIN RESULTS

We begin this section with making the following hypotheses on the mappings T, L, H and K.

HYPOTHESIS 2. $\langle T(v^i), \psi^i \rangle = 1$, $i = 1,\dots, q$ and $\langle T(v^1), \psi^j \rangle = 0$, $j = 2,\dots, q$.

HYPOTHESIS 3. There is a real number b such that $\alpha L(\bar{\lambda}, v) = L(\alpha^b \bar{\lambda}, v)$ holds for all $\alpha \in [0,1]$, $v \in \bar{D}$.

HYPOTHESIS 4. $\langle H(\bar{\lambda}, v^1), \psi^1 \rangle \neq 0$ and $\langle H(\bar{\lambda}, v^1), \psi^j \rangle = 0$, $j = 2,\dots, q$.

HYPOTHESIS 5.

$$\Gamma = \det \left(\langle H(\bar{\lambda}, v^1), \psi^1 \rangle \delta_{ki} - \langle D_v H(\bar{\lambda}, v^1) v^k, \psi^i \rangle \right)_{k,i=2,\dots,q} \neq 0,$$

where D_v denotes the partial Fréchet derivative with respect to $v \in D$.

HYPOTHESIS 6. $\alpha^{-a} P_Y K(\bar{\lambda}/(1+\alpha^{a-1})^b, \alpha v)$ and $\alpha^{1-a} P_Y D_v K(\bar{\lambda}/(1+\alpha^{a-1})^b\, \alpha v)$ tend to zero as $\alpha \to 0$ uniformly in $v \in \bar{D}$, where a, b are taken from Hypotheses 1, 3, respectively.

THEOREM 7. *Under Hypotheses 1 — 6. $(\bar{\lambda}, 0)$ is a bifurcation point of Equation* (2) *provided that either 1) a is even or, 2) a is odd and $\langle H(\bar{\lambda}, v^1), \psi^1 \rangle > 0$. Furthermore, in Case 2) there exist at least two distinct parameter families of nontrivial solutions in a neighbourhood of $(\bar{\lambda}, 0)$. More precisely, in Case 1/ (in Case 2/, for $\sigma = \pm$) there are neighbourhoods I of zero in R, U (U^σ) of the point $(\bar{x}_1, 0,\dots, 0)$ ($(\bar{x}_1^\sigma, 0, \dots, 0)$) in R^q with $\bar{x}_1 = 1/(\langle H(\bar{\lambda}, v^1) \psi^1 \rangle 1)^{1/(a-1)}$ and ($\bar{x}_1^\sigma = \sigma 1/(\langle H(\bar{\lambda}, v^1), \psi^1 \rangle)^{1/(a-1)}$), V of the origin in R^{p-q}, and q ($2q$, respectively) continuous funstions $x_i : I \times V \to R$ with $x_1(0,\dots,0) = \bar{x}_1$, $x_j(0,\dots,0) = 0$, $j=2,\dots, q$ $(x_1(y),\dots, x_q(y)) \in U$ ($x_i^\sigma : I \times V \to R$ with $x_i^\sigma(0,\dots, 0) = \bar{x}_1^\sigma$, $x_j^\sigma(0,\dots, 0) = 0$, $j = 2,\dots, q$; $(x_1^\sigma(y),\dots, x_q^\sigma(y)) \in U^\sigma$), $i = 1,\dots, q$, and a continuous mapping w (w^σ) $: I \times V \to X_1$ such that $(\lambda(\alpha), v(y))$ ($(\lambda(\alpha), v^\sigma(y))$) with*

$$\lambda(\alpha) = \bar{\lambda}/(1 + \alpha^{a-1})^b$$

and

$$v(y) = \sum_{j=1}^{q} \alpha x_j(y) v^j + \sum_{k=q+1}^{p} \alpha x_k v^k + w(y),$$

$$(v^\sigma(Y) = \sum_{j=1}^{q} \alpha x_j^\sigma(Y) v^j \sum_{k=q+1}^{q} \alpha x_k v^k + w^\sigma(Y)),$$

$Y = (\alpha, x_{q+1}, \dots, x_p) \in I \times V$, satisfies Equation (2), $\lambda(\alpha) \to \bar{\lambda}$, $v(Y)$ $(v^\sigma(Y)) \to 0$ as $\alpha \to 0$ for any $Y = (\alpha, x_{q+1}, \dots, x_p) \in I \times V$; $v(Y)(v^\sigma(Y)) \neq 0$ for $Y = (\alpha, x_{q+1}, \dots, x_p) \in I \times V$ with $\alpha \neq 0$, the family $(\lambda(\alpha), v(Y))$ is called a parameter family of nontrivial solutions of Equation (2) in a neighbourhood of $(\bar{\lambda}, 0)$).

Proof. Since $\bar{\lambda}$, T, L, H and K satisfy Hypotheses 1 — 4,6 , it then follows that they fulfil Hypotheses 1 and 2 of Theorem 6 in [5]. Further, we take $x = (\bar{x}_1, 0, \dots, 0)$ in the case where a is even and $\bar{x}^\sigma = (\bar{x}_1^\sigma, \dots, 0)$ in the case where a is odd and $\langle H(\bar{\lambda}, v^1), \psi^1\rangle > 0$ with $\bar{x}_1$ and $\bar{x}_1^\sigma$ as above. By a simple calculation we can easily verify that $\mathcal{A}(\bar{x}) = \mathcal{A}(\bar{x}^\sigma) = 0$, $\bar{x}$, $\bar{x}^\sigma \neq 0$ and

$$\det\left(\frac{\partial \mathcal{A}_i}{\partial x_j}(\bar{x})\right)_{i,j=1,\dots,q} = \det\left(\frac{\partial \mathcal{A}_i}{\partial x_i}(\bar{x}^\sigma)\right)_{i,\ j=1,\dots,q}$$
$$= (1-a)\left(\langle H(\bar{\lambda}, v^1), \psi^1\rangle\right)^{1-q}\Gamma \neq 0,$$

where Γ is from Hypothesis 5. Thus, $\bar{x}$ and $\bar{x}^\sigma$ satisfy Conditions (3) and (5) of Theorem 6 in [5]. Therefore, to complete the proof of the theorem, it remains to apply this theorem to $\bar{x}$ ($\bar{x}^\sigma$, respectively).

Remark 8. If in Theorem 7 Hypotheses 2, 4 and 5 are replaced by $\langle T(v^i), \psi^i\rangle = 1$, $i = 1, \dots, q$, $\langle T(v^k), \psi^j\rangle = 0$, $k = 1, \dots, l$, $l \leq q$, $j = 1, \dots, q$, $k \neq j$, and $\langle H(\bar{\lambda}, v^k), \psi^k\rangle \neq 0$, $\langle H(\bar{\lambda}, v^k), v^k), \psi^j\rangle = 0$, $j = 1, \dots, q$, $k = 1, \dots, l$, $l \leq q$, $k \neq j$, and

$$\det\left(\langle H(\bar{\lambda}, v^k), \psi^k\rangle \delta_{ij} - \langle H_v(\bar{\lambda}, v^k) v^i, \psi^j\rangle\right)_{i,j=1,\dots,q,\ i,j\neq k} \neq 0,$$

$k = 1, \dots, e$, then there exist at least l $(2l)$ distinct parameter families of nontrivial solutions in a neighbourhood of $(\bar{\lambda}, 0)$.

Moreover, we can also describe these families in an analytical form as in Theorem 7, remarking that the points $(\bar{x}_1, 0, \dots, 0)$, $\bar{x}_1^\sigma, \dots, 0)$ are replaced by $\bar{x}^k = (0, \dots, 0, \bar{x}_k, 0, \dots, 0)$ and $\bar{x}^{k\sigma} = (0, \dots, \bar{x}_k^\sigma, 0, \dots, 0)$, respectively, where

$$\bar{x}_k = 1/(\langle H(\bar{\lambda}, v^k), \psi^k\rangle)^{1/(a-1)}$$

and

$\bar{x}_k^{\sigma} = \sigma 1 / (\langle H(\bar{\lambda}, v^k), \psi^k \rangle)^{1/(a-1)}$, $k = 1, \ldots, l$, (see the prrof in [5, Remark 7]).

