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ON CUBIC METACIRCULANT GRAPHS

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1. INTRODUCTION

In 1982 Alspach B. and Parsons T.D. defined a class of vertex—transitive graphs called metacirculant [1]. This class includes as a special case the class of circulant graphs which has been intensively investigated in the last two decades.

Let n be a positive integer. Denote by Z_n the ring of integers modulo n and by Z_n^* the multiplicative group of units in Z_n. Let $S \subseteq Z_n \setminus \{0\}$ be such that $i \in S$ implies $-i \in S$. The circulant graph $C(n,S)$ is a graph whose vertex set is $\{v_0, v_1, \ldots, v_{n-1}\}$, and an edge joins v_i and v_j if and only if $(j-i) \in S$, where we take $(j-i)$ modulo n. The set S is called the symbol of $C(n, S)$.

Suppose that m, n are integers, $m \geqslant 1$, $n \geqslant 2$, $\alpha \in Z_n^*$, $\mu = \left\lfloor \frac{m}{2} \right\rfloor$ (i.e. μ is the greatest integer not exceeding $\frac{m}{2}$) and $S_0, S_1, \ldots, S_\mu$ are subsets of Z_n = satisfying:

(1) $0 \notin S_0 = -S_0$;

(2) $\alpha^m S_r = S_r$ for $0 \leqslant r \leqslant \mu$;

(3) if m is even, then $\alpha^\mu S_\mu = -S_\mu$.

We define the (m, n)—metacirculant graph $G = MC(m,n, \alpha, S_0, S_1, \ldots, S_\mu)$ to be a graph with vertex set

$$V(G) = \left\{ v_j^i \;\middle|\; i \in Z_m, j \in Z_n \right\}.$$

and edge set

$$E(G) = \left\{ (v_j^i, v_h^{i+r}) \;\middle|\; 0 \leqslant r \leqslant \mu,\ i \in Z_m,\ h, j \in Z_n \text{ and } (h-j) \in \alpha^i S_r \right\},$$

where superscripts and subscripts are always reduced modulo m and n respectively.

A graph G is called a **metacirculant** graph if it is an (m, n)-metacirculant graph MC $(m, n, \alpha, S_o, S_1, \ldots, S_\mu)$ for some $m, n, \alpha, S_o, S_1, \ldots, S_\mu$.

We define two permutations ρ and τ on the vertex set $V(G)$ of an (m, n)—metacirculant graph G by

$$\rho(v^i_j) = v^i_{j+1} \text{ and}$$

$$\tau(v^i_j) = v^{i+1}_{\alpha j}.$$

It is easy to see that ρ and τ generate the transitive subgroup $\langle \rho, \tau \rangle$ of the symmetric group on $V(G)$. Moreover, as it has been proved in [1], the subgroup $\langle \rho, \tau \rangle$ is contained in the automorphism group Aut G of the graph G. Conversely, any graph G' with vertex set $V(G') = V(G)$ and $\langle \rho, \tau \rangle \leqslant$ Aut G' is an (m, n)—metacirculant graph.

In this paper we consider cubic (m, n)—metacirculant graphs with $S_o \neq \varnothing$, i.e. (m, n)-metacirculant graphs with $S_o \neq \varnothing$, each vertex of which has degree 3. We describe connected components and determine automorphism groups for these graphs. We also give an answer to the question: which graphs in this class are Hamiltonian?

2. CONNECTED COMPONENTS

In this section we describe connected components of cubic (m, n)-metacirculant graphs with $S_o \neq \phi$. First we prove the following lemma.

LEMMA 1. *An (m, n)-metacirculant graph $G = MC\ (m, n, \alpha, S_o, S_1, \ldots, S_\mu)$ has $S_o \neq \phi$ and is a cubic graph if and only if one of the following conditions is satisfied:*

1) n is even, $|S_o| = 3$, $S_i = \phi$ for all $i \in \{1, 2, \ldots, \mu\}$;

2) m is odd, n is even, $|S_o| = 1$, $|S_i| = 1$ for some $i \in \{1, 2, \ldots, \mu\}$, $S_j = \phi$ for all $j \neq i$;

3) m is even, n is even, $|S_o| = 1$, $|S_i| = 1$ for some $i \in \{1, 2, \ldots, \mu - 1\}$ and $S_j = S_\mu = \phi$ for $i \neq j \in \{1, 2, \ldots, \mu - 1\}$;

4) m is ven, n is even, $|S_o| = 1$, $|S_\mu| = 2$ and $S_i = \phi$ for all $i \in \{1, 2, \ldots, \mu - 1\}$;

5) m is even, $|S_o| = 2$, $|S_\mu| = 1$ and $S_i = \phi$ for all $i \in \{1, 2, \ldots, \mu - 1\}$.

