ACTA MATHEMATICA VIETNAMICA

Vol. 15 No 2 (1990)

ON THE FOLIATIONS FORMED BY THE GENERIC K-ORBITS OF THE MD4-GROUPS

LE ANH VU

INTRODUCTION

In general, the leaf space of a foliation with the quotient topology is a fairly untractable topological space. To improve upon this shortcoming, A. Connes [2] associates to each foliation a C^*-algebra. And, in the case of Reeb foliations (see [11]), the method of K-functors first used by D.N. Diep [3], has been is proved very effective in describing the Connes' C^*-algebras

On the other hand, the Kirillov's Orbit Method allows us to obtain foliations by the generic K-orbits of solvable Lie groups. Combining these methods of Kirillov and Connes, we consider the foliations formed by the maximal-dimensional K-orbits of all indecomposable MD4-groups.

In [12] and [13] we have considered a special interesting case, of the real diamond foliation associated to the real diamond group $\mathbb{R}.H_3$. The present paper is concerned with a similar problem for the remaining indecomposable *MD4* – groups and is a detailed exposition of the results submitted for publication in Comptes Rendus Acad. Sci. Paris.

An *MD4*—group in terms of D.N. Diep is a four-dimensional solvable Lie group whose orbits in the coadjoint representation (i.e. the *K*-representation) are the orbits of zero or maximal dimension. The corresponding Lie algebra is also called an MD4-algebra. Let us recall that a Lie algebra $\mathcal{G}$ is decomposable if $\mathcal{G}$ is the direct product $\mathcal{G}_1 \times \mathcal{G}_2$ of its nontrivial ideals.

We begin our discussion in Section §1 by giving a complete classification of all MD4-algebras. This result is a complement of an unpublished paper of D.V. Tra.

In Section §2 we describe geometrically the K-orbits of all indecomposable MD4-groups.

Section §3 is devoted to the discussion of the foliations formed by the maximal-dimensional *K*-orbits of all indecomposable *MD4*-groups. These foliations are also called *MD4*-foliations. We shall also a topological classification of all *MD4*-foliations.

Finally in Section §4 we give a characterization of the Connes' C^*-algebras associated to all MD4-foliations.

The author expresses his deep gratitude to Dr. D.N. Diep and Dr. Ho Huu Viet for suggesting this study and for many helpful discussions. It is a particular pleasure for the author to acknowledge many valuable comments of Prof. P. Cartier. The author would also like to thank Dr. D.V. Tra for helpful suggestions.

§1. THE CLASSIFICATION OF THE MD4-ALGEBRAS.

The MD-algebras have been first considered by V.M. Son and H.H. Viet [9]. Who have given a necessary condition for a solvable Lie algebra to be an MD-algebra and have considered an interesting example of the real diamond algebra (see [9, Sect. 4]).

In 1984, D.V. Tra made a preliminary classification of the MD4-algebras, which was been carefully discussed at the Seminar on Harmonic Analysis and announced in the Annual Scientific Conference at the Hanoi Institute of Mathematics in November, 1984.

In this section we complete Tra's result by a final classification of all MD4-algebras.

First let us recall Tra's result. Let $\mathcal{G}$ be an MD4-algebra with basis $\{X, Y, Z, T\}$ and $\mathcal{G}^1 = [\mathcal{G}, \mathcal{G}]$, $\mathbb{R}^n$ be the commutative Lie algebra of dimension n, Tra has proved the following:

PROPOSITION 1. I. Assume that $\mathcal{G}$ is decomposable. Then $\mathcal{G} = \mathbb{R}^n \times \tilde{\mathcal{G}}$, where $\tilde{\mathcal{G}}$ is an indecomposable ideal of $\mathcal{G}$, $1 \leqslant n \leqslant 4$.

II. Assume that $\mathcal{G}$ is indecomposable. Then $\mathcal{G}$ is isomorphic to one of the following Lie algebra

1. $\mathcal{G}^1 = \text{gen}(Z) \cong \mathbb{R}$, $[X, Y] = aZ$, $[Y, T] = bZ$, $[T, X] = cZ$, $[X, Z] = xZ$, $[Y, Z] = yZ$, $[T, Z] = tZ$, where $a,b,c,x,y,t \in \mathbb{R}$, $a^2 + b^2 + c^2 + x^2 + y^2 + t^2 \neq 0$ and $at + bx + cy = 0$.

2. $\mathcal{G}^1 = \text{gen}(Y, Z) \cong \mathbb{R}^2$, $[T, X] = 0$

2.1. $ad_T \in \text{Aut}_{\mathbb{R}}\, \mathcal{G}^1 \cong GL_2(\mathbb{R})$, $ad_X = \alpha ad_T, \alpha \in \mathbb{R}$

2.2. $\mathcal{G} = \text{Lie}(Aff\, \mathbb{C})$: the complex affine Lie algebra.

3. $\mathcal{G}^1 = \text{gen}(X, Y, Z) \cong \mathbb{R}^3$, $ad_T \in Aut_{\mathbb{R}}\, \mathcal{G}^1 \cong GL_3(\mathbb{R})$

4. $\mathcal{G}^1 = \text{gen}(X, Y, Z = \text{Lie}(H_3)$: the 3-dimensional Heisenberg algebra,

$$\mathrm{ad}_T = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & -a_{11} & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} \in \mathrm{End}_{\mathbb{R}}\, \mathcal{G}^1 \cong \mathrm{Mat}_3(\mathbb{R})$$

where $a_{11}^2 + a_{12}\, a_{21} \neq 0$.

The main result of this section is the following complete classification

THEOREM 1. **Under the above notation,**

I. Assume that $\mathcal{G}$ *is decomposable, then* $\mathcal{G} = \mathbb{R}^n \times \widetilde{\mathcal{G}}$, *where* $\widetilde{\mathcal{G}}$ *is an indecomposable ideal of* $\mathcal{G}$, $1 \leqslant n \leqslant 4$.