Next, we prove some results on the existence of bifurcation points of Equation (2) involving potential operators. Let us first recall that an operator $A: X \to X^*$ is said to be a potential operator iff there exists a Gateaux differentiable functional $h: X \to R$ such that $h'(u) = A(u)$, i. e.,

$$\lim_{t \to o} t^{-1} (h(u + tv) - h(u)) = \langle v, A(u) \rangle$$

for all $u, v \in X$. The functional h is said to be a potential of A and is uniquely determined by the requirement $h(0) = 0$. If A is a continuous potential operator with potential h, then we have

$$h(u) = \int_0^1 \langle u, A(tu) \rangle \, dt,$$

for all $u \in X$, and h is continuous. Therefore, if in addition, A is an a-homogeneous operator, $a \geq 0$, then it follows

$$h(u) = \frac{1}{a+1} \langle u, A(u) \rangle, \; u \in X.$$

Further, let Ω be a subset of R^q and $f: R^q \to R$ a function. A point $\bar{y} \in \Omega$ is said to be a local relative maximum (minimum) of f on Ω if there is a neighbourhood U of $\bar{y}$ in R^q such that $f(x) \leq f(\bar{y})$ ($f(x) \geq f(\bar{y})$, respectively) for all $x \in U \cap \Omega$.

Now, let A be an a-homogeneous, $a \geq 2$, continuous potential operator with potential h. In what follows, by $[\,.\,]$ we denote the scalar product in R^q and by S^q the unit sphere in R^q. Further, we put

$$E = \{z \in S^q \,/\, z \text{ is a local relative extremum of } h \text{ on } S^q, \; h(z) \neq 0\} \text{ and}$$

$$E_+ = \{z \in E \,/\, h(z) > 0\}.$$

By $|E|$, $|E_+|$, we denote the number of elements in E, E_+, respectively. One can easily verify that if a is even, then h is odd function and then $|E|$ is even; $|E| = 2\,|E_+|$.

LEMMA 9. *If A is an a-homogeneous, $a \geqslant 2$, continuous potential operator from R^q into itself with potential h and $h(\bar{x}) \neq 0$ for some $\bar{x} \in R^q$, then $E \neq \emptyset$.*

Proof. It follows from the continuity of h and the compactness of S^q that there exist two points $z^1, z^2 \in S^q$ with $h(z^1) = \max_{x \in S^q} h(x)$ and $h(z^2) = \min_{x \in S^q} h(x)$.

Consequently, we obtain

$$h(z^1) \geq h\left(\frac{\bar{x}}{\|\bar{x}\|}\right) = \int_0^1 \left[\frac{\bar{x}}{\|\bar{x}\|},\ A\left(t\frac{\bar{x}}{\|\bar{x}\|}\right)\right] dt = \frac{1}{(a+1)\,|\bar{x}|^{a+1}} h(\bar{x}) > 0,$$

if $h(\bar{x}) > 0$, and

$$h(z^2) \leq h\left(\frac{\bar{x}}{|\bar{x}|}\right) = \int_0^1 \left[\frac{\bar{x}}{|\bar{x}|},\ A\left(t\,\frac{\bar{x}}{|\bar{x}|}\right)\right] dt = \frac{1}{(a+1)\,|\bar{x}|^{a+1}}\ h(\bar{x}) < 0,$$

f $h(\bar{x}) < 0$. Thus, we deduccie ther $z^1 \in E$ or, $z^2 \in E$. This completes the proof of the lemma.

Assume now that A is an a-homogeneous, $a \geq 2$, continuously differentiable potential operator from R^q into itself with potential h and z^1 is a local relative extremum of h on S^q with $h(z^1) \neq 0$, i. e., $z^1 \in E$. Setting

$$\bar{R}^{q-1} = \{x \in R^q \ / \ [x, z^1] = 0\},$$

one can easily verify that $\bar{R}^{q-1}$ is a $(q-1)$ -dimensional subspace of R^q and $R^q = \{z^1\} \oplus \bar{R}^{q-1}$. Further, we put

$$B\ (z^1) = \left(\frac{\partial^2 h}{\partial x_i\, \partial x_j}(z^1)\right)_{i,j=1,\dots,q}$$

and define the function $g : R^q \to R^q$ by

$$g(x) = \frac{1}{2}\,[B(z^1)\,x, x]\ ,\quad x \in R^q\,.$$

Let z^2 be a local relative extremum of g on the unit sphere S^{q-1} of $\bar{R}^{q-1}$. Setting

$$\bar{R}^{q-2} = \{x \in \bar{R}^{q-1} \ / \ [x, z^2] = 0\},$$

we conclude that $\bar{R}^{q-1}$ is a $(q-2)$ - dimensional subspace of $\bar{R}^{q-1}$ and $\bar{R}^{q-1} = \{z^2\} \oplus \bar{R}^{q-2}$. Let z^3 be a local relative extremum of g on the unit sphere S^{q-2} of $\bar{R}^{q-2}$. We put

$$\bar{R}^{q-3} = \{x \in \bar{R}^{q-2} \ / \ [x, z^3] = 0\}\,.$$

It then follows that $\bar{R}^{q-3}$ is a $(q-3)$ - dimensional subspace of $\bar{R}^{q-2}$.
In this manner we define inductively the sequence of the spaces
$R^q \supset \bar{R}^{q-1} \supset \bar{R}^{q-2} \supset \dots \supset \bar{R}^1$ and the points $z^1, \dots, z^q$.

LEMMA 10. *$\{z^1, \ldots, z^q\}$ is an orthonormal basis for R^q with the following properties:*

$$1/\ [z^1, A(z^1)] \neq 0,$$

$$2/\ [z^j, A(z^1)] = 0, \qquad j = 2, \ldots, q,$$

$$3/\ [B(z^1)z^i, z^j] = 0, \ i,j = 1,\ldots,q,\ j \neq i.$$

Furthermore, if z^1 is a local relative maximum of h on S^q and z^j is a local relative maximum of g on S^{q-j+1}, $j = 2, \ldots, q$, then

4/ $\{g(z^j)\}_{j=2}^q$ is a descending sequence with

$$g(z^2) \leq \frac{a+1}{2}\, h(z^1).$$

Analogously, for minima we conclude that

4'/ $\{g(z^j)\}_{j=2}^q$ is an increasing sequence with

$$g(z^2) \geq \frac{a+1}{2}\, h(z^1).$$

Proof. It is obvious that $\{z^1, \ldots, z^q\}$ is an orthonormal basis for R^q. Since $0 \neq h(z^1) = [z^1, A(z^1)]/2$, 1/ follows. Therefore, we need only to prove 2/—4/ and 4'/. The proof proceeds similarly as the ones of Lemma 2 and Lemma 3 in [4]. Let $\sigma(\theta) = -\theta/(1 + (1 - \sigma\theta^2))^{1/2}$, $0 < |\theta| < 1$. One can easily verify that $\sigma(\theta)$ is a root of the equation $\theta\sigma^2 + 2\sigma + \theta = 0$ and $\sigma(\theta)$ tends to zero as θ approaches zero. We put

$$z^{ij} = z^i + \theta(\sigma z^i + z^j),\ i,\ j = 1, \ldots, q,\ i < j.$$

One can easily see that $z^{ij} \in S^{q-i+1}$, $i, j = 1, \ldots, q$, $i < j$. Now, we prove 2/ and 3/. For the sake of simplicity of notations we assume that z^1 is a local relative maximum of h on S^q and z^j, $j = 2, \ldots, q$, are local relative maxima of g on S^{q-j+1}, (the proof for the other case is similar). It then follows that there are neighbourhoods U^1 of z^1 in R^q and U^j of z^j in R^{-q-j+1} such that $h(x) \leq h(z^1)$ for all $x \in U^1 \cap S^q$ and $g(x) \leq g(z^j)$ for all $x \in U^j \cap S^{q-j+1}$, $j = 2, \ldots$, q. Hence,

$$0 \geqq h(z^{1j}) - h(z^1) = \theta[\sigma z^1 + z^j,\ A(z^1)] + r^1(z^1, \theta(\sigma z^1 + z^j)),$$

and

$$0 \geq g(z^{ij}) - g(z^i) = \theta[\sigma z^i + z^j,\ B(z^1)z^i] + r^i(z^i, \theta(\sigma z^i + z^j)),\ i, j = 1, \ldots, q,$$

$i < j$. Since, $\lim_{\theta \to 0_{\pm}} r^k (z^k, \theta(\sigma z^k + z^j))/\theta = 0$ for all $k = 1,..., q$, by considering the case $\theta > 0$ and $\theta < 0$ separately, dividing by θ, and letting $\theta \to 0$ we obtain 2 and 3/ for $i < j$. Since $[B(z^1) z^i, z^j] = [B(z^1) z^j, z^i]$, we conclude that 3/ holds for all $i, j = 1,..., q$; $i \neq j$. To complete the proof of the lemma, it remains to show 4/ and 4'/.

