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SOME FIXED POINT THEOREMS FOR MAPPINGS OF CONTRACTION TYPE IN QUASIMETRIC SPACES

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1. INTRODUCTION

In [6] we have defined the quasimetric spaces and studied their elementary properties. In this paper we shall prove two fixed point theorems of mappings of contraction type in these spaces. First, recall that a quasimetric space [6] is an ordered pair (X,d) consisting of a set X and a non-negative real function $d : X \times X \to R$ satisfying the following conditions:

QM_1) $d(x, y) = 0$ iff $x = y$,

QM_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

QM_3) For each $\varepsilon > 0$ there exists $\delta > 0$ such that

$|d(x, y) - d(x', y')| \leqslant \varepsilon$ whenever $d(x, x') \leqslant \delta$ and $d(y, y') \leqslant \delta$.

Note that our definition differs from the one of quasi-metric spaces defined in [2].

For every nonnegative number r, the set $B(x, r) = \{y \in X / d(x,y) < r\}$ is said to be an open ball. A set A in a quasimetric space (X, d) is said to be open if for each $x \in A$ there exists an open ball $B(x, r) \subset A$ with $r > 0$. Open sets form a Hausdorff topology on X. A sequence $\{x_n\}$ is said to be convergent to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$ and then we write $x_n \to x$ or $\lim x_n = x$. Let (X, d), (Y, d') be quasimetric spaces, $f : X \to Y$ be a mapping, then f is continuous at x iff $fx_n \to fx$ for every sequence $\{x_n\}$ converging to x. A sequence $\{x_n\}$ is called a Cauchy sequence if for each $\varepsilon > 0$ there exists a positive integer $n \in N$ such that $d(x_p, x_q) < \varepsilon$ for all $p, q \geqslant n$. (X,d) is said to be complete if every Cauchy sequence converges in it. It is shown [6] that every metric space is quasimetric but the converse is not true and so the class of quasimetric spaces is larger than the class of metric spaces.

2. MAIN RESULTS

THEOREM 1. Let (X, d) be a complete quasimetric space, $f : X \to X$ a continuous mapping, $q(x, y) = \max\{d(x, y), d(x, fx), d(y, fy)\}$. If for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(fx, fy) < \varepsilon$ whenever $q(x, y) < \varepsilon + \delta$ then f has a unique fixed point x_* and $f^n x_0 \to x_*$ as $n \to \infty$ for each $x_0 \in X$.

Proof. First, we show that

$$d(fx, fy) < q(x, y) \tag{1}$$

for all $x \neq y$ in X. Indeed, if $x_1 \neq y_1$ we take $\varepsilon_1 = q(x_1, y_1) > 0$. For this ε_1 by the assumption of the theorem there exists $\delta_1 > 0$ such that $d(fx, fy) < \varepsilon_1$ whenever $q(x, y) < \varepsilon_1 + \delta_1$. Since $q(x_1, y_1) < \varepsilon_1 + \delta_1$ we have $d(fx_1, fy_1) < \varepsilon_1 = q(x_1, y_1)$. So (1) is proved.

Fix $x_0 \in X$ and denote $x_{n+1} = fx_n$, $n = 0,1,2\ldots$ If $x_{m+1} = x_m$ for some m then x_m is a fixed point and $f^n x_m = x_m$ for each n. Therefore we may suppose that $x_{n+1} \neq x_n$ for each $n \in N$. Then we have $d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) < q(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$ for each $n \in N$. From this it follows that $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for each $n \in N$. This means that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of positive numbers, hence it converges to some non-negative ε. Denote

$$d(x_n, x_{n+1}) \downarrow \varepsilon \geqslant 0 \tag{2}$$

We show that $\varepsilon = 0$. Indeed, if $d(x_n, x_{n+1}) \downarrow \varepsilon > 0$, then by the assumption of the theorem there exists $\delta > 0$ such that