II. Assume that $\mathcal{G}$ *is indecomposable, then* $\mathcal{G}$ *is isomorphic to one of the following Lie algebras:*

1. $\mathcal{G}^1 = gen(Z) \cong \mathbb{R}$.

1.1. $\mathcal{G}_{4,1,1}$: $[T, X] = Z$, $[T, Y] = [T, Z] = [X, Y] = [X, Z] = [Y, Z] = 0$.

1.2. $\mathcal{G}_{4,1,2}$: $[T, Z] = Z$, $[T, X] = [T, Y] = [X, Y] = [X, Z] = [Y, Z] = 0$.

2. $\mathcal{G}^1 = gen\ (Y, Z) \cong \mathbb{R}^2$, $[T, X] = 0$, $ad_X \in End_{\mathbb{R}}\, \mathcal{G}^1 \cong Mat_2\ (\mathbb{R})$, *and* $ad_T \in Aut_{\mathbb{R}}\, \mathcal{G}^1 \cong GL_2\ (\mathbb{R})$.

2.1. $\mathcal{G}_{4,2,1(\lambda)}$: $ad_X = 0$, $ad_T = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, $\lambda \in \mathbb{R}^*$

2.2. $\mathcal{G}_{4,2,2}$: $ad_X = 0$, $ad_T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

2. 3 $\mathcal{G}_{4,2,3(\varphi)}$: $ad_X = 0$, $ad_T = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$, $\varphi \in (0, \pi)$

2.4. $\mathcal{G}_{4,2,4} = Lie(Aff\,C)$: $ad_X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $ad_T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3. $\mathcal{G}^1 = gen\ (X, Y, Z) \cong \mathbb{R}^3$, $ad_T \in Aut_{\mathbb{R}}\, \mathcal{G}^1 \cong GL_3(\mathbb{R})$

3.1. $\mathcal{G}_{4,3,1}(\lambda_1, \lambda_2)$: $ad_T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\lambda_1, \lambda_2 \in \mathbb{R}^*$

3.2. $\mathcal{G}_{4,3,2(\lambda)}$: $ad_T = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\lambda \in \mathbb{R}^*$

3.3. $\mathcal{G}_{4,3,3}$: $ad_T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

3.4. $\mathcal{G}_{4,3,4}(\lambda,\varphi)$: $ad_T \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & \lambda \end{bmatrix}$, $\lambda \in \mathbf{R}^*$, $\varphi \in (0,\pi)$

4. $\mathcal{G}^1 = gen(X,Y,Z) \cong Lie(H_3)$, $ad_T \in \underset{\mathbf{R}}{End}\, \mathcal{G}^1 \cong Mat_3(\mathbf{R})$.

4.1. $\mathcal{G}_{4,4,1}$: $ad_T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4.2. $\mathcal{G}_{4,4,2} = Lie(\mathbf{R}.H_3)$

(*the real diamond algebra*) : $ad_T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Remark 1. For convenience, from now on each indecomposable simply connected *MD4*-groups is also denoted by the same indices as the corresponding *MD4*-algebra.

For example, $G_{4,4,1}$ is the simply connected *MD4*-group corresponding to $\mathcal{G}_{4,4,1}$.

Proof of Theorem 1. Clearly it suffices to prove Part II of the theorem.

1. Assume that $\mathcal{G}$ is the algebra given in Proposition 1, II.1, i. e. $\mathcal{G} = \text{gen}(X, Y, Z, T)$ with $[X, Y] = aZ$, $[Y, T] = bZ$, $[T, X] = cZ$, $[X, Z] = xZ$, $[Y, Z] = yZ$, $[T, Z] = tZ$, $a^2 + b^2 + c^2 + x^2 + y^2 + t^2 \neq 0$ and $at + bx + cy = 0$.

Let $t \neq 0$. By changing T by $T' = \frac{1}{t}T$, X by $X' = X - \frac{x}{t}T - \frac{1}{t}cZ$, $Y' = Y - \frac{y}{t}T + \frac{1}{t}bZ$, we get $[T', Z] = Z$, $[T', X'] = [T', Y'] = [X', Y'] = [X', Z] = [Y', Z] = 0$. Hence $\mathcal{G} \cong \mathcal{G}_{4,1,2}$.

The same arguments show that $\mathcal{G} \cong \mathcal{G}_{4,1,2}$ if $x^2 + y^2 + t^2 \neq 0$ and $\mathcal{G} = \mathcal{G}_{4,1,1}$ if $x = y = t = 0$, $a^2 + b^2 + c^2 \neq 0$.

2. Assume that $\mathcal{G}$ is the algebra given in Proposition 1, II. 2 and $\mathcal{G}$ is not Lie (AffC) $\cong \mathcal{G}_{4,2,4}$. Then $\mathcal{G} = \text{gen}(X, Y, Z, T)$ with $\mathcal{G}^1 = \text{gen}(Y, X) \cong \mathbf{R}^2$, $[T, X] = 0$, $ad_T \in \text{Aut}_{\mathbf{R}}\mathcal{G}^1 \cong GL_2(\mathbf{R})$, $ad_X = \alpha ad_T$, $\alpha \in \mathbf{R}$. By changing X by $X' = X - \alpha T$ if necessary, we may assume that $ad_X = 0$. By elementary transformations of matrices, we get the similar classification of ad_T as follows:

$\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}$, $\lambda \in \mathbf{R}^*$, $\varphi \in (0, \pi)$. Thus $\mathcal{G}$ is isomorphic to one of the algebras $\mathcal{G}_{4,2,1(\lambda)}$, $\mathcal{G}_{4,2,2}$ and $\mathcal{G}_{4,2,3(\varphi)}$, $\lambda \in \mathbf{R}^*$, $\varphi \in (0, \pi)$.