By the mean value Theorem we have.

$$0 \geqq g(z^{ij}) - g(z^i) = \int_0^1 [B(z^1) (z^i + t\theta (\sigma z^i + z^j)), \theta(\sigma z^i + z^j))]\, dt$$

$$= \frac{-\theta^2}{(1+(1-\theta^2))^{1/2}} \int_0^1 [B(z^1) (z^i + t\theta (\sigma z^i + z^j)), z^j]\, dt$$

$$+ \int_0^1 [B(z^1) (t\theta (\sigma z^i + z^j), z^j]\, dt + 0(|\theta^2|),$$

$$i, j = 1,..., q, \ i < j.$$

and

$$0 \geq h(z^{12}) - h(z^1) = \int_0^1 [\theta(\sigma z^1 + z^2), A(z^1 + t\theta(\sigma z^1 + z^2)]\, dt$$

$$= \frac{-\theta^2}{(1+(1-\theta^2))^{1/2}} \int_0^1 [z^1, A(z^1 + t\theta(\sigma z^1 + z^2)]\, dt$$

$$+ \theta \int_0^1 [B(z^1)\, t\theta(\sigma z^1 + z^2), z^2]\, dt + 0(|\theta^2|).$$

Hence, dividing by θ^2 and letting $\theta \to 0$ we obtain

$$[B(z^1) z^j, z^j] \leq [B(z^1) z^i, z^i] \quad \text{for all} \quad i, j = 2,..., q, \ i < j,$$

and

$$[B(z^1) z^2, z^2] \leq [z^1, A(z^1)], \text{ or, } g(z^2) \leq \frac{a+1}{2} h(z^1).$$

Thus, we obtain 4/. The proof of 4'/ is similar. This completes the proof of the lemma.

Let $\overline{\lambda}$, $\{v^1,..., v^p\}$ and $\{\varphi^1,..., \varphi^q\}$ be as above and the mapping H satisfy Hypothesis 1. We define the mapping $A: R^q \to R^q$, $A = (A_1,..., A_q)$, by

$$A_i(x) = \langle H(\overline{\lambda}, \sum_{j=1}^{q} x_j v^j), \varphi^i \rangle, \ i = 1,..., q, \ x = (x_1,..., x_q) \in R^q$$

and assume that A is a potential operator. Then the potential h of A is given by

$$h(x) = \frac{1}{a+1}[x, A(x)] = \frac{1}{a+1} \langle H(\bar{\lambda}, \sum_{j=1}^{q} x_j v^j), \sum_{k=1}^{q} x_k \varphi^k \rangle,$$

$$x = (x_1, \dots, x_q) \in R^q .$$

Let z^1 be a local relative extremum of h on S^q with $h(z^1) \neq 0$. we put

$$B(z)^1) = \left(\frac{\partial^2 h}{\partial x_i \, \partial x_k} (z^1) \right)_{i, k = 1, \dots, q}$$

Further, we assume that z^2 is a local relative extremum of the function $g(x) = [B(z^1)x, x]/2$ on the unit sphere S^{q-1} in the space $\overline{R}^{q-1} = \{x \in R^q / [x, z^1] = 0\}$.

THEOREM 11. *Let $\bar{\lambda}$, L, H and K satisfy Hypotheses 1,3 and 6. Let $Ker(T - L(\bar{\lambda},.)) = [v^1, \dots, v^p]$ and $Ker\,(T\text{-}L(\bar{\lambda},.))^* = [\psi^1, \dots, \psi^q]$ with $\langle T(v^i), \psi^j \rangle = \delta_{ij}$, $i, j = 1, \dots, q$, and z^1, z^2 as above. If z^1, z^2 are local relative maxima (minima) and $q(z^2) < (a+1)\, h(z^1)/2$ $(g)(z^2) > (a+1)\, h(z^1)/2$, respecitvely), then $(\bar{\lambda}, 0)$ is a bifurcation point of Equation (2) provided that either 1/ a is even or, 2/ a is odd and $h(z^1) > 0$. Moreover, the parameter families of nontrivial solutions in a neighbourhood of $(\bar{\lambda}, \theta)$ can be desoribed in an analytical form as in Theorem 7 with $x_j(0, \dots, \theta) = z^1_j/((a+1)h(z^1))^{1/(a-1)}$ and $x^\sigma_j(0, \dots, 0) = \sigma z^1_j/((a+1)\, h(z^1))^{1/(a-1)}$, $j = 1, \dots, q$, respectively.*

Proof. For the sake of simplicity we prove only the case when z^1 is a local relative maximum of h on S^q and z^2 is a local relative maximum of g on the unit sphere S^{q-1} in $\overline{R}^{q-1}$. Let z^j, $j = 3, \dots q$, be local relative maxima of g on S^{q-j+1}, constructed as in Lemma 10. A use of this lemma yields

$$[B(z^1)\, z^i, z^j] = 0, \; i, \; j = 1, \dots, q, \; i \neq j$$

and

$$\prod_{k=2}^{q} ((a+1) h(z^1) - 2g(z^k)) > 0.$$

Putting $\bar{v}^i = \sum_{j=1}^{q} z^i_j v^j$ and $\bar{\psi}^i = \sum_{j=1}^{q} z^i_j \psi^j$, $i = 1, \dots, q$, and $\bar{v} = v^i$, $i = q+1, \dots, p$;

we conclude that Ker $(T - L(\bar{\lambda},.)) = [\bar{v}^1, \dots \bar{v}^p]$ and $Ker(T - L(\bar{\lambda},.))^* = [\bar{\psi}^1, \dots, \bar{\psi}^q]$; $\langle T(\bar{v}^i), \bar{\psi}^j \rangle = \delta_{ij}$, i, $j = 1, \dots, q$. By Lemma 10 we have $\langle H(\bar{\lambda}, \bar{v}^1), \bar{\psi}^1 \rangle > 0$ and $\langle H(\bar{\lambda}, \bar{v}^1), \bar{\psi}^j \rangle = [z^j, A(z^1)] = 0$ for $j = 2, \dots, q$. Further, since

$$\det\left(\langle H(\overline{\lambda}, \overline{v}^1), \overline{\psi}\rangle \overline{\delta}_{ij} - \langle H_v(\overline{\lambda}, \overline{v}^1)\overline{v}^i, \overline{\psi}^j\rangle\right)_{i,\ j=2,\dots,q}$$

$$= \det\left((a+1)h(z^1)\delta_{ij} - [B(z^1)z^i, z^j]\right)_{i,\ j=2,\dots,q}$$

$$= \prod_{k=2}^{q} \left((a+1)h(z^1) - 2g(z^k)\right) > 0,$$

it then follows that $\overline{\lambda}$, T, L, H and K satisfy Hypotheses 1 — 6. Therefore, to complete the proof of the theorem, it remains to apply Theorem 7.

COROLLARY 12. *Let $\overline{\lambda}$ T, L, H, K, $\{v^1,\dots, v^p\}$ and $\{\psi^1,\dots, \psi^q\}$ be as in Theorem 11. Let z^1, z^2 be local relative maxima (minima) of h on S^q with $h(z^1) \neq 0$ and of g on S^{q-1}, respectively. Further, assume that one of the following conditions is satisfied:*

1. There is a local relative maximum (minimum) $\overline{z}^2$ of g on S^{q-1} with $g(\overline{z}^2) \neq g(z^2)$.

2. $(a+1)h(z^1)$ is not an eigenvalue of the restriction of $B(z^1)$ to $\overline{R}^{q-1}$.