Proof. Let G be an (m, n)-metacirculant graph $MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ and let degv be the degree of a vertex v of G. It is easy to see that

$$\deg v_0^0 = |S_0| + 2|S_1| + 2|S_2| + \ldots + 2|S_\mu| \tag{1}$$

if m is odd and

$$\deg v_0^0 = |S_0| + 2|S_1| + 2|S_2| + \ldots + 2|S_{\mu-1}| + |S_\mu| \tag{2}$$

if m is even.

Let now $S_0 \neq \phi$ and G be a **cubic graph.** Then we have $\deg v_0^0 = 3$ and $0 \neq |S_0| \leqslant 3$. There are three possibilities.

a) $|S_0| = 3$. Since the subgraph G_0 induced by G on $\{v_0^0, v_1^0, \ldots, v_{n-1}^0\}$ is a circulant graph the with symbol S_0, n has to be even and $\frac{n}{2} \in S_0$. By (1), (2) and $\deg v_0^0 = 3$, we have $S_i = \phi$ for all $i \in \{1, 2, \ldots, \mu\}$. This is just Condition 1 in the lemma.

b) $|S_0| = 2$. From (1) it is easy to see that m cannot be odd. So m is even and from (2) it follows that $|S_\mu| = 1$, $S_i = \varnothing$ for all $i \in \{1, 2, \ldots, \mu-1\}$. This is Condition 5.

c) $|S_0| = 1$. This case can happen only if n is even and $S_0 = \left\{\frac{n}{2}\right\}$, because the subgraph G_0 induced by G on $\{v_0^0, v_1^0, \ldots, v_{n-1}^0\}$ is a circulant graph with the symbol S_0.

If m is odd, then from (1) we have $|S_i| = 1$ for some $i \in \{1, \ldots, \mu\}$, $S_j = \varnothing$ for all $j \neq i$ and Condition 2 follows.

If m is even, then from (2) we have either Condition 3 or Condition 4.

Conversely, if one of Conditions 1-5 is satisfied, then $S_0 \neq \varnothing$ and from (1) or (2) it follows that G is a cubic graph. The lemma is proved.

Now for any integers n and k with $n \geqslant 2$, $1 \leqslant k \leqslant n-1$ we define a generalized Petersen graph $GP(n, k)$ to be a graph with vertex set

$$V(GP(n, k)) = \{u_0, u_1, \ldots, u_{n-1}, y_0, y_1, \ldots, y_{n-1}\}$$

and edge set

$$E(GP(n, k)) = \{(u_i, u_{i+1})\ (u_i, y_i), (y_i, y_{i+k}) | i \in Z_n\},$$

where subscripts are always reduced modulo n.

Thus $GP(n, k)$ is a graph of order $2n$ and $GP(5, 2)$ is the well-known Petersen graph.

We formulate and prove now the main result of the section.

THEOREM 1. *Let G be a cubic (m, n)-metacirculant graph with $S_0 \neq \varnothing$. Then its connected components are isomorphic to each other and to one of the following graphs:*

1) a circulant graph $C(2l, S)$ on $2l$ vertices with the symbol $S = \{1, -1, l\}$, where l is an integer $\geqslant 1$;

2) a generalized Petersen graph $GP(d, k)$, where $k \in Z_d^*$ and $k^2 \equiv \pm 1 \pmod d$.

Proof. Let G be a cubic (m, n)-metacirculant graph with $S_0 \neq \varnothing$. Since G is vertex-transitive, its connected components are isomorphic to each other. Therefore to prove Theorem 1, it is sufficient to show that some of its connected components is isomorphic to one of the graphs of the form 1) or 2).

By Lemma 1, one of Conditions 1-5 is satisfied. We consider all these possibilities in turn.

1) n is even, $|S_0| = 3$, $S_i = \varnothing$ for all $i \in \{1,..., \mu\}$.