The remaining assertions of the theorem can be obtained in a similar way.

§2. THE GEOMETRIC PICTURE OF THE K-ORBITS OF THE INDECOMPOSABLE SIMPLY CONNECTED MD4—GROUPS.

By Theorem 1, the study of the decomposable $MD4$-groups can be directly reduced to the cases of the MD—groups of dimension 3 or less. Hence, from now on, we are concerned with the indecomposable simply connected $MD4$—groups. In this section we describe geometrically the K—orbits of all such groups.

Our method is analogous to that used in the case of the real diamond group $\mathbf{R}.H_3$ (see [12]). For each $MD4$-group G, we denote by Ω_F the K-orbit including F of the dual space $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$ corresponding to G. It should be noted that for $G \neq G_{4,2,3}(\frac{\pi}{2})$, $G_{4,2,4}$, $G_{4,3,4(\lambda, \frac{\pi}{2})}$, $G_{4,4,1}$, $\exp : \mathcal{G} \to G$ is surjective (see [8]), hence Ω_F can be given by:

$$\Omega_F = \{F_U \in \mathcal{G}^*, U \in \mathcal{G}\}. \tag{2.1}$$

where F_U is the linear form on $\mathcal{G}$ as follows:

$$\langle F_U , X\rangle = \langle F, \exp(\mathrm{ad}_U)(X)\rangle, X \in \mathcal{G}, U \in \mathcal{G}. \tag{2.2}$$

For the remaining groups, (2.1) is easily verified by direct computations. The dual space $\mathcal{G}^*$ of $\mathcal{G} = \mathrm{gen}\,(X, Y, Z, T)$ can be identified to $\mathbf{R}^4$ by means of the dual basis $\{X^*, Y^*, Z^*, T^*\}$. Unless otherwise stated, throughout this section, F always denotes a point $(\alpha, \beta, \gamma, \delta)$ in the $\mathcal{G}^* \cong \mathbf{R}^4$.

As an application of (2.1) and (2.2) to the indecomposabe and simply connected $MD4$-groups, we obtain the following results the proof of which is analogous to the case of $\mathbf{R}.H_3$ (see [12]) and is therefore omited.

THEOREM 2. (The picture of the K—orbits)

1. $G = G_{4,1,1}$.

(i) $\gamma = 0$ *then* $\Omega_F = \{F\}$. *(the 0-dimensional orbit)*

(ii) $\gamma \neq 0$ *then* $\Omega_F = \{(x, \beta, \gamma, t), x, t \in \mathbf{R}\}$ *: a plane.*

(the 2-dimensional orbit)

2. $G = G_{4,1,2}$.

(i) $\gamma = 0$ then $\Omega_F = \{F\}$, (the 0-dimensional orbit)

(ii) $\gamma \neq 0$ then $\Omega_F = \{(\alpha, \beta, z, t), \gamma z > 0\}$: a half-plane.

(the 2-dimensional orbit)

3. G is one of $G_{4,2,1}(\lambda)$, $G_{4,2,2}$, $\lambda \in \mathbb{R}^*$.

(i) $\beta = \gamma = 0$ then $\Omega_F = \{F\}$. (the o-dimensional orbit)

(ii) $\beta^2 + \gamma^2 \neq 0$ then

$$\Omega_F = \begin{cases} \{(\alpha, \beta e^{s\lambda}, \gamma e^s, t), s, t \in \mathbb{R}\} \text{ when } G = G_{4,2,1,(\lambda)}, \lambda \in \mathbb{R}^*, \\ \{(\alpha, \beta e^s, \beta s e^s + \gamma e^s, t), s, t \in \mathbb{R}\} \text{ when } G = G_{4,2,2}, \end{cases}$$

(a vertical cylinder) (the 2-dimensional orbit)

4. $G = G_{4,2,3(\varphi)}$, $\varphi \in (0, \pi)$. Let us identify $\mathcal{G}^*_{4,2,3(\varphi)}$ with $\mathbb{R} \times \mathbb{C} \times \mathbb{R}$, F with $(\alpha, \beta + i\gamma, \delta) \in \mathbb{R} \times \mathbb{C} \times \mathbb{R}$, $\varphi \in (0, \pi)$.

(i) $\beta + i\gamma = 0$ then $\Omega_F = \{F\}$. (the 0-dimensional orbit)

(ii) $\beta + i\gamma \neq 0$ then $\Omega_F = \{(\alpha, (\beta + i\gamma)\, e^{se^{i\varphi}}, t)\, s, t \in \mathbb{R}\}$: a vertical cylinder.

(the 2-dimensional orbit)

5. $G = G_{4,2,4} = \widetilde{Aff\,\mathbb{C}}$.

(i) $\beta = \gamma = 0$ then $\Omega_F = \{F\}$. (the 0-dimensional orbit)

(ii) $\beta^2 + \gamma^2 \neq 0$ then $\Omega_F = \{(x, y, z, t), y^2 + z^2 \neq 0\} \cong \mathbb{R} \times (\mathbb{R}^2)^* \times \mathbb{R}$.

(the unique 4 dimensional orbit)

6. G is one of $G_{4,3,1(\lambda_1, \lambda_2)}$, $G_{4,3,2(\lambda)}$, $G_{4,3,3}$, $\lambda_1, \lambda_2, \lambda \in \mathbb{R}^*$.