Then the conclusions of Theorem 11 continue to hold.

Proof. If 1. holds, then the proof follows immediately from Theorem 11, with the remark that either $(a+1)h(z^1)/2 \neq g(z^2)$ or, $(a+1)h(z^1) \neq g(\overline{z}^2)$. Now, we assume that 2, holds. Let $z^3,\dots, z^q$ be constructed by Lemma 10. For arbitrary $y \in R^q$ we can write $y = \sum_{i=1}^{q} \alpha_i z^i$. Using Lemma 10, we conclude

$$[B(z^1)z^2, y] = \alpha_2 [B(z^1)z^2, z^2] = [B(z^1)z^2, z^2][z^2, y]$$
$$= [[B(z^1)z^2, z^2]z^2, y].$$

It then follows that

$[B(z^1)z^2, y] = [[B(z^1)z^2, z^2]z^2, y]$ for all $y \in \overline{R}^{q-1}$ or, $B(z^1)z^2 = [B(z^1)z^2, z^2]z^2$ in $\overline{R}^{q-1}$. Consequently, $[B(z^1)z^2, z^2]$ is an eigenvalue of the restriction of $B(z^1)$ to $\overline{R}^{q-1}$. Therefore, we deduce that $(a+1)h(z^1) \neq [B(z^1)z^2, z^2]$ or, $(a+1)h(z^1)/2 \neq g(z^2)$. Hence, to complete the proof of the corollary, it remains to apply Theorem 11.

In the following corollary we consider a special case:

Ker $(T - L(\bar{\lambda},.)) = [v^1, v^2]$, Ker $(T - L(\bar{\lambda},.))^* = [\psi^1, \psi^2]$ and there is an a-multilinear mapping $F: X \times X \times ,..., \times X$ (a-times) $\to Y$ such that $P_Y H(\bar{\lambda}, v) = P_Y F(v,..., v)$. For the sake of simplicity of notations we investigate only the case $a = 2$. Assume that the mapping $A: R^2 \to R^2$, $A = (A_1, A_2)$ with

$A_i(x) = \langle H(\bar{\lambda}, x_1 v^1 + x_2 v^2), \psi^i \rangle, i = 1, 2; x = (x_1, x_2) \in R^2$ is a continuosly differentiable potential operator. It then follows that the potential h of A is given by

$$h(x) = \frac{1}{3} \langle H(\bar{\lambda}, x_1 v^1 + x_2 v^2), (x_1 \psi^1 + x_2 \psi^2) \rangle.$$

Setting

$$\alpha_{ijk} = \langle F(v^i, v^j), \psi^k \rangle, \; i, k, j = 1, 2,$$

we can see that

$$A_1(x) = \alpha_{111} x_1^2 + (\alpha_{121} + \alpha_{211}) x_1 x_2 + \alpha_{221} x_2^2,$$

$$A_2(x) = \alpha_{112} x_1^2 + (\alpha_{122} + \alpha_{212}) x_1 x_2 + \alpha_{222} x_2^2,$$

and

$$h(x) = \frac{1}{3} \Big\{ \alpha_{111} x_1^3 + (\alpha_{211} + \alpha_{121} + \alpha_{112}) x_1^2 x_2 + (\alpha_{122} + \alpha_{212} + \alpha_{221}) x_1 x_2^2 + \alpha_{222} x_2^3 \Big\}.$$

Let $z^1 = (z_1, z_2)$ be a local relative extremum of h on S^2. It implies that $\overline{R}^1 = \{ x \in R^2 \,/\, x = \beta(-z_2, z_1), \beta \in R \}$ and $S^1 = \{(-z_2, z_1)\} \cup \{(z_2, -z_1)\}$. Let $B(z^1)$ and g be defined as above. It is a simple matter to show that the function g is constant on S^1 and

$$g(z^2) = \frac{1}{2} \Big\{ (\alpha_{122} + \alpha_{212}) z_1^3 + (2\alpha_{222} - \alpha_{121} - \alpha_{211} - 2\alpha_{112}) z_1^2 z_2 + (2\alpha_{111} - \alpha_{212} - \alpha_{122} - \alpha_{221}) z_1 z_2^2 + (\alpha_{211} + \alpha_{121}) z_2^3 \Big\}.$$

Lastly, we put

$$D_{ij} = \alpha_{iii} - \alpha_{ijj} - \alpha_{jij}$$

and

$$C_{ij} = 3\alpha_{iij} + 2\alpha_{jii} + 2\alpha_{jii} - 2\alpha_{iii}, \; i,j = 1,2.$$

COROLLARY 13. *Let T, L, H, D_{ij}, C_{ij}, $i, j=1,2$, $i \neq j$, be as above and K satisfy Hypothesis 6 and $z^1 = (z_1, z_2) \in R^2$ be a local relative extremum of h on S_2. In addition, assume that $z_j \neq 0$ ($j = 1$ or 2) and $\bar{t} = z_i / z_j$, $i \neq j$, is not a solution of the equation*

$$D_{ij} t^3 + C_{ij} t^2 + C_{ji} t + D_{ji} = 0.$$

Then $(\bar{\lambda}, 0)$ is a bifarcation point of Equation (2)

The parameter families of nontrivial solutions of (2) in a neighbourhood of $(\bar{\lambda}, 0)$ can be written in an analytical form as in Theorem 11 with $p = q = 2$.

Proof. By a simple calculation we obtain

$$3h\ (z^1)/2 - g(z^2) = \frac{1}{2}\{D_{ij} z_i^3 + C_{ij}\ z_i^2 z_j + C_{ji} z_i\ z_j^2 + D_{ji} z_j^3\}$$

$$= \frac{1}{2 z_j^3}\{D_{ij}\left(\frac{z_i}{z_j}\right)^3 + C_{ij}\left(\frac{z_i}{z_j}\right)^2 + C_{ji}\left(\frac{z_i}{z_j}\right) + D_{ji}\} \neq 0.$$

Therefore, to complete the proof, it remains to apply Theorem 11.

Remark 14. 1/ Let $(\lambda(\alpha), v(y))$, $((\lambda(\alpha), v^{\sigma}(y)), \sigma = \pm)$, $y \in I \times V$, be a parameter family of nontrivial solutions in a neighbourhood of $(\bar{\lambda}, 0)$ which exists by Theorem 11. We can easily verify that

$$v\ (y) = \sum_{j=1}^{q} \alpha\ (z_j^1/((a+1)\ h(z^1))^{1/(a-1)})\ v^j + \bar{0}\ (|y|).$$

$$(v^{\sigma}(y) = \sum_{j=1}^{q} \sigma\alpha(z_j^1/((a+1)\ \ h\ \ (z^1))^{1/(a-1)})\, v^j + \bar{0}\ \ (|y|)),$$

where $y = (\alpha, x_{q+1}, \ldots, x_p)$ and $\bar{0}(|y|)/\alpha \to 0$ as $\alpha \to 0$, $(x_{q+1}, \ldots, x_p) \to 0$.

2/ Let the same assumptions of Theorem 11 be satisfied for z^1, z^2 and $\bar{z}^1, \bar{z}^2$, respectively, where $z^1, \bar{z}^1$ are local relative extrema of h on S^q and $z^2, \bar{z}^2$ are local relative extrema of g on S^{q-1} and $\widehat{S}^{q-1}$, respectively, with $\widehat{S}^{q-1}$ denoting the unite sphere in the space $\widehat{R}^{q-1} = \{x \in R^q / [x, \bar{z}^1] = 0\}$.