$$d(fx, fy) < \varepsilon \tag{3}$$

whenever $q(x, y) \leqslant \varepsilon + \delta$. Since $d(x_n, x_{n+1}) \downarrow \varepsilon$ there exists $n_0 \in N$ such that $d(x_n, x_{n+1}) < \varepsilon + \delta$ for each $n \geqslant n_0$. We have $q(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_n, x_{n+1}) < \varepsilon + \delta$ for each $n \leqslant n_0$. Consequently, by (3) we get $d(x_{n+1}, x_{n+2}) = d(fx_n, fx_{n+1}) < \varepsilon$ for each $n \geqslant n_0$. This contradicts (2) and in fact

$$d(x_n, x_{n+1}) \downarrow 0. \tag{4}$$

We now prove that $\{x_n\}$ is a Cauchy sequence. Suppose the contrary that $\{x_n\}$ is not Cauchy. Then there exists $\varepsilon_0 > 0$ such that for each $n \in N$ there exist $p, q \in N, q > p \geqslant n$ such that $d(x_p, x_q) > 2\varepsilon_0$. By the assumption of the theorem for this ε_0 there exists $\delta_0 > 0$ such that

$$d(fx, fy) < \varepsilon_0 \tag{5}$$

whenever $q(x, y) < \varepsilon_0 + \delta_0$. Denote $\delta_1 = \min\{\varepsilon_0, \delta_0\}$. By QM_3 for $\frac{\delta_1}{4}$ there exists $\delta_2 > 0$ such that

$$|d(x, y) - d(x', y')| \leqslant \frac{\delta_1}{4} \tag{6}$$

whenever $d(x, x') \leqslant \delta_2$, $d(y, y') \leqslant \delta_2$. In view of (4) there exists $n_1 \in N$ such that $d(x_n, x_{n+1}) < \min\left\{\delta_2, \frac{\delta_1}{4}\right\}$ for each $n \geqslant n_1$. For this n_1, by the assumption there exist $p, q \in N$, $q > p \geqslant n_1$ such that $d(x_p, x_q) > 2\varepsilon_0$. Then by (6) for each $i \in \{p, p+1, \ldots q\}$ we have $|d(x_p, x_i) - d(x_p, x_{i+1})| \leqslant \frac{\delta_1}{4}$.
Since $d(x_p, x_{p+1}) < \varepsilon_0$ and $d(x_p, x_q) > 2\varepsilon_0 \geqslant \varepsilon_0 + \delta_1$, it follows that there exists $k \in \{p, p+1, \ldots, q\}$ such that

$$\varepsilon_0 + \frac{\delta_1}{2} \leqslant d(x_p, x_k) \leqslant \varepsilon_0 + \frac{3\delta_1}{4}.$$

We now show that $q(x_p, x_k) < \varepsilon_0 + \delta_0$. Observe first that

$$d(x_p, x_k) \leqslant \varepsilon_0 + \frac{3\delta_1}{4} < \varepsilon_0 + \delta_1 \leqslant \varepsilon_0 + \delta_0,$$

$$d(x_p, x_{p+1}) < \min\left\{\delta_2, \frac{\delta_1}{4}\right\} \leqslant \frac{\delta_1}{4} < \varepsilon_0 + \delta_0,$$

$$d(x_k, x_{k+1}) < \min\left\{\delta_2, \frac{\delta_1}{4}\right\} \leqslant \frac{\delta_1}{4} < \varepsilon_0 + \delta_0.$$

Consequently, $q(x_p, x_k) < \varepsilon_0 + \delta_0$ and then by (5), on one hand, we have

$$d(x_{p+1}, x_{k+1}) = d(fx_p, fx_k) < \varepsilon_0 \tag{7}$$

On the other hand by (6) we have

$$|d(x_p, x_k) - d(x_{p+1}, x_{k+1})| \leqslant \frac{\delta_1}{4}.$$

From this it follows that $d(x_{p+1}, x_{k+1}) \geqslant d(x_p, x_k) - \frac{\delta_1}{4} \geqslant \varepsilon_0 + \frac{\delta_1}{2} - \frac{\delta_1}{4} =$

$= \varepsilon_0 + \frac{\delta_1}{4} > \varepsilon_0$, contradicting (7). So $\{x_n\}$ is Cauchy. Since X is complete, $x_n \to x_*$ for some x_* for some $x_* \in X$.