(i) $\alpha = \beta = \gamma = 0$ then $\Omega_F = \{F\}$, (the o-dimensional orbit)

(ii) $\alpha^2 + \beta^2 + \gamma^2 \neq 0$ then

$$\Omega_F = \begin{cases} \{(\alpha e^{s\lambda_1}, \beta e^{s\lambda_2}, \gamma e^s, t), s, t \in \mathbb{R}\} \text{ when } G = G_{4,3,1,(\lambda_1,\lambda_2)}, \lambda_1, \lambda_2 \in \mathbb{R}^* \\ \{(\alpha e^{s\lambda}, \alpha s e^{s\lambda} + \beta e^{s\lambda}, \gamma e^s, t), s, t \in \mathbb{R}\} \text{ when } G = G_{4,3,2(\lambda)}, \lambda \in \mathbb{R}^* \\ \{(\alpha e^s, \alpha s e^s + \beta e^s, \frac{1}{2}\alpha s^2 e^s + \beta s e^s + \gamma e^s, t), s, t \in \mathbb{R}\} \text{ when } G = G_{4,3,3} \end{cases}$$

(a vertical cylinder). (the 2-dimensional orbit)

7. $G = G_{4,3,4(\lambda,\varphi)}$, $\lambda \in \mathbb{R}^*$, $\varphi \in (0,\pi)$. *Let us identify* $\mathcal{G}^*_{4,3,4(\lambda,\varphi)}$ *with* $\mathbb{C} \times \mathbb{R}^2$ *and* F *with* $(\alpha + i\beta, \gamma, \delta)$, $\lambda \in \mathbb{R}^*$, $\varphi \in (0,\pi)$.

(i) $\alpha + i\beta = \gamma = 0$ *then* $\Omega_F = \{F\}$. *(the 0-dimensional orbit)*

(ii) $|\alpha + i\beta|^2 + \gamma^2 \neq 0$ *then* $\Omega_F = \{((\alpha + i\beta)e^{se^{i\varphi}}, \gamma e^{s\lambda}, t),\ s, t \in \mathbb{R}\}$ *(a vertical cylinder)* *(the 2-dimensional orbit)*

8. $G = G_{4,4,1}$.

(i) $\alpha = \beta = \gamma = 0$ *then* $\Omega_F = \{F\}$. *(the 0-dimensional orbit)*

(ii) $\alpha^2 + \beta^2 \neq 0 = \gamma$ *then* $\Omega_F = \{(x, y, 0, t), x^2 + y^2 = \alpha^2 + \beta^2\}$: *a vertical cylinder of revolution* *(the 2-dimensional orbit)*

(iii) $\gamma \neq 0$ *then* $\Omega_F = \{(x, y, \gamma, t), x^2 + y^2 - \alpha^2 - \beta^2 = 2\gamma(t - \delta)\}$: *a paraboloid of revolution.* *(the 2-dimensional orbit)*

9. $G = G_{4,4,2} = \mathbb{R} \cdot H_3$

(i) $\alpha = \beta = \gamma = 0$ *then* $\Omega_F = \{F\}$. *(the 0-dimensional orbit)*

(ii) $\alpha = 0 \neq \beta$, $\gamma = 0$ *then* $\Omega_F = \{(0, y, 0, t), \beta y > 0\}$: *a coordinate half-plane.* *(the 2-dimensional orbit)*

(iii) $\alpha \neq 0 = \beta = \gamma$ *then* $\Omega_F = \{(x, 0, 0, t), \alpha x > 0\}$: *a coordinate half-plane.* *(the 2-dimensihnal orbit)*

(iv) $\alpha\beta \neq 0 = \gamma$ *then* $\Omega_F = \{(x, y, 0, t), xy = \alpha\beta, \alpha\beta, \alpha x > 0, \beta y > 0\}$: *a vertical hyperbolic cylinder.* *(the 2-dimensional orbit)*

(v) $\gamma \neq 0$ *then* $\Omega_F = \{(x, y, \gamma, t), xy - \alpha\beta = \gamma(t - \delta)\}$: *a hyperbolic paraboloid.* *(the 2-dimensional orbit)*

§3. THE MD4—FOLIATIONS AND THEIR CLASSIFICATION

This section is devoted to the *MD4*—foliations, i. e. the foliations formed by the *K*—orbits of maximal dimension of the indecomposable simply connected *MD4*—groups. The main results of the section is the following proposition and theorem.

PROPOSITION 2. Let G be an indecomposable simply connected *MD4*—group, $\mathcal{F}_G$ be the family of all its K—orbits of maximal dimension and $V_G = \cup\{\Omega, \Omega \in \mathcal{F}_G\}$. Then $(V_G, \mathcal{F}_G)$ is a measured foliation in the Connes' sense. We call this foliation *MD4*—foliation associated to G.

Remark and Notation. It should be noted that V_G is an open submanifold in the dual space $\mathcal{G}^* \cong \mathbf{R}^4$ of the Lie algebra $\mathcal{G}$ corresponding to G. Furthermore, for all $MD4$—groups of the forms $G_{4,\,n,\dots}$ $(1 \leqslant n \leqslant 4)$, the manifolds V_G are diffeomorphic to each other. Hence, for convenience, we will write

$$(V_n, \mathcal{F}_{n,\dots}) \text{ for } (V_{G_{4,\,n,\dots}}, \mathcal{F}_{G_{4,n,\dots}}).$$

THEOREM 3. (The topological classification of the MD4-foliations)