In addition, assume that one of the following conditions is fulfilled: i/ $z^1 \neq \bar{z}^1$; ii/ $|h(z^1)| \neq |h(\bar{z}^1)|$; iii/ $z_{j_0}^1 / h(z^1) \neq \bar{z}_{j_0}^1 / h(\bar{z}^1)$ for some $j_0 = 1, \ldots, q$. Let $(\lambda(\alpha), v(y))$ and $(\lambda(\alpha), \bar{v}(y))$ exist by Theorem 11 corresponding to z^1, z^2 and $\bar{z}^1, \bar{z}^2$, respectively. Then there exist neighbourhoods I_0 of zero in R and V_0 of the origin in R^{p-q} such that $v(y) \neq \bar{v}(y)$ for all $y \in I_0 \times V_0$, $y = (\alpha, x_{q+1}, \ldots, x_p)$ with $\alpha \neq 0$. Indeed, by contradiction, we take sequences $\{I_n\}$, $\{V_n\}$ of neighbourhoods of zero in R and the origin in R^{p-q}, respectively, with $I_{n+1} \subset I_n$, $\cap I_n = \{0\}$, and $V_{n+1} \subset V_n$, $\cap V_n = \{0\}$, and assume that for any $n = 1, \ldots$, there exists $y_n \in I_n \times V_n$, $y_n = (\alpha_n, x^n_{q+1}, \ldots, x^n_p)$ with $\alpha_n \neq 0$ such that $v(y_n) = \bar{v}(y_n)$. It then follows that

$$\sum_{j=1}^{q} \alpha_n (z_j^1 / ((a+1)\, h\, (z^1))^{1/(1a-1)}) v^j + \bar{o}(|y_n|) =$$

$$\sum_{j=1}^{q} \alpha_n\, \bar{z}_j^1/((a+1)\, h\, (\bar{z}^1))^{1/(a-1)}) v^j + \bar{o}(|y_n|).$$

Dividing both sides by α_n and letting $n \to +\infty$, we obtain

$$\sum_{j=1}^{q} z_j/((a+1)\, h\, (z^1))^{1/(a-1)} - \bar{z}_j^1/ (h\, (\bar{z}^1))^{1/(a-1)}\, v^j = 0.$$

Since $v^1, \ldots, v^q$ are linear independent, it follows

$$z_j^1 /(\, (a+1)\, h\, (z^1)\,)^{(1/a-1)} = \bar{z}_j^1/\, (a+1)\;\; h\;\; (\bar{z}^1)\, 1/(a-1) \quad , \tag{9}$$

for all $j = 1, \ldots, q$. We observe that $1 = \sum_{j=1}^{q} (z_j^1)^2 = \sum_{j=1}^{q} (\bar{z}_j^1)^2$

and deduce from (9) that $|h\,(z^1)| = |h\;\;(\bar{z}^1)|$. If $h\,(z^1) = h\;\;(\bar{z}^1)\,(h\,(z^1) = - h\,(\bar{z}^1))$, then (9) Yields $z_j^1 = \bar{z}_j^1$ ($z_j^1 = - \bar{z}_j^1$) for all $j = 1, \ldots, q$, and hence $z^1 = \pm \bar{z}^1$. Thus, in any case we have a contradiction

Analogously, one can show that there exist neighbourhoods I_0 or zero in R and V_0 of the origin in R^{p-q} such that $v^\sigma\,(Y) \neq \bar{v}^\sigma(y)$ for all $y = (\alpha, x_{k+1}, \ldots, x_p)$ $\in I_0 \times V_0$ with $\alpha \neq 0$, where $v^\sigma\,(y)$ and $\bar{v}^\sigma\,(y)$ are from Theorem 11 corresponding to z^1, z^2 and $\bar{z}^1$, $\bar{z}^2$, respectively. This remark shows that the number of parameter families of nontrivial solutions of Equation (2) in a neighbourhood of $(\bar{\lambda}, 0)$ depends on the number of local relative extrema z^1 of h on S^q with $h(z^1) \neq 0$, which satisfy the condition $(a+1)\; h\;(z^1) \neq g\;(z^2)$ with z^2 being a local relative extremum of g on S^{q-1}.

Next, we make additional hypotheses on the mappings L and K.

HYPOTHESIS 15. There is $\bar{\beta} \in \Lambda$ such that $\langle D_\lambda L(\bar{\lambda}, v^i)(\bar{\beta}), \psi^i \rangle \neq 0$ $i = 1,\dots, q$, and $\langle D_\lambda L(\bar{\lambda}, v^1)(\bar{\beta}), \psi^j \rangle = 0$, $j = 2,\dots, q$.

HYPOTHESIS 16. $\alpha^{-a} P_Y K(\bar{\lambda} - \bar{\beta}\alpha^{a-1}/(1+\alpha^{a-1}), \alpha v)$ and

$\alpha^{1-a} D_v P_Y K(\bar{\lambda} - \bar{\beta}\alpha^{a-1}/(1+\alpha^{a-1}), \alpha v)$ tend to zero as

$\alpha \to 0$ uniformly in $v \in D$, where a is from

Hypothesis 1 and $\bar{\beta}$ is from Hypothesis 15.

THEOREM 17. *Under Hypotheses, 1, 4, 5 and 15, 16, the conclusions of Theorem 7 continue to hold with* $\langle T(v^1), \psi^1 \rangle$ *replaced by* $\langle D_\lambda L(\bar{\lambda}\, v^1)(\bar{\beta}), \psi^1 \rangle$ *and* $\lambda(\alpha)$ *by*

$$\lambda(\alpha) = \bar{\lambda} - \bar{\beta}\alpha^{a-1}/(1+\alpha^{a-1}).$$

Proof. Setting $\Lambda_0 = \bar{\lambda} - \bar{\beta} + \Lambda$ and $\beta = \lambda - \bar{\lambda} + \bar{\beta}$, we define the mappings $\bar{T} : X \to Y$, $\bar{L}, \bar{H}, \bar{K} : \Lambda_0 \times \bar{D} \to Y$ by

$$\bar{T}(v) = T(v) - L(\bar{\lambda}, v) + D_\lambda L(\bar{\lambda}, v)(\bar{\beta}),$$

$$\bar{L}(\beta, v) = D_\lambda L(\bar{\lambda}, v)(\beta),$$

$$\bar{H}(\beta, v) = H(\beta + \bar{\lambda} - \bar{\beta}, v),$$

and

$$\bar{K}(\beta, v) = K(\beta + \bar{\lambda} - \bar{\beta}, v) + L(\bar{\beta} + \bar{\lambda} - \beta, v) - L(\bar{\lambda}, v) - D_\lambda L\bar{\lambda}, v)\, \beta - \bar{\beta}),$$
$$(\beta, v) \in \Lambda_0 \times \bar{D}.$$

It then follows that Equation (2) is equivalent to the equation

$$\bar{T}(v) = \bar{L}(\beta, v) + \bar{H}(\beta, v) + \bar{K}(\beta, v), \ (\beta, v) \in \Lambda_0 \times \bar{D}. \tag{10}$$

It is clear that $\bar{\beta}$ is a characteristic value of the pair $(\bar{T}, \bar{L})$ and

$$\mathrm{Ker}\,(\bar{T} - \bar{L}(\bar{\beta},.)) = \mathrm{Ker}\,(T - L(\bar{\lambda},.)) = [v^1,\dots, v^p],$$

$$\mathrm{Ker}\,(\bar{T} - \bar{L}(\bar{\beta},.))^* = \mathrm{Ker}\,(T - L(\bar{\lambda}\,.))^* = [\psi^1,\dots, \psi^q].$$

Further, we can easily verify that Hypotheses 1 — 6 are satisfied with $b = 1$ and T, L, H and K replaced by $\bar{T}, \bar{L}, \bar{H}$ and $\bar{K}$, respectively. Therefore, to complete the proof of the theorem, it remains to apply Theorem 7 to Equation (10).

THEOREM 18. *Let the assumptions of Theorem 11 satisfy with Hypotheses 3, 6 replaced by Hypothesis 16 and* $\langle T(v^1), \psi^j \rangle = \delta_{ij}$ *replaced by* $\langle D_\lambda L(\bar{\lambda}, v^i)(\bar{\beta}), \psi^j \rangle = \delta_{ij}$. *Then the conclusions of Theorem 11 continue to hold with* $\lambda(\alpha)$ *replaced by*

$$\lambda(\alpha) = \bar{\lambda} - \bar{\beta}\alpha^{a-1}/(1+\alpha^{a-1}).$$

Proof. The proof is similar to the one of Theorem 11, with the use of Theorem 11 instead of Theorem 7.