By the continuity of f, $x_* = \lim x_{n+1} = \lim fx_n = fx_*$ and so x_* is a fixed point of f.

The uniqueness of x_* is obvious. In fact, suppose the contrary that there is $y_* \neq x_*$ and $y_* = fy_*$, then by (1) we have $q(x_*, y_*) = d(fx_*, fy_*) < q(x_*, y_*)$, a contradiction. The theorem is proved.

COROLLARY. *Let* (X, d) *be a complete quasimetric space,* $f: X \to X$. *If for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $d(fx, fy) < \varepsilon$ *whenever* $d(x, y) < \varepsilon + \delta$, *then* f *has a unique fixed point* x_* *and* $f^n x_0 \to x_*$ *for each* $x_0 \in X$.

Proof. It is clear that f satisfies the condition $d(fx, fy) \leqslant d(x, y)$ for all $x, y \in X$. Hence it is continuous and all the conditions of Theorem 1 are satisfied. The corollary is then evident.

In [3] Meir and Keeler have proved the following theorem which generalizes the Banach contraction principle [1].

THEOREM. *Let* (X, d) *be a complete metric space,* $f: X \to X$. *If for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\varepsilon \leqslant d(x, y) < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon \qquad (8)$$

then f *has a unique fixed point* x_* *and* $f^n x_0 \to x_*$ *for each* $x_0 \in X$.

In [4] the authors have shown that Condition (8) is equivalent to the following:

$$d(x, y) < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon .$$

This is true also for the case of quasimetric spaces. Therefore, our corollary is a generalization of the above-mentioned theorem.

Note that without the assumption on the continuity of f Theorem 1 does not hold even in the case of metric spaces ([4], Remark 1.1).

Also in [4] the authors have proved that the last theorem remains valid if in (8) $d(x, y)$ is replaced by $r(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}(d(x, fy) + d(y, fx))\}$. Moreover, by an example they have shown that it does not hold if $d(x, y)$ is replaced by $\max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}$.

It is interesting to know whether Theorem 1 remains valid if $q(x, y)$ is replaced by $r(x, y)$. Here is the answer.

THEOREM 2. *Assume that (X, d) is a quasimetric space and $f: X \to X$ satisfies the following conditions:*

1) There exist $q_1 \in [0, +\infty)$, $q_2 \in [0, 1)$ such that
$$d(fx, fy) \leqslant q_1 d(x, y) + q_2 \max\{d(x, fx) + d(y, fy), d(x, fy) + d(y, fx)\}$$
for all $x, y \in X$,

2) There exists $x_0 \in X$ such that the sequence $\{f^n x_0\}$ has two successive subsequences converging to some $x_* \in X$
$$f^{n_i} x_0 \to x_* \text{ and } f^{n_i+1} x_0 \to x_* \text{ as } i \to \infty.$$
Then x_* is a fixed point of f.

Proof. If q_1, q_2 satisfy Condition 1) and $q_1' \geqslant q_1$, $q_2' \geqslant q_2$ then q_1', q_2' also satisfy 1). Therefore we may assume that $q_1 \in (0, +\infty)$, $q_2 \in (0, 1)$. Suppose the contrary that $x_* \neq fx_*$. Then by QM_1 we have $d(x_*, fx_*) = r > 0$. By QM_3; for $\varepsilon = \frac{(1-q_2)r}{4}$ there exists $\delta > 0$ such that
$$| d(x, y) - d(x', y') | \leqslant \frac{(1-q_2)r}{4} \tag{9}$$
whenever $d(x, x') \leqslant \delta$, $d(y, y') \leqslant \delta$. Denote
$$\delta_1 = \min\left\{ \delta, \frac{(1-q_2)r}{4}, \frac{(1-q_2)r}{4q_1} \right\} \tag{10}$$
and $x_{n+1} = fx_n$, $n = 0, 1, 2 \ldots$.