Since $0 \notin S_0 = -S_0$ and $|S_0| = 3$, S_0 must have the form $S_0 = \{s, -s, \frac{n}{2}\}$, $0 \neq s \neq \frac{n}{2}$. Let's consider the subgraph G_0 induced by G on $\{v_0^0, v_1^0, ..., v_{n-1}^0\}$. By definition of the (m, n)-metacirculant graph G, the subgraph G_0 is the circulant graph $C(n, S_0)$ on $\{v_0^0, v_1^0, ..., v_{n-1}^0\}$. Then $(v_0^0, v_s^0), (v_s^0, v_{2s}^0)$, (v_{2s}^0, v_{3s}^0),... are edges of G_0. There exists a smallest positive integer d such that $v_{ds}^0 = v_0^0$. Since the permutation

$$\rho : v_j^i \mapsto v_{j+1}^i$$

is an automorphism of G, the permutation ρ_0 on $V(G_0)$ defined by

$$\rho_0 (v_j^0) = \rho(v_j^0) = v_{j+1}^0$$

is an automorphism of G_0. So, if X_0 denotes the cycle

$$v^0_0\, v^0_s\, v^0_{2s} \dots v^0_{(d-1)s}\, v^0_0$$

and $\rho^t_0(X_0)$ denotes the cycle

$$\rho^t_0(v^0_0)\ \rho^t_0(v^0_s)\,\rho^t_0(v^0_{2s})\dots\ \rho^t_0(v^0_{(d-1)s})\,\rho^t_0(v^0_0)\ ,$$

then either

$$\rho^t_0(X_0) \cap \rho^u_0(X_0) = \varnothing$$

or $$\rho^t_0(X_0) = \rho^u_0(X_0)$$

for any integers t, u. Therefore $dx = n$ and d is a divisor of n.

There are two possibilities to consider.

a) $v^0_{\frac{n}{2}}$ is a vertex of the cycle X_0. In this case the subgraph induced by G on $\{v^0_0, v^0_s, v^0_{2s}, \dots, v^0_{(d-1)s}\}$ is isomorphic to a cubic circulant graph $C(d, S)$. Hence $d = 2l$, $l \geqslant 1$, $S = \{1, -1, l\}$ and this subgraph is a connected component of G.

b) $v^0_{\frac{n}{2}}$ is not a vertex of the cycle X_0. Then the subgraph induced by G on $\{v^0_0 v^0_s, v^0_{2s}, \dots, v^0_{(d-1)s}, v^0_{\frac{n}{2}}, v^0_{\frac{n}{2}+s}, v^0_{\frac{n}{2}+2s}, \dots, v^0_{\frac{n}{2}+(d-1)s}\}$ is isomorphic to the generalized Petersen graph $GP(d, 1)$ and is a connected component of G.

2) m is odd, n is even, $|S_0| = 1$, $|S_i| = 1$ for some $i \in \{1, 2, \dots, \mu\}$, $S_j = \varnothing$ for all $j \neq i$.

Since the subgraph induced by G on $\{v^0_0, v^0_1, \dots, v^0_{n-1}\}$ is a circulant graph with the symbol S_0, the set S_0 must be $\left\{\frac{n}{2}\right\}$. Let $S_i = \{s\}$. Then

$$(v^0_0, v^i_s),$$

$$(v^i_s, v^{2i}_{\alpha^i s+s}),$$

$$(v^{2i}_{\alpha^i s+s}, v^{3i}_{\alpha^{2i} s+\alpha^i s+s}), \dots,$$

are edges of G. There exists a smallest positive integer d such that

$$v^{di}_{\alpha^{(d-1)i}s+\alpha^{(d-2)i}s+\dots+\alpha^{i}s+s} = v^{o}_{o}.$$

Hence

$$Y_o = v^o_o\, v^i_s\, v^{2i}_{\alpha^i s+s} \dots v^{(d-1)i}_{\alpha^{(d-2)i}s+\dots+s}\, v^o_o$$

is a cycle of G. Since $|S_i| = 1$ and ρ is an automorphism of G, we have either

$$\rho^t(Y_o) \cap \rho^u(Y_o) = \emptyset$$

or

$$\rho^t(Y_o) = \rho^u(Y_o)$$

for any integers t, u. Therefore $dx = mn$ and d is a divisor of nm.

Here we have also 2 possibilities to consider.

a) $v^o_{\frac{n}{2}}$ is a vertex of the cycle Y_o.