Next, let $F: \Lambda \times \bar{D} \to Y$ be a C^k — mapping, $k \geqq 3$, with $F(\lambda, 0) = 0$ for all $\lambda \in \Lambda$. We consider the equation

$$F(\lambda, v) = 0, (\lambda, v) \in \Lambda \times \bar{D}. \tag{11}$$

Let $\bar{\lambda} \in \Lambda$ be a point such that $D_v F(\bar{\lambda}, 0)$ is a Fredholm mapping with nullity p and index s, $p > s \geq 0$. Further, let

$$X_0 = \operatorname{Ker} D_v F(\bar{\lambda}, 0) = [v^1, \ldots, v^p]$$

and

$$\operatorname{Ker} (D_v F(\bar{\lambda}, 0))^* = [\psi^1, \ldots, \psi^q],\ q = p - s.$$

In what follows, X_1, Y_0, Y_1, $P_Y$etc. are supposed to be defined as in Introduction.

Denote by $D_v^j F$ ($D_\lambda^j F$), $j = 2, 3, \ldots$, the j-th partial Fréchet derivative of F with respect to $v \in D$ ($\lambda \in \Lambda$, respectively). We make the following hypotheses on the mapping F:

HYPOTHESIS 19. There exists $\bar{\beta} \in \Lambda$ such that

$$\langle D_\lambda(D_v F(\bar{\lambda}, 0)(v^i)(\bar{\beta}), \psi^j\rangle = \delta_{ij},\ i, j = 1, \ldots, q.$$

HYPOTHESIS 20. There is a natural number a, $2 \leq a < k$ such that

$$P_Y(D_v^j F(\bar{\lambda}, 0)(v, \ldots, v) = 0,\ \ j = 1, \ldots, a-1, \ \text{ for all }\ v \in X_0 \ \text{ and}$$

$$P_Y(D_v^a F(\bar{\lambda}, 0)(X_0)) \neq 0 \text{ with } X_0^a = X_0 \times X_0 \times X_0 \text{ (a times)}.$$

HYPOTHESIS 21. The mapping $A: R^q \to R^q$, $A = (A_1, \ldots, A_q)$ with

$$A_i(x) = \frac{1}{a!} \langle D_v^a F(\lambda, 0)(\sum_{j=1}^q x_j v^j, \ldots, \sum_{j=1}^q x_j v^j), \psi^i\rangle, \qquad i = 1, \ldots, q;$$

$x = (x_1, \ldots, x_q) \in R^q$, is a potential operator with potential h and there are local relative maxima (minima) z^1, z^2 of h on S^q and of g on S^{q-1}, respectively, with

$(a+1)\, h(z^1)/2 \neq g(z^2)$, where S^q, S^{q-1}, g are defined as in Lemma 10.

THEOREM 22. *Under Hypotheses 19 – 21, the conclusions of Theorem 18 continue to hold for Equation (11).*

Proof. By Taylor's Theorem we can write

$$F(\lambda, v) = D_v F(\lambda, 0)(v) + \ldots + D_v^a F(\lambda, \bar{v})(v, \ldots, v) + 0(\|v\|^{a+1})$$

$$= D_v F(\bar{\lambda}, 0)(v) + D_\lambda (D_v F(\bar{\lambda}, 0)(v))(\lambda - \bar{\lambda}) + \ldots +$$

$$\frac{1}{a!} D_v^a F(\bar{\lambda}, 0)(v, \ldots, v) + \frac{1}{a!} D(D_v^a F(\bar{\lambda}, 0)(v, \ldots, v)(\lambda - \bar{\lambda})$$

$$+ \ldots + 0(\|v\|^{a+1}).$$

Further, we define the mappings $T: X \to Y$, L, H, $K: \Lambda \times \bar{D} \to Y$ by

$$T(v) = -\bar{D}_v F(\lambda, 0)(v) + D_\lambda (\bar{D}_v F(\lambda, \overline{0)(v)})(\bar{\lambda}),$$

$$L(\lambda, v) = D_\lambda (D_v F(\bar{\lambda}, 0)(v))(\lambda),$$

$$H(\lambda, v) = \frac{1}{2} D_v^2 F(\bar{\lambda}, 0)(v, v) + \ldots + \frac{1}{a!} D_v^a F(\bar{\lambda}, 0)(v, \ldots, v),$$

and

$$K(\lambda, v) = F(\lambda, v) + T(v) - L(\lambda, v) - H(\lambda, v), \ (\lambda, v) \in \Lambda \times \bar{D}.$$

It then follows that Equation (11) is equivalent to the following

$$T(v) = L(\lambda, v) + H(\lambda, v) + K(\lambda, v), \ (\lambda, v) \in \Lambda \times \bar{D}. \tag{12}$$

Since $T(v) - L(\bar{\lambda}, v) = -D_v F(\bar{\lambda}, 0)(v)$, we conclude that $T - L(\bar{\lambda}, .)$ is also a Fredholm mapping with the same nullity and index as $D_v F(\bar{\lambda}, 0)$. It can be verified that all assumptions of Theorem 18 for Equation (12) are satisfied. Therefore, to complete the proof of the theorem, it remains to apply Theorem 18.

3. APPLICATIONS

Application 23. *We begin the applications of the results in the previous section by considering bifurcation points.* Consider the problem of the small amplitude free vibrations of a thin rectangular plate which is formulated as follows:

$$D\Delta^2 w + p\,\frac{\partial^2 w}{\partial t^2} = \frac{6D}{Ah^2}\left(\int_\Omega \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial Y}\right)^2\right] dxdy\right) \Delta w \text{ in } (0, T) \times \Omega \tag{13}$$

$$w(t,x,y) = \frac{\partial^2 w}{\partial x^2}(t, x, y) = \frac{\partial^2 w}{\partial Y^2}(t, x, y) = 0 \text{ in } [0, T] \times \partial\Omega$$

and

$w(0, x, y) = w(T, x,y), \frac{\partial w}{\partial t}(0, x, y) = \frac{\partial w}{\partial t}(T, x, y)$ $(x, y) \in \Omega$, in which $\Omega =$

$= \{(x, y) \in R^2 / 0 < x < a, 0 < y < b\}$, with the boundary $\partial\Omega$. D is the flexural rigidity of the plate given by $Eh^2 / 12(1 - \nu^2)$ where E is the modulus of elasticity, h is the thickness of the plate, ν is Poisson's ration $(0 < \nu < 1/2)$, p is the mass density, A is the plate area, and $w(t, x, y)$ is the deflection normal to the middle surface.

Putting $V = \left\{ u \in C^4(\bar{\Omega}) / \ u = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } \partial\Omega \right\}$, we define the inner product $(u, v)_o = \int_\Omega \Delta u \Delta v d\Omega$, and the norm $\| u \|_o = (u, u)_o^{1/2}$. Let H_o be the completion of V in the $\| . \|_o$-topology. It then follows that H_o is a linear and closed subspace of the Sobolev space $H^2(\Omega)$, (see, for example, [1]), hence it is a Hilbert space. Furthermore, the system of functions

$$W_{mn} = \frac{2\sqrt{2}}{\sqrt{ab}\left(a_m^2 + b_n^2\right)} \sin a_m x \sin b_n y, \ m, n = 1, 2, ..$$

with $a_m = \frac{m\pi}{a}, b_n = \frac{n\pi}{b}$, forms a complete system in H_o (see [1, Lemma 1]).

Further, we set

$$X = \{f \in L^2(o, T, H_o) / f(o,.,.) = f(T,.,.), \frac{\partial f}{\partial t}(o,.,.) = \frac{\partial f}{\partial t}(T,...)$$

$$\frac{\partial^2 f}{\partial t^2}, \frac{\partial^2 f}{\partial x \partial y} \in L^2(o, T, L^2(\Omega)), \ |\alpha| \leq 2\}$$

and define the inner product $<, >$ and the norm $\| . \|$ on X by

$$\langle f, g \rangle = \int_o^T (f, g)_o dt = \int_o^T (\int_\Omega \Delta f \Delta g d\Omega) dt$$

and

$$\| f \| = (\langle f, f \rangle)^{1/2}.$$

One can easily verify that X is a Hilbert space.

Now put $\lambda = \rho/D$ and consider it as a parameter. A point $(\lambda, v) \in R \times X$ is called a week solution of Equation (13) if

(14)

$$\int_0^T (\int_\Omega \Delta u \Delta v d\Omega)\, dt = \lambda \int_0^T (\int_\Omega \frac{\partial v}{\partial t} \frac{\partial u}{\partial t}\, d\Omega)\, dt + \frac{6}{Ah^2} \int_0^T \left(\int_\Omega [(\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2] d\Omega \right). \int_\Omega \Delta v\, u d\Omega\, dt$$

holds for all $u \in X$.