By the same reason for this δ_1 there exists $\delta_2 > 0$ such that
$$| d(x, y) - d(x', y') | \leqslant \delta_1 \tag{11}$$
whenever $d(x, x') \leqslant \delta_2$, $d(y, y') \leqslant \delta_2$. Since $x_{n_i} \to x_*$ and $x_{n_i+1} \to x_*$, for $\delta_0 = \min\{\delta_1, \delta_2\}$ there exist $i_0 \in N$ such that $\max\{d(x_{n_i+1}, x_*), d(x_{n_i}, x_*)\} < \delta_0$ for all $i \geqslant i_0$. Then for $i \geqslant i_0$, by (11), taking $x' = y' = x_*$, $x = x_{n_i+1}$, $y = x_{n_i}$ we have $d(x_{n_i+1}, x_{n_i}) \leqslant \delta_1$. So we obtain the inequality
$$\max\{d(x_{n_i}, x_*), d(x_{n_i+1}, x_*), d(x_{n_i}, x_{n_i+1})\} \leqslant \delta_1 \tag{12}$$
for all $i \geqslant i_0$. By (9) we have
$$| d(x_{n_i}, fx_*) - d(x_*, fx_*) | \leqslant \frac{(1-q_2)r}{4}$$

and $$|d(x_{n_i+1}, fx_*) - d(x_*, fx_*)| \leqslant \frac{(1-q_2)r}{4} \text{ for all } i \geqslant i_0 .$$

Consequently,

$$d(x_{n_i}, fx_*) \leqslant \frac{(1-q_2)r}{4} + r \tag{13}$$

and

$$d(x_*, fx_*) \leqslant \frac{(1-q_2)r}{4} + d(x_{n_i+1}, fx_*) \tag{14}$$

for all $i \geqslant i_0$. Using Condition 1), Inequalities (14), (13), (12) and (10) for $i \geqslant i_0$ we get

$$0 < r = d(x_*, fx_*) \leqslant \frac{(1-q_2)r}{4} + d(x_{n_i+1}, fx_*) =$$

$$= \frac{(1-q_2)r}{4} + d(fx_{n_i}, fx_*) \leqslant \frac{(1-q_2)r}{4} + q_1 d(x_{n_i}, x_*) +$$

$$q_2 \max\{d(x_{n_i}, x_{n_i+1}) + d(x_*, fx_*),\ d(x_{n_i}, fx_*) + d(x_{n_i+1}, x_*)\}$$

$$\leqslant \frac{(1-q_2)r}{4} + q_1 \frac{(1-q_2)r}{4q_1} + q_2 \max\left\{\frac{(1-q_2)r}{4} + r, \frac{(1-q_2)r}{4} + r + \frac{(1-q_2)r}{4}\right\}$$

$\leqslant \frac{(1-q_2)r}{2} + q_2\left(r + \frac{(1-q_2)r}{2}\right)$. From the fact that $q_2 \in (0,1)$ it follows $0 < r <$ $< \frac{(1-q\ )r}{2} + q_2\left(r + \frac{(1-q_2)r}{2q_2}\right) = r$, a contradiction. The theorem is proved.

Note that if (X, d) is a metric space this theorem can be derived, from Theorem 1 of [5].

Remark 1. In Theorem 2, in general the point x_* need not be a unique fixed point of f. For example, the identity $I : X \to X$ satisfies all the conditions of the theorem with $q_1 = 1$, $q_2 = 0$ and each $x \in X$ is a fixed point of I.

Remark 2. From our proof it follows that Theorem 2 remains valid if Condition 2) is replaced by a weaker condition:

2') For each $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that

$$\max\{d(x_*, x_\varepsilon),\ d(x_*, fx_\varepsilon)\} < \varepsilon.$$

Indeed, for each $n \in N$ take $\varepsilon = \frac{1}{n}$. Then there exists $y_n \in X$ such that $\max\{d(y_n, x_*), d(fy_n, x_*)\} < \frac{1}{n}$. Denote $x_{2n} = y_n$ and $x_{2n+1} = fy_n$. Then $\lim x_{2n} = \lim x_{2n+1} = x_*$ and the proof of Theorem 2 goes through.

Finally, note that Theorem 2 does not hold even in the case of metric spaces if q_2 is replaced by 1 (see Example 1 of [5]).

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