1. There exist exactly 9 topological types of the MD4-foliations: $\{(V_1, \mathcal{F}_{1,1})\}$, $\{(V_1, \mathcal{F}_{1,2})\}$, $\{(V_2, \mathcal{F}_{2,1}(\lambda), (V_2, \mathcal{F}_{2,2}), \lambda \in \mathbf{R}^*\}$, $\{(V_2, \mathcal{F}_{2,3}(\varphi)), \varphi \in (0, \pi)\}$, $\{(V_2, \mathcal{F}_{2,4})\}$, $\{V_3, \mathcal{F}_{3,1}(\lambda_1, \lambda_2)), (V_3, \mathcal{F}_{3,2}(\lambda)), (V_3, \mathcal{F}_{3,3}),$ $\lambda \in \mathbf{R}^*, \lambda_1, \lambda_2 \in \mathbf{R}^*\}$, $\{(V_3, \mathcal{F}_{3,4}(\lambda,\varphi)), \lambda \in \mathbf{R}^*, \varphi \in (0,\pi)\}$, $\{(V_4, \mathcal{F}_{4,1})\}$, $\{(V_4, \mathcal{F}_{4,2})\}$. *We denote these types by* $\mathcal{F}_1, \dots, \mathcal{F}_9$ *respectively,*

2. (i) The MD4—foliations of the types $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6$ *are trivial fibrations with connected fibers on* $\mathbf{R}\times\mathbf{R}^*$, $\mathbf{R}^2 \cup \mathbf{R}^2$, $\mathbf{R} \times S^1$, $\mathbf{R}_+\times\mathbf{R}$, $\{pt\}$, S^2 *respectively, where* $\{pt\}$ *is the one-point space.*

(ii) The MD4-foliations of types $\mathcal{F}_7, \mathcal{F}_8, \mathcal{F}_9$ *can be given by suitable actions of* $\mathbf{R}^2$ *on the foliated manifolds* $V_3 \cong (\mathbf{C}\times\mathbf{R}^*)\times\mathbf{R}$, $V_4 \cong (\mathbf{R}^3)^* \times \mathbf{R}$.

Proof of Proposition 2. The proof is analogous to the case (R.H_3 (see [12, Th.2]). First we need to define a system of smooth vector fields on the manifold V_G such that each K-orbit Ω from $\mathcal{F}_G$ is a maximal connected integrated submanifold corresponding to that system. As the next step, we have to show that the Lebegues measure is invariant for the polyvector field generating the above system. The last step is a simple matter and can be verified by simple computations. Now we introduce the systems (of smooth vector fields) corresponding to each of the $MD4$-groups. For convenience, we denote by $\mathcal{S}_{n,\dots}$ the system (of smooth vector fields) corresponding to $G_{4,\,n,\dots}$. By direct computations, we get:

$$1.\ \mathcal{S}_{1,1}: \begin{cases} \mathcal{X}_1(x, y, z, t) = (z, 0, 0, 0) \\ \mathcal{X}_2(x, y, z, t) = (0, 0, 0, -z) \end{cases}$$

$$\mathcal{S}_{1,2}: \begin{cases} \mathcal{X}_1\ (x, y, z, t) = (0, 0, z, 0 \\ \mathcal{X}_2\ (x, y, z, t = (0, 0, 0, -z) \end{cases}$$

on the manifold $V_1 \cong \mathbf{R}^2 \times \mathbf{R}^* \times \mathbf{R}$.

$$2.\ \mathcal{S}_{2,1}(\lambda): \begin{cases} \mathcal{X}_1\ (x, y, z, t) = (0, \lambda y, z, 0) \\ \mathcal{X}_2\ (x, y, z, t) = (-\lambda y, 0, 0, 0) \\ \mathcal{X}_3\ (x, y, z, t) = (-z, 0, 0, 0), \ \lambda \in \mathbf{R}^*. \end{cases}$$

$$\mathcal{S}_{2,2}: \begin{cases} \mathcal{X}_1(x, y, z, t) = (0, y, y + z, 0) \\ \mathcal{X}_2(x, y, z, t) = (-y, 0, 0, 0) \\ \mathcal{X}_3(x, y, z, t) = (-y - z, 0, 0, 0) \end{cases}$$

on the manifold $V_2 \cong \mathbf{R} \times (\mathbf{R}^2)^* \times \mathbf{R}$.

$$\mathcal{S}_{2,3(\varphi)}: \begin{cases} \mathcal{X}_1(x, y + iz, t) = (0, (y + iz)e^{i\varphi}, 0) \\ \mathcal{X}_2(x, y + iz, t) = (-y\cos\varphi + z\sin\varphi, 0, 0) \\ \mathcal{X}_3(x, y + iz, t) = (-y\sin\varphi - z\cos\varphi, 0, 0), \quad \varphi \in (0, \pi) \end{cases}$$

on the manifold $V_2 \cong \mathbf{R} \times \mathbf{C}^* \times \mathbf{R}$.

$$\mathcal{S}_{2,4}: \begin{cases} \mathcal{X}_1(x, y, z, t) = (0, 0, 0, 1) \\ \mathcal{X}_2(x, y, z, t) = (1, 0, 0, 0) \\ \mathcal{X}_3(x, y, z, t) = (0, y, z, 0) \\ \mathcal{X}_4(x, y, z, t) = (0, -z, y, 0) \end{cases}$$

on the manifold $V_2 \cong \mathbf{R} \times (\mathbf{R}^2)^* \times \mathbf{R}$.