Further, we define the mappings $L; H; X \to X$ by

$$\langle L(v), u \rangle = \int_o^T \left(\int_\Omega \frac{\partial v}{\partial t} \cdot \frac{\partial u}{\partial t}\, d\Omega \right) dt$$

$$\langle H(v)\, u \rangle = \frac{6}{Ah^2} \int_o^T \left(\int_\Omega \left[(\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 \right] d\Omega \right) \int_\Omega \Delta v\ u d\Omega\, dt.$$

It then follows that (14) is equivalent to the equation

$$v = \lambda L(v) + H(v),\ (\lambda, v) \in R \times X. \qquad (15)$$

Let $\overline{\gamma}$ be an eigenvalue of the problem

$$-u'' = \gamma u$$

$$u(o) = u(T),\ u'(o) = u'(T)$$

with a corresponding eigenvector φ. We can easily see that for any $m,n = 1,2...;$

$$\lambda_{mn} = (a_m^2 + b_n^2)^2 / \overline{\gamma}$$

is a characteristic value of the pair (id, L) and

$$U_{mn} = \varphi(t) \frac{2\sqrt{2}}{\sqrt{ab}\,(a_m^2 + b_n^2)} \sin a_m \times \sin b_n y$$

is a corresponding eigenvector.

Now, assume that $\overline{\lambda}$ is a characteristic value of the pair (id, L) with multiplicity p. It then follows that there exist p pairs (m_i, n_i), $i = 1,\dots p$ with $(a^2_{m_i} + a^2_{n_i})^2 = \overline{\lambda}\overline{\gamma}$. Putting $v^i = U_{m_i n_i}$. We deduce

$$\langle v^i, v^j \rangle = (\int_o^T \varphi^2(t)\, dt)\, \delta_{ij},\ i,j = 1,\dots, p,$$

and

$$\langle H(v^i), v^j \rangle = \left(\frac{3}{8Ah^2} a^2 b^2 \overline{\lambda}\overline{\gamma} \int_0^T \varphi^4(t)\, dt \right) . \delta_{ij},\ i, j = 1,\dots, p.$$

Consequently, $\overline{\lambda}$, $\overline{T} = \mathrm{id}$, L, H and $K = 0$ satisfy Hypotheses 2 – 6.

Setting $\overline{x}^{i\sigma} = (\overline{x}_1^{i\sigma},\dots,\overline{x}_p^{i\sigma})$ with

$$\overline{x}_i^{i\sigma} = \sigma \left| \frac{8Ah^2 \int_0^T \varphi^2(t)\,dt}{3a^2b^2\overline{\lambda}\overline{\gamma}\int_0^T \varphi^4(t)\,dt} \right|^{1/2}, \quad i = 1, 2,\dots, p$$

and

$$\overline{x}_j^{i\sigma} = 0, \quad i \neq j,$$

we have

THEOREM 24. *If $\overline{\lambda}$ is as above, then $(\overline{\lambda}, 0)$ is a bifurcation point of Equation (15) and there exist at least 2p distinct parameter families of nontrivial solutions of (20) in a neighbourhood of $(\overline{\lambda}, 0)$. More precisely, for any $\sigma = \pm$, $i = 1,\dots, p$, there exist neighbourhoods $U^{i\sigma}$ of the point $\overline{x}^{i\sigma}$, $I^{i\sigma}$ of zero in R, p continuous functions $x_j^{i\sigma}: I^{i\sigma} \to R$, $x_j^{i\sigma}(0) = \overline{x}_j^{i\sigma}$, $j = 1,\dots, p(x_1^{i\sigma}(\alpha),\dots, x_p^{i\sigma}(\alpha)) \in U^{i\sigma}$ and a continuous mapping $w^{i\sigma}: I^{i\sigma} \to [v^1,\dots, v^p]^1$, $\| w^{i\sigma}(\alpha)\| = 0(|\alpha|)$ as $\alpha \to 0$ such that $(\lambda^{i\sigma}(\alpha), v^1(\alpha))$ with*

$$\lambda^{i\sigma}(\alpha) = \frac{\overline{\lambda}}{1+\alpha^2}$$

and

$$v^{i\sigma}(\alpha) = \sum_{j=1}^{p} \alpha x_j^{i\sigma}(\alpha) v^j + w^{i\sigma}(\alpha),$$

$$= \overline{x}_i^{i\sigma} v^i + o(\alpha), \ \alpha \in I^i, \ as\ \alpha \to 0$$

satisfies Equation (15), $\lambda^{i\sigma}(\alpha) \to \overline{\lambda}$, $v^{i\sigma}(\alpha) \to 0$ as $\alpha \to 0$, $v^{i\sigma}(\alpha) \neq 0$ for $\alpha \neq 0$.

Proof. The result follows immediately from Theorem 7 and Remark 8.

Application 25. Let us study bifurcation points of boundary value problem of nonlinear elliptic differential equation:

$$Av = \lambda Bv + c(v, v)Cv + D(v) \ in\ \Omega, \tag{16}$$

$$B_k v = 0, \ 0 \leq k \leq m-1, \ in\ \partial\Omega,$$

where Ω denotes a bounded domain in R^n with the infinitely differentiable boundary $\partial\Omega$ which is a linear (n−1) manifold and Ω lies locally on one side of $\partial\Omega$, $\partial\Omega \in C^\infty$, A is a uniform elliptic differential operator in Ω of order $2m$:

$$Av = \sum_{|\alpha|\,|\beta| \leq m-1} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta(v)) \ \text{with}\ a_{\alpha\beta} \in C^\infty(\overline{\Omega}).$$

B and C are uniform elliptic operators in Ω but at most of the order $2m-2$ and given by

$$Bv = \sum_{|\alpha|\,|\beta| \leq m-1} (-1)^{|\alpha|} D^{\alpha} (b_{\alpha\beta}(x) D^{\beta}(v)) \text{ with } b_{\alpha\beta} \in C^{\infty}(\overline{\Omega}.$$

$$Cv = \sum_{|\alpha|\,|\beta| \leq m-1} (-1)^{|\alpha|} D^{\alpha}(c_{\alpha\beta}(x) D^{\beta} v), \text{ with } c_{\alpha\beta} \in C^{\infty}(\overline{\Omega}),$$

$$c(v, v) = \sum_{|\alpha|\,|\beta| \leq m} \int_{\Omega} c_{\alpha\beta}(x) D^{\alpha} v \; D^{\beta} v \, d\Omega,$$

and D is an operator with « higher order » that we shall describe later in Hypothesis 29. B_k, for k, $0 \leq k \leq m-1$, are linear homogeneous differential operators, which are defined in a neighbourhood of $\partial\Omega$. For simplicity we assume that the order of B_k is less than m and the boundary operators are the elliptic operators A, B, C with the property that one may associate with A, B, C symmetric bilinear forms

$$a(v,u) = \int_{\Omega} u \; Av \; d\Omega = \sum_{|\alpha|\,|\beta| \leq m} \int_{\Omega} a_{\alpha\beta} D^{\alpha} v \; D^{\alpha} u \; d\Omega,$$

$$b(v,u) = \int_{\Omega} u \; B \, v \; d\Omega = \sum_{|\alpha|\,|\beta| \leq m-1} \int_{\Omega} b_{\alpha\beta} D^{\alpha} v \; D^{\alpha} u \; d\Omega \, ,$$

and

$$c(v,u) = \int_{\Omega} u \; Cv \; d\Omega = \sum_{|\alpha|\,|\beta| \leq m-1} \int_{\Omega} c_{\alpha\beta} D^{\alpha} v^{\alpha} D \; u \; d\Omega,$$

for all $u, v \in V = \{f \in C(\overline{\Omega} \mid B_k f = 0 \text{ on } \partial\Omega, 0 \leq k \leq m-1\}$.