For $i = 0,1,\dots, m-1$, the subgraph G_i induced by G on $\left\{v^i_o, v^i_1, \dots, v^i_{n-1}\right\}$ is a circulant graph with the symbol $\alpha^i S_o$. Moreover, they are isomorphic to each other because $\alpha \in Z^*_n$. Consequently, $\alpha^i S_o = S_o = \left\{\frac{n}{2}\right\}$ for any $i \in Z_m$ and

$$(v^o_o, v^o_{\frac{n}{2}}),$$

$$(v^i_s, v^i_{\frac{n}{2}+s}), \dots,$$

are edges of G. So the subgraph induced by G on

$$\left\{ v^o_o, v^i_s, v^{2i}_{\alpha^i s+s}, \dots, v^o_{\frac{n}{2}}\, v^i_{\frac{n}{2}+s}, v^{2i}_{\frac{n}{2}+\alpha^i s+s}, \dots \right\}$$

is isomorphic to a cubic circulant graph $C(d, S)$. From this we have $d=2l$, $l \geqslant 1$, $S = \{1, -1, l\}$ and this subgraph is a conneted component of G.

b) $v^o_{\frac{n}{2}}$ is not a vertex of the cycle Y_o. Then the subgraph induced by G on

$$\left\{ v^o_o, v^i_s, v^{2i}_{\alpha^i s+s}, \dots, v^{(d-1)i}_{\alpha^{(d-2)i}s+\alpha^{(d-3)i}s+\dots+s}, \right.$$

$$v^{o}_{\frac{n}{2}},\ v^{i}_{\frac{n}{2}+s},\ v^{2i}_{\frac{n}{2}+\alpha^{i}s+s},\ \dots,\ v^{(d-1)i}_{\frac{n}{2}+\alpha^{(d-2)i}s+\alpha^{(d-3)i}s+\dots+s}\Big\}$$

is isomorphic to the generalized Petersen graph $GP\ (d,\ 1)$ and is a connected component of G.

3) m is even, n is even, $|S_o| = 1$, $|S_i| = 1$ for some $i \in \{1, 2, \dots, \mu - 1\}$, $S_j = S_\mu = \varnothing$ for all $i \neq j \in \{1, 2, \dots, \mu - 1\}$.

This case is completely similar to 2.

4) m is even, n is even, $|S_o| = 1$, $|S_\mu| = 2$ and

$$S_i = \varnothing \text{ for all } i \in \{1, 2, \dots, \mu - 1\}.$$

As in 2) we can prove that $S_o = \alpha^{\mu} S_o = \left\{ \frac{n}{2} \right\}$. Let $S_\mu = \{s, r\}$, $s \neq r \pmod n$. Then

$$(v^o_o,\ v^{\mu}_s),$$

$$(v^{\mu}_s,\ v^o_{s-r}),$$

$$(v^o_{s-r},\ v^{\mu}_{2s-r}),$$

$$(v^{\mu}_{2s-r},\ v^o_{2s-2r}),\ \dots$$

are edges of G. Since $|S_\mu| = 2$, there exists a smallest positive integer d such that

$$v^o_{d(s-r)} = v^o_o.$$

Therefore

$$W_o = v^o_o\ v^{\mu}_s\ v^o_{s-r}\ v^{\mu}_{2s-r}\ \dots\ v^o_{d(s-r)}$$

is a cycle of G. It is easy to see that for any integer t and u either.

$$\rho^t(W_o) \cap \rho^u(W_o) = \varnothing$$

or

$$\rho^t(W_o) = \rho^u(W_o)$$

and each vertex from $\left\{ v^o_o,\ v^o_1, \dots, v^o_{n-1},\ v^{\mu}_o,\ v^{\mu}_1, \dots, v^{\mu}_{n-1} \right\}$ belongs to some $\rho^t(W_o)$. Consequently, $2dx = 2n$ and d is a divisor of n.

As in 1) and 2) there are two possibilities to consider.

a) $v^{o}_{\frac{n}{2}}$ is a vertex of W_o. Then

$$W_o = v^o_o \; v^{\mu}_s \; v^o_{s-r} \dots v^o_{\frac{n}{2}} \; v^{\mu}_{\frac{n}{2}+s} \; v^o_{\frac{n}{2}+s-r} \dots$$

Since $S_0 = \alpha^{\mu} S_0 = \left\{ \frac{n}{2} \right\}$, the pairs

$$(v^o_o, \; v^o_{\frac{n}{2}}),$$

$$(v^{\mu}_s, \; v^{\mu}_{\frac{n}{2}+s}),$$

$$(v^o_{s-r}, \; v^o_{\frac{n}{2}+s-r}), \dots,$$

are edges of G. So, the subgraph induced by G on

$$\left\{ v^o_o, \; v^{\mu}_s, \; v^o_{s-r}, \dots, \; v^{\mu}_{ds-(d-1)r} \right\}$$

is isomorphic to a cubic circulant graph $C(2d, S)$ with $S = \{1, -1, d\}$ and so this subgraph is a connected component of G.