3. $$\mathcal{S}_{3,1(\lambda_1, \lambda_2)}: \begin{cases} \mathcal{X}_1(x, y, z, t) = (\lambda_1 x, \lambda_2 y, z, 0) \\ \mathcal{X}_2(x, y, z, t) = (0, 0, 0, -\lambda_1 x) \\ \mathcal{X}_3(x, y, z, t) = (0, 0, 0, -\lambda_2 y) \\ \mathcal{X}_4(x, y, z, t) = (0, 0, 0, -z), \quad \lambda_1, \lambda_2 \in \mathbf{R}^* \end{cases}$$

$$\mathcal{S}_{3,2(\lambda)}: \begin{cases} \mathcal{X}_1(x, y, z, t) = (\lambda x, x + \lambda y, z, 0) \\ \mathcal{X}_2(x, y, z, t) = (0, 0, 0, -\lambda x) \\ \mathcal{X}_3(x, y, z, t) = (0, 0, 0, -x - \lambda y) \\ \mathcal{X}_4(x, y, z, t) = (0, 0, 0, -z), \quad \lambda \in \mathbf{R}^* \end{cases}$$

$$\mathcal{S}_{3,3}: \begin{cases} \mathcal{X}_1(x, y, z, t) = (x, x + y, y + z, 0) \\ \mathcal{X}_2(x, y, z, t) = (0, 0, 0, -x) \\ \mathcal{X}_3(x, y, z, t) = (0, 0, 0, -x - y) \\ \mathcal{X}_4(x, y, z, t) = (0, 0, 0, -y - z) \end{cases}$$

on the manifold $V_3 \cong (\mathbb{R}_3)^* \times \mathbb{R}$.

$$\mathcal{S}_{3,4}(\lambda, \varphi): \begin{cases} \mathcal{X}_1(x+iy, z, t) = ((x+iy)e^{i\varphi}, \lambda z, 0 \\ \mathcal{X}_2(x+iy, z, t) = (0, 0, -x\cos\varphi + y\sin\varphi) \\ \mathcal{X}_3(x+iy, z, t) = (0, 0, -x\sin\varphi - y\cos\varphi) \\ \mathcal{X}_4(x+iy, z, t) = (0, 0, -\lambda z),\ \lambda \in \mathbb{R}^*, \varphi \in (0, \pi) \end{cases}$$

on the manifold $V_3 \cong (\mathbb{C} \times \mathbb{R})^* \times \mathbb{R}$.

$$4.\ \mathcal{S}_{4,1}: \begin{cases} \mathcal{X}_1(x, y, z, t) = (-y, x, 0, 0) \\ \mathcal{X}_2(x, y, z, t) = (0, z, 0, y) \\ \mathcal{X}_3(x, y, z, t) = (-z, 0, 0, -x) \end{cases}$$

$$\mathcal{S}_{4,2}: \begin{cases} \mathcal{X}_1(x, y, z, t) = (-x, y, 0, 0) \\ \mathcal{X}_2(x, y, z, t) = (0, z, 0, x) \\ \mathcal{X}_3(x, y, z, t) = (-z, 0, 0, -y) \end{cases}$$

on the manifold $V_4 \cong (\mathbb{R}^3)^* \times \mathbb{R}$.

It can be easily verified that each K — orbit Ω from $\mathcal{F}_{n,\dots}$ is a maximal connected integrated submanifold corresponding to $\mathcal{S}_n$,... $(1 \leqslant n \leqslant 4)$. This completes our proof.

Proof of Theorem 3. 1. Let us recall that two foliation $(V, \mathcal{F})$ $(V, \mathcal{F}')$ are said to be topologically equivalent if there exists a homeomorphism $h: V \to V'$ which takes leaves of $\mathcal{F}'$ onto leaves of $\mathcal{F}'$.

Let $h_{2,1(\lambda)}$, $h_{2,2}: V_2 \to V_2$ $(\lambda \in \mathbb{R}^*)$ be the following maps:

$$h_{2,1(\lambda)}(x, y, z, t) = (x, \operatorname{sign}(y).\ |y|^{1/\lambda}, z, t)$$

$$h_{2,2}(x, y, z, t) = \begin{cases} (x, y, z - y\ln|y|, t) & y \neq 0 \\ (x, 0, z, t) & y = 0 \end{cases}$$

where $(x, y, z, t) \in V_2 \cong \mathbb{R} \times (\mathbb{R}^2)^* \times \mathbb{R}$, $\lambda \in \mathbb{R}^*$, and let $h_{2,3(\varphi)}$: $V_2 \to V_2$ $(\varphi \in (0, \pi))$ be the following map

$$h_{2,3(\varphi)}(x, re^{i\theta}, t) = (x, e^{(\ln r + i\theta)ie^{-\varphi}}, t)$$

where $(x, re^{i\theta}, t) \in V_2 \cong \mathbb{R} \times \mathbb{C}^* \times \mathbb{R}$, $\varphi \in (0, \pi)$.

It is easy to prove that $h_{2,1(\lambda)}$, $h_{2,2}$ are homeomorphisms which take leaves of $\mathcal{F}_{2,1(\lambda)}$, $\mathcal{F}_{2,2}$ onto leaves of $\mathcal{F}_{2,1(1)}$ $(\lambda \in \mathbb{R}^*)$ and $h_{2,3(\varphi)}$ is homeomorphism which takes leaves of $\mathcal{F}_{2,3(\varphi)}$ onto leaves of $\mathcal{F}_{2,3}\left(\frac{\pi}{2}\right)$, $\varphi \in (0, \pi)$. Thus the foliations $(V_2, \mathcal{F}_{2,1(\lambda)})$, $(V_2, \mathcal{F}_{2,2})$ $(\lambda \in \mathbb{R}^*)$ are topologically equivalent to each other. Similarly for the foliations $(V_2, \mathcal{F}_{2,3(\varphi)})$, $\varphi \in (0, \pi)$.