We provide V with the standard $\|\cdot\|_m$ — topology of the Sobolev space $H^m(\Omega)$, and the completion of $(V, \|\cdot\|_m)$ becomes a linear and closed subspace of $H^m(\Omega)$ and is denoted by X. The restricted norm and the inner product of $H^m(\Omega)$ to X are denoted by $\|\cdot\|$ and $\langle \, , \rangle$, respectively. Hence, X is a Hilbert space.

HYPOTHESIS 26. There is a constant $k_0 > 0$, $a(v,v) \geq k_0 \|v\|^2$ for all $v \in X$.

HYPOTHESIS 27. $b(v,v) \geq 0$ for all $v \in X$ and $b(v,v) = 0$ implies $v = 0$.

HYPOTHESIS 28. $c(v,v) = 0$ implies $v = 0$, $v \in X$.

HYPOTHESIS 29. D is a continuously differentiable mapping from X into $L^2(\Omega)$ and $\alpha^{-3} D(\alpha v)$, $\alpha^{-2} D_v(\alpha v)$ tend to zero uniformly in v in a bounded neighbourhood of zero in X as $\alpha \to 0$.

We say that $(\lambda, v) \in \Lambda \times X$ is a weak solution of the boundary value problem (16) if

$$a(v,u) = \lambda \, b(v,u) + c(v,v) \, c(v,u) \int_{\Omega} u \; Dv \; d\Omega \tag{17}$$

holds for all $u \in X$.

It is easy to verify that the linear boundary value problem

$$\mathbf{A}u = f \quad \text{in } \Omega \text{ , } f \in L^2 (\Omega) \text{ ,}$$
$$\mathbf{B}_k u = u \text{ , } 0 \leq k \leq m-1 \text{ , in } \partial \Omega$$

always posesses a weak solution denoted by $G(f)$ (see, for example [2]). The operator $G : L2(\Omega) \to X$ is called a Green operator. Consequently, (17) is equivalent to

$$v = \lambda L(v) + H(v) + K(v) \text{ , } (\lambda, v) \in R \times X \tag{18}$$

with $L(v) = G(\mathbf{B}v)$, $H(v) = G(c(v,v)\, \mathrm{C}(v))$ and $K(v) = G(\mathbf{D}v)$.
Therefore, to study the bifurcation points of Equation (17) we need only to consider the ones of Equation (18).

By Lemma 2.1 in [3] the pair (id, L) has only positive enumerable characteristic values. Each characteristic value has a finite multiplicity, i.e., they can be ordered as $0 < \lambda_1 \leqq \lambda_2 \dots < +\infty$. Moreover, the eigenfunctions u^m and u^n corresponding to the characteristic values $\lambda_n \neq \lambda_m$ are orthogonal w.r.t the bilinear form $b(.,.)$ i.e. $b(u^m, u^n) = 0$ for $m \neq n$.

Let $\bar{\lambda}$ be a characteristic value of the pair (id, L) with multiplicity p. One can easily verify that $(id - \bar{\lambda}L) = (id - \bar{\lambda}L)^*$. Suppose that $v^1 \dots, v^p$ are eigenfunctions corresponding to $\bar{\lambda}$. Without loss of generality we may assume $\langle v^i , v^j \rangle = \delta_{ij}$, $i, j = 1,\dots, p$. Define the mapping $A : R^p \to R^p$, $A = (A_1 \dots, A_p)$ by

$$A_i(x) = \langle H (\sum_{j=1}^{p} x_j v^j), v^i \rangle = \langle G(c(\sum_{j=1}^{p} x_j v^j, \sum_{j=1}^{p} x_j v^j)\, \mathrm{C}(\sum_{j=1}^{p} x_j v^j)), v^j \rangle$$
$$= c(\sum_{j=1}^{p} x_j v^j \sum_{j=1}^{p} x_j v^j) \int_\Omega v^i\, \mathrm{C}(\sum_{j=1}^{p} x_j v^j)\, d\Omega.$$
$$= c(\sum_{j=1}^{p} x_j v^j, \sum_{j=1}^{p} x_j v^j) \cdot c(v^i, \sum_{j=1}^{p} x_j v^j).$$

By a simple calculation we conclude that A is a potential operator with potential

$$h(x) = \frac{1}{4}\left(c(\sum_{j=1}^{p} x_j v^j, \sum_{j=1}^{p} x_j v^j)\right)^2 = \frac{1}{4}\left(\sum_{k,l=1}^{p} x_k x_k\, c(v^k, v^l)\right)^2.$$

Let E_+ be defined as in Section 2 associated with h. Hypothesis (28) implies $h(x_o) > 0$ for $x_o = (1, 0,\dots,0)$. By Lemma 9 we conclude $E_+ \neq \phi$.

Now, let $z^1 \in E_+$. We can write

$$R^p = \{z^1\} \oplus \overline{R}^{q-1}$$

and set

$$B(z^1) = \left(\frac{\partial h}{\partial x_i\, \partial x_k}(z^1)\right)_{i,k=1,\dots,p}$$

with

$$\frac{\partial h}{\partial x_i\, \partial x_k}(z^1) = 2c\,(v^k, \sum_{j=1}^{p} z_j^1 v^j)\, c\,(v^i, \sum_{j=1}^{p} z_j^1 v^j) + c(\sum_{j=1}^{p} z_j^1 v^j, \sum_{j=1}^{p} z_j^1 v^j) c(v^k, v^i).$$

Let z^2 be a local extremum of $g(x) = \frac{1}{2}((B(z^1)\,x, x))$ on the unit sphere S^{p-1}, of $\overline{R}^{p-1}$.

We make the following hypothesis :

HYPOTHESIS 30. z^1, z^2 are local relative maxima (minima) of h and g, respectively with $h(z^1) > \frac{1}{2} g(z^2)$ ($h(z^1) < \frac{1}{2} g(z^2)$, respectively).

THEOREM 31. *Under Hypotheses 26 — 30, $(\overline{\lambda}, 0)$ is a bifurcation point of Equation (23). Moreover, there exist at least two distinct parameter families of non-trivial solutions in a neighbourhood of $(\overline{\lambda}, 0)$ and we can describe these families in an analytical form as in Theorem 11.*

Proof. This follows immediately from Theorem 11.

As an illustration of the above Theorem, we consider Equation (18) in the special case: $c(v^i, v^j) = 0$ for $i \neq j$, $i, j = 1, \dots, p$; $0 < c(v^1, v^1) < \dots < c(v^p, v^p)$. For the sake of simplicity of notations we put $c_k = c(v^k, v^k)$; $k = 1, \dots, p$. One can see

$$h(x) = \frac{1}{4}\left(\sum_{k=1}^{p} c_k x_k^2\right)^2.$$

By a simple calculation we conclulde that h possesses local relative minima $z_\sigma^1 = (\sigma 1, 0, \dots, 0)$, $\sigma = \pm$ on S^p.

We have $h(z^1) = \frac{1}{4} c_1^2$ and

$$B(z_\sigma^1) = \left(\frac{\partial h}{\partial x_i\, \partial x_k}(z_\sigma^1)\right)_{i,k=1,\dots,p} = (a_{ik})_{i,k=1,\dots,p}$$

with

$$a_{ik} = \begin{cases} 0, & i \neq k \\ 3c_1^2, & i = 1 \\ c_1 c_i, & i \neq 1. \end{cases}$$

It then follows $g(x) = \frac{1}{2}(3\ c_1^2 x_1^2 + \sum_{k=2}^{p} c_1 c_k x_k^2)$. We can see that

$$\overline{R}^{p-1} = \{x = (x_1, x_2, \ldots, x_p)\, R^p / x_1 = 0\}$$

and g possesses local relative minima $z_\sigma^2 = (0, \sigma 1, 0, \ldots, 0)$ on S^{p-1},

$$\sigma = \pm\, g(z_\sigma^2) = \frac{1}{2} c_1 c_2 \neq \frac{1}{2} c_1^2 = \frac{(a+1)}{2} h(z_\sigma^1) .$$

Applying Theorem 11 and Remark 14, we conclude that in this special case $(\overline{\lambda}, 0)$ is a bifurcation point of Equation (18) and there exist at least four distinct parameter families of nontrivial solutions in a neighbourhood of $(\overline{\lambda}, 0)$. Moreover, it is possible to describe these families in an analytical form as in Theorem 11.

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