b) $v^o_{\frac{n}{2}}$ is not a vertex of the cycle W_o. Since $S_0 = \alpha^{\mu} S_0 = \left\{ \frac{n}{2} \right\}$, the subgraph induced by G on the vertices $v^o_o, v^{\mu}_s, v^o_{s-r}, \dots, v^{\mu}_{ds-(d-1)r}, v^o_{\frac{n}{2}}, v^{\mu}_{\frac{n}{2}+s}, v^o_{\frac{n}{2}+s-r}, \dots, v^{\mu}_{\frac{n}{2}+ds-(d-1)r}$ is isomorphic to the generalized Petersen graph $GP(2d, 1)$ and is a connected component of G.

5) m is even, $|S_o| = 2$, $|S_{\mu}| = 1$ and $S_i = \emptyset$ for all $i \in \{1, 2, \dots, \mu - 1\}$.

Let $S_o = \{s, -s\}$, $s \neq -s \neq 0$ and $S_{\mu} = \{r\}$. Then

$$(v^o_o, \; v^o_s), (v^o_s, v^o_{2s}), (v^o_{2s}, v^o_{3s}), \dots$$

are edges of G. Let d be the smallest positive integer such that $v^o_{ds} = v^o_o$. Then

$$Z_o = v^o_o \; v^o_s \; v^o_{2s} \dots v^o_{(d-1)s} \; v^o_o$$

is a cycle. For any integers t, u either

$$\rho^t(Z_o) \cap \rho^u(Z_o) = \phi$$

or

$$\rho^t(Z_o) = \rho^u(Z_o),$$

and any vertex from $\{v_0^o, v_1^o, \dots, v_{n-1}^o\}$ is in some $\rho^t(Z_o)$. So $dx = n$ and d is a divisor of n.

Let $k = \alpha^{\mu}$. Then

$$(v_r^{\mu}, v_{r+ks}^{\mu}),$$

$$(v_{r+ks}^{\mu}, v_{r+2ks}^{\mu}),$$

$$(v_{r+2ks}^{\mu}, v_{r+3ks}^{\mu}),$$

$$\dots\dots$$

are edges of G. Since d is the smallest positive integer such that $ds \equiv 0 \pmod n$, the walk

$$v_r^{\mu}\ v_{r+ks}^{\mu}\ v_{r+2ks}^{\mu} \dots v_{r+(d-1)ks}^{\mu}\ v_{r+dks}^{\mu}$$

is a cycle of length d, too.

We have

$$\{0, ks, 2ks, \dots, (d-1)ks\} \subseteq \{0, s, 2s, 3s, \dots, (d-1)s\}$$

(all numbers are considered modulo n), because

$$\{0, s, 2s, 3s, \dots, (d-1)s\}$$

is the set of all multipliers of s modulo n. So

$$|\{0, ks, 2ks, \dots, (d-1)ks\}| = |\{0, s, 2s, 3s, \dots, (d-1)s\}| = d$$

implies

$$\{0, ks, 2ks, \dots, (d-1)ks\} = \{0, s, 2s, 3s, \dots, (d-1)s\}$$

and $\{v_0^0, v_s^0, v_{2s}^0, \dots, v_{(d-1)s}^0, v_r^{\mu}, v_{r+ks}^{\mu}, v_{r+2ks}^{\mu}, \dots, v_{r+(d-1)ks}^{\mu}\}$

$$= \{v_0^0, v_s^0, v_{2s}^0, \dots, v_{(d-1)s}^0, v_r^{\mu}, v_{r+s}^{\mu}, v_{r+2s}^{\mu}, \dots, v_{r+(d-1)s}^{\mu}\}.$$

Further, we consider k as an element of the ring Z_d. Since $k = \alpha^{\mu} \in Z_n^*$; it is relatively prime to n. Therefore k is relatively prime to d because d is a divisor of n. It means that $k \in Z_d^*$.