The topological equivalence of the foliations $(V_3, \mathcal{F}_{3,1(\lambda_1,\lambda_2)})$, $V_3, \mathcal{F}_{3,2(\lambda)})$ and $(V_3, \mathcal{F}_{3,3})$, $\lambda_1, \lambda_2, \lambda \in \mathbf{R}^*$ or the foliations $(V_3, \mathcal{F}_{3,4(\lambda,\varphi)})$, $\lambda \in \mathbf{R}^*$, $\varphi \in (0,\pi)$ is also verified similarly by considering the following maps $V_3 \to V_3$:

$$h_{3,1(\lambda_1,\lambda_2)}(x, y, z, t) = (\text{sign}(x) \,.\, |x|^{1/\lambda_1},\ \text{sign}(y) \,.\, |y|^{1/\lambda_2}, z, t)$$

$h_{3,2(\lambda)}(x, y, z, t) = (\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{t})$ with

$$\widetilde{x} = \text{sign}(x) \,.\, |x|^{1/\lambda}$$

$$\widetilde{y} = \begin{cases} \text{sign}\left(y - \dfrac{1}{\lambda} x \ln |x|\right) \left| y - \dfrac{1}{\lambda} x \ln |x| \right|^{1/\lambda} & x \neq 0 \\ \text{sign}(y) \,.\, |y|^{1/\lambda} & x = 0 \end{cases}$$

$$\widetilde{z} = z$$

$$\widetilde{t} = t$$

$h_{3,3}(x, y, z, t) = (\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{t})$ with

$$\widetilde{x} = x$$

$$\widetilde{y} = \begin{cases} y - x \ln |x| & , x \neq 0 \\ y & , x = 0 \end{cases}$$

$$\widetilde{z} = \begin{cases} z - \dfrac{1}{2} y \ln |x| - \dfrac{1}{2}(y - x \ln |x|) \ln \left| y - x \ln |x| \right|, & x \neq 0,\ y \neq x \ln |x| \\ z - \dfrac{1}{2} y \ln |x| & x \neq 0,\ y = x \ln |x| \\ z & x = 0 \end{cases}$$

$$\widetilde{t} = t$$

where $(x, y, z, t) \in V_3 \cong (\mathbf{R}^3)^* \times \mathbf{R}$, $\lambda_1, \lambda_2, \lambda \in \mathbf{R}^*$, and

$$h_{3,4(\lambda,\varphi)}(re^{i\theta}, z, t) = (e^{(\ln r + i\theta) i e^{-i\varphi}}, \text{sign}(z) \,.\, |z|^{1/\lambda}, t)$$

where $(re^{i\theta}, z, t) \in V_3 \cong (\mathbf{C} \times \mathbf{R})^*, \times \mathbf{R}$, $\lambda \in \mathbf{R}^*$, $\varphi \in (0,\pi)$.

2. (i): The triviality of the foliations of the types $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_5$ is obvious. It should be noted that $\left(V_2, \mathcal{F}_{2,1(1)}\right)$, $\left(V_2, \mathcal{F}_{2,3(\frac{\pi}{2})}\right)$ and $(V_3, \mathcal{F}_{3,1(1,1)})$ are the

foliations of the types $\mathfrak{F}_3, \mathfrak{F}_4$ and $\mathfrak{F}_6$ respectively. It is easily seen that the following submersions:

$$P_{2,1(1)}: V_2 \cong \mathbf{R} \times (\mathbf{R}^2)^* \times \mathbf{R} \cong \mathbf{R} \times S^1 \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R} \times S^1$$

$$P_{2,1(1)}(x, s, u, v) = (x, s), \ (x, s, u, v) \in \mathbf{R} \times S^1 \times \mathbf{R}_+ \times \mathbf{R}$$

$$P_{2,3(\frac{\pi}{2})}: V_2 \cong \mathbf{R} \times \mathbf{C}^* \times \mathbf{R} \to \mathbf{R}_+ \times \mathbf{R}$$

$$P_{2,3(\frac{\pi}{2})}(x, re^{i\theta}, t) = (r, x), \ (x, re^{i\theta}, t) \in \mathbf{R} \times \mathbf{C}^* \times \mathbf{R}$$

and

$$P_{3,1(1,1)}: V_3 \cong (\mathbf{R}^3)^* \times \mathbf{R} \cong S^2 \times \mathbf{R}_+ \times \mathbf{R} \to S^2$$

$$P_{3,1(1,1)}(s, u, v) = s, \ (s, u, v) \in S^2 \times \mathbf{R}_+ \times \mathbf{R}$$

define the foliations $(V_2, \mathfrak{F}_{2,1(1)})$, $(V_2, \mathfrak{F}_{2,3(\frac{\pi}{2})})$ and $(V_3, \mathfrak{F}_{3,1(1,1)})$ respectively. Hence the foliations of $\mathfrak{F}_3, \mathfrak{F}_4, \mathfrak{F}_6$ are trivial fibrations. Clearly, the fibers of the foliations of $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_6$ are simply connected, but the fibers of the foliations of $\mathfrak{F}_4, \mathfrak{F}_5$ are connected and nonsimply connected. (iii) Let us consider the following actions of $\mathbf{R}^2$ on $V_3 \cong V_4$:

$$\rho_{3,4}: \mathbf{R}^2 \times V_3 \to V_3$$

$$\rho_{3,4}((r, s), (x + iy, z, t)) = ((x + iy)e^{is}, ze^s, t + r) \tag{3.1}$$

$$(r, s) \in \mathbf{R}^2, \ (x + iy, z, t) \in V_3 \cong (\mathbf{C} \times \mathbf{R})^* \times \mathbf{R}$$

$$\rho_{4,1}: \mathbf{R}^2 \times V_4 \to V_4$$

$$\rho_{4,1}((r, s), (x, y, z, t)) = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$$

$$\begin{aligned} \tilde{x} &= x\cos r - y\sin r - sz \\ \tilde{y} &= x\sin r + y\cos r - sz \\ \tilde{z} &= z \\ \tilde{t} &= t - s(x + y)\cos r + s(y - x)\sin r + s^2 z \end{aligned} \tag{3.2}$$

$$(r, s) \in \mathbf{R}^2, \ (x, y, z, t) \in V_4 \cong (\mathbf{R}^3)^* \times \mathbf{R}$$