Let G_o be the subgraph induced by G on

$$\left\{ v_0^0, v_s^0, v_{2s}^0, \ldots, v_{(d-1)s}^0, v_r^\mu, v_{r+s}^\mu, v_{r+2s}^\mu, \ldots, v_{r+(d-1)s}^\mu \right\} = V(G_o)$$

and let f be the following mapping from $V(G_o)$ to $V(GP(d,k))$:

$$f(v_{is}^0) = u_i$$

and
$$f(v_{r+is}^\mu) = y_i \quad \text{for all } i \in Z_d .$$

It is easy to verify that f is an isomorphism of the graphs G_0 and $GP(d,k)$ and G_0 is a connected component of G.

By definition, the graph G is vertex-transitive, i. e. its automorphism group is a transitive permutation group on its vertex set. So if G_0 is a connected component of G, then it is a vertex transitive graph, too. Therefore, $GP(d, k)$ is a vertex-transitive graph. Using the results in [3] and noting that $k \in Z_d^*$ we conclude $k^2 \equiv \pm 1 \pmod d$.

Theorem 1 is completely proved.

3. EXISTENCE OF HAMILTONIAN CYCLES

In this section we give an answer on the question: which cubic (m, n) — metacirculant graphs with $S_o \neq \emptyset$ are Hamiltonian?

THEOREM 2. *A connected cubic (m,n)-metacirculant graph G with $S_o \neq \emptyset$ is Hamiltonian if and only if G is not isomorphic to the Petersen graph $GP(5,2)$.*

Proof. Let G be a connected cubic (m, n)-metacirculant graph with $S_o \neq \emptyset$. By Theorem 1, the graph G must be isomorphic either to a circulant graph $C(2l, S)$ on $2l$ vertices with the symbol $S = \{1, -1, l\}$, where l is an integer ≥ 1 or to a generalized Petersen graph $GP(d, k)$, where $k \in Z_d^*$ and $k^2 \equiv \pm 1 \pmod d$.

Suppose that G is Hamiltonian. It is clear that G can not be isomorphic to the Petersen graph $GP(5,2)$ because it is well-known that the Petersen graph $GP(5,2)$ is not Hamiltonian.

Conversely, suppose that G is not isomorphic to the Petersen graph $GP(5,2)$. If G is isomorphic to a circulant graph $C(2l, S)$ with $S = \{1, -1, l\}$, then of course G is Hamiltonian. If G is isomorphic to a generalized Petersen graph $GP(d,k)$, $k \in Z_d^*$ and $k^2 \equiv \pm 1 \pmod d$, then by the results obtained in [2], G is Hamiltonian if and only if the ordered pair (d,k) is not among the following ones:

(i) $(n, 2)$, $(n, n-2)$, $\left(n, \frac{n-1}{2}\right)$, $\left(n, \frac{n+1}{2}\right)$, where $n \equiv 5 \pmod 6$;

(ii) $\left(n, \frac{n}{2}\right)$, where $n \equiv 0 \pmod 4$, $n \geqslant 8$.

Among the pairs (d, k) listed in (i) and (ii), only the pairs $(5,2)$ and $(5,3)$ can satisfy the conditions $k \in Z_d^*$, $k^2 \equiv \pm 1 \pmod d$. It is easy to see that $GP(5,3)$ is isomorphic to $GP(5,2)$. Hence, if G is isomorphic to a generalized Petersen graph $GP(d,k)$, $k \in Z_d^*$, $k^2 \equiv \pm 1 \pmod d$ and $(d,k) \neq (5,2), (5,3)$, then G is Hamiltonian. Theorem 2 is proved.

4. AUTOMORPHISM GROUPS

In this section we describe automorphism groups of cubic (m, n)-metacirculant graphs with $S_0 \neq \varnothing$. The following result shows that the above problem can be reduced to the case of connected graphs.

THEOREM 3. *If k is the number of connected components of a vertex-transitive graph G and C is one of them. Then the automorphism group Aut G of G is the wreath product of the automorphism group Aut C of C with the symmetric group S_k of degree k.*

Proof. Since G is a vertex-transitive graph, all the connected components of G are isomorphic to each other. Hence the automorphism group Aut G is the wreath product of Aut C with the symmetric group S_k of degree k ([4], Theorem 14.5). The theorem is proved.

In view of the above theorem, in order to describe the automorphism group of a vertex-transitive graph G, in particular, that of a cubic (m, n)-metacirculant graph with $S_0 \neq \varnothing$, it is sufficient to describe the automorphism group of any connected component C of G.

If $a, b, c, \ldots$ are permutations on the vertex set $V(G)$ of a graph G, then we write $\langle a, b, c, \ldots \rangle$ for the subgroup generated by $a, b, c, \ldots$ in the symmetric group of $V(G)$.

THEOREM 4. *If l is an integer $\geqslant 4$ and G is the circulant graph $C(2l, S)$ on $2l$ vertices with the symbol $S = \{1, -1, l\}$, then Aut G is the dihedral group $D_{2l} = \langle a, b \rangle$, where a and b are the following permutations on the vertices of G:*

$$a = (v_0, v_1, \dots, v_{2l-1}),$$
$$b = (v_0, v_{2l-1})(v_1, v_{2l-2}) \dots (v_{l-1}, v_l).$$

Proof. Let $v_0, v_1, \dots, v_{2l-1}$ be vertices of G and G' be the circulant graph on $\{v_0, v_1, \dots, v_{2l-1}\}$ with the symbol $S' = \{1, -1\}$. It is well-known that

$$\mathrm{Aut}\, G' = D_{2l} = \langle a, b \rangle.$$

Since G is a circulant graph on $\{v_0, v_1, \dots, v_{2l-1}\}$, the permutation a is also an automorphism of G. We have

$$b(v_j) = v_{2l-(j+1)}$$

for any $j \in Z_n$. Consequently, if some edge of G is of the form (v_j, v_{j+l}), then

$$b(v_j, v_{j+l}) = (v_{2l-(j+1)}, v_{2l-(j+l+1)}) = (v_{2l-(j+1)}, v_{l-(j+1)})$$

is also an edge of G. This means that b is an automorphism of G and

$$D_{2l} = \mathrm{Aut}\, G' \leqslant \mathrm{Aut}\, G.$$

Since $l \geqslant 4$, the girth of G is 4 and all shortest cycles of G have the form

$$v_k\ v_{k+1}\ v_{k+1+l}\ v_{k+l}\ v_k,$$

where $k \in Z_n$. If some edge (v_k, v_{k+1}) is transformed into $(v_{k'}, v_{k'+l})$ by an automorphism α of G, then either

$$\alpha(v_k) = v_{k'}\ ,\ \alpha(v_{k+1}) = v_{k'+l}$$

or

$$\alpha(v_k) = v_{k'+l}\ ,\ \alpha(v_{k'+1}) = v_{k'}.$$

We can suppose without loss of generality that

$$\alpha(v_k) = v_{k'}\ ,\ \alpha(v_{k+1}) = v_{k'+l}.$$

There are two cases to consider.

1) $\alpha(v_{k+l}) = v_{k'+1}$. Then $\alpha(v_{k-1}) = v_{k'-1}$. Hence

$$\alpha(v_{k-1}\, v_k\, v_{k+l}\, v_{k-1+l}\, v_{k-1}) = v_{k'-1}\, v_{k'}\, v_{k'+1}\, \alpha(v_{k-1+l})\, v_{k'-1}$$

which is not a cycle if $l \geqslant 4$. This is impossible because α is an automorphism of G.

2) $\alpha(v_{k+l}) = v_{k'-1}$. Then $\alpha(v_{k-1}) = v_{k'+1}$ and we can obtain a contradiction as in 1).

Thus if α is an automorphism of G, then

$$\alpha(v_k, v_{k+1}) = (v_{k'}, v_{k'+1})$$

for any $k \in Z_n$, i. e. α maps edges of G' onto edges of G'. Hence α is an automorphism of G' and Aut $G \leqslant$ Aut G'.

Consequently Aut G = Aut G' = D_{2l} and Theorem 4 is proved.

THEOREM 5. *If G is the circulant graph $C(6,S)$ on vertices $v_0, v_1, v_2, v_3, v_4, v_5$ with the symbol $S = \{1, -1, 3\}$ and a, b, c are the following permutations on vertices:*

$$a = (v_0, v_1, v_2, v_3, v_4, v_5),$$

$$b = (v_0, v_5)(v_1, v_4)(v_2, v_3),$$

$$c = (v_1, v_5),$$

then Aut $G = < a, b, c >$.

If G is the circulant graph $C(4, S)$ with $S = \{1,-1,2\}$ or G is the circulant graph $C(2, S)$ with $S = \{1,-1\}$, then Aut G is a symmetric group on its vertices.

Proof. As in Theorem 4, the group $D_6 = \langle a, b\rangle$ is in Aut G. If some edge (v_k, v_{k+1}) is transformed into $(v_{k'}, v_{k'+3})$ by an automorphism α' of G, then we have either

$$\alpha'(v_k) = v_{k'}, \; \alpha'(v_{k+1}) = v_{k'+3}$$

or

$$\alpha'(v_k) = v_{k'+3}, \; \alpha'(v_{k+1}) = v_{k'} .$$

We can suppose without loss of generality that

$$\alpha'(v_k) = v_{k'}, \; \alpha'(v_{k+1}) = v_{k'+3} .$$

Since $a \in$ Aut G, $a^k(v_0) = v_k$ and $a^{-k'}(v_{k'}) = v_0$, the permutation $\alpha = a^{-k'} \alpha' a^k$ is an automorphism of G and

$$\alpha(v_0) = v_0 , \; \alpha(v_1) = v_3 .$$

There are two cases to consider:

1) $\alpha(v_3) = v_1$. Then $\alpha(v_5) = v_5$.

If $\alpha(v_4) = v_4$, then $\alpha(v_2) = v_2$ and $\alpha = (v_1, v_3)$.

If $\alpha(v_4) = v_2$, then $\alpha(v_2) = v_4$ and

$$\alpha = (v_1, v_3)(v_2, v_4) .$$

2) $\alpha(v_3) = v_5$. Then $\alpha(v_5) = v_1$.

If $\alpha(v_4) = v_4$, then $\alpha(v_2) = v_2$ and

$$\alpha = (v_1, v_3, v_5).$$

If $\alpha(v_4) = v_2$, then $\alpha(v_2) = v_4$ and

$$\alpha = (v_1, v_3, v_5)(v_2, v_4).$$

It is easy to see that in both cases $\alpha \in \langle a, b, c \rangle$. Hence $\alpha \in \langle a, b, c \rangle$ and Aut $G \leqslant \langle a, b, c \rangle$. But $\langle a, b, c \rangle \leqslant$ Aut G. So Aut $G = \langle a, b, c \rangle$.

The last assertions are trivial. Theorem 5 is proved.

THEOREM 6. *Let G be a generalized Petersen graph GP (d, k) where* $k \in Z_d^*$, $k^2 \equiv \pm 1 \pmod d$ *and*

(*i*) *if the ordered pair (d, k) is not among* (4,1), (4,3), (5,2), (5,3), (8,3), (8,5), (10,3), (10,7), (12,5), (12,7), (24,5), (24,19), *and if* $k^2 \equiv 1 \pmod d$, *then*

$$\text{Aut } G = \langle \rho, \delta, \tau \rangle, \text{ where}$$

$$\rho(u_i) = u_{i+1}, \qquad \rho(y_i) = y_{i+1}, \text{ all } i,$$

$$\delta(u_i) = u_{-i}, \qquad \delta(y_i) = y_{-i}, \text{ all } i,$$

$$\tau(u_i) = y_{ki}, \qquad \tau(y_i) = u_{ki}, \text{ all } i;$$

(*ii*) *if the ordered pair (d, k) is as in (i) and if* $k^2 \equiv -1 \pmod d$, *then Aut* $G = \langle \rho, \tau \rangle$, *where* ρ *and* τ *are the permutations defined in (i).*

(*iii*) *if G is one of GP (4,1), GP (4,3), GP (8,3), GP (8,5), GP (12,5), GP (12,7) GP (24,5), GP (24,19), then Aut* $G = \langle \rho, \delta, \sigma \rangle$, *where* ρ, δ *are as in (i) and*

$$\sigma(u_{4i}) = u_{4i}, \qquad \sigma(y_{4i}) = u_{4i+1},$$

$$\sigma(u_{4i+1}) = u_{4i-1}, \qquad \sigma(u_{4i-1}) = y_{4i},$$

$$\sigma(u_{4i+2}) = y_{4i-1}, \qquad \sigma(y_{4i-1}) = y_{4i+5};$$

$$\sigma(y_{4i+1}) = u_{4i-2}, \qquad \sigma(y_{4i+2}) = y_{4i-6}, \text{ all } i,$$

(*iv*) *if G is GP (10,3) or GP (10,7), then Aut* $G = \langle \rho, \mu \rangle$, *where* ρ *is as in (i) and*

$$\mu = (u_2, y_1)(u_3, y_4)(u_7, y_6)(u_8, y_9)(y_2, y_8)(y_3, y_7):$$

(*v*) *if G is GP (5,2) or GP (5,3), then Aut* $G = \langle \rho, \mu \rangle$, *where* ρ *is as in (i) and*

$$\mu = (u_2, y_1)(u_3, y_4)(y_2, y_3).$$

The proof of these assertions can be found in [3].

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