$$\rho_{4,2} : \mathbf{R}^2 \times V_4 \longrightarrow V_4$$

$$\rho_{4,2}\ ((r, s), (x, y, z, t)) = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$$

$$\begin{aligned}
\tilde{x} &= e^{-s}\left(x + r \cdot \frac{yz}{x^2+y^2+z^2}\right) \\
\tilde{y} &= e^{-s}\left(y + r \cdot \frac{xz}{x^2+y^2+z^2}\right) \\
\tilde{z} &= z \\
\tilde{t} &= t + r \cdot \frac{x^2+y^2}{x^2+y^2+z^2} + r^2 \cdot \frac{xyz}{(x^2+y^2+z^2)^2}
\end{aligned} \tag{3.3}$$

$$(r, s) \in \mathbf{R}^2 \ , \ (x, y, z, t) \in V_4 \cong (\mathbf{R}^3)^* \times \mathbf{R} \, .$$

It can be verified that the above actions $\rho_{3,4}$, $\rho_{4,1}$, $\rho_{4,2}$ generate the foliations $\left(V_3, \mathcal{F}_{3,4}\left(1, \frac{\pi}{2}\right)\right)$, $(V_4, V_{4,1})$, $(V_4, \mathcal{F}_{4,2})$ respectively. Consequently the foliations $(V_3, \mathcal{F}_{3,4(\lambda,\varphi)})$, $\lambda \in \mathbf{R}^*$, $\varphi \in (0,\pi)$ also have the similar properties. The proof is complete.

§ 4. THE C*- ALGEBRA ASSOCIATED TO THE MD4 — FOLIATIONS

In this section, we determine all C^* — algebras associated to the $MD4$-foliations. It should be noted that if the foliation $(V, \mathcal{F})$ comes from a fibration (with connected fibers) $p : V \to M$, then $C^*(V, \mathcal{F}) \cong C_0(M) \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the C^*-algebra of compact operators on an (infinite dimensional) separable Hilbert space (see [2, Sect. 5]). Furthermore, topologically equivalent foliations yield isomorphic C^*-algebras (see [11, Sect. 2]).

The next proposition and theorems are fundamental in the section. The results on the real diamond foliation $(V_4, \mathcal{F}_{4,2})$ are included here for the sake of completeness.

PROPOSITION 3.

$C^*(V_1, \mathcal{F}_{1,1}) \cong C_0(\mathbf{R} \times \mathbf{R}^*) \otimes \mathcal{K}$; $C^*(V_1, \mathcal{F}_{1,2}) \cong C_0(\mathbf{R}^2 \cup \mathbf{R}^2) \otimes \mathcal{K}$;

$C^*(V_2, \mathcal{F}_{2,1(\lambda)}) \cong C^*(V_2, \mathcal{F}_{2,2}) \cong C_0(\mathbf{R} \times S^1) \otimes \mathcal{K}$, $\lambda \in \mathbf{R}^*$;

$C^*(V_2\ \mathcal{F}_{2,3(\varphi)}) \cong C_0(\mathbf{R}_+ \times \mathbf{R}) \otimes \mathcal{K}$, $\varphi \in (0, \pi)$;

$C^*(V_2, \mathcal{F}_{2,4}) = \mathbf{C} \otimes \mathcal{K}$;

$C^*\ V_3, \mathcal{F}_{3,1(\lambda_1, \lambda_2)}) \cong C^*(V_3, \mathcal{F}_{3,2(\lambda)}) = C^*(V_3, \mathcal{F}_{3,3}) = C(S^2) \otimes \mathcal{K}$,
$\lambda_1, \lambda_2, \lambda \in \mathbf{R}^*$.

THEOREM 4. *1. We have the following canonical extensions:*

$(\gamma_1): 0 \to C_0(\mathbf{R}^2 \cup \mathbf{R}^2) \otimes \mathcal{K} \to C^*(V_3, \mathcal{F}_{3,4(\lambda,\varphi)}) \to C_0(\mathbf{R}_+) \otimes \mathcal{K} \to 0,$

$(\gamma_2): 0 \to C_0(\mathbf{R}^* \times \mathbf{R}) \otimes \mathcal{K} \to C^*(V_4, \mathcal{F}_{4,1}) \to C_0(\mathbf{R}_+) \otimes \mathcal{K} \to 0,$

where $\lambda \in \mathbf{R}^*$, $\varphi \in (0, \pi)$.

2. We have the following canonical repeated extensions:

$(\gamma_3): 0 \to C_0(\mathbf{R}^* \times \mathbf{R}) \otimes \mathcal{K} \to C^*(V_4, \mathcal{F}_{4,2}) \to C_0((\mathbf{R}^2)^* \times \mathbf{R}) \rtimes_\rho \mathbf{R}^2 \to 0,$

$(\gamma_4): 0 \to C_0(\mathbf{R}^* \cup \mathbf{R}^*) \otimes \mathcal{K} \to C_0((\mathbf{R}^2)^* \times \mathbf{R}) \rtimes_\rho \mathbf{R}^2 \to \mathbf{C}^4 \otimes \mathcal{K} \to 0,$ *where* $\rho = \rho_{4,2}$ *is the action given in (3.3).*

THEOREM 5. *The C^*-algebras associated to the nontrivial MD4-foliations are characterized by the topological invariants in KK-theory as follows.*

1. (i) Index $C^*(V_3, \mathcal{F}_{3,4(\lambda,\varphi)}) = \gamma_1 = (1,1)$ *in the KK-group:*

$$Ext(C_0(\mathbf{R}_+)\otimes\mathcal{K}, C_0(\mathbf{R}^2 \cup \mathbf{R}^2)\otimes\mathcal{K}) \cong \mathbf{Z}^2$$

where $\lambda \in \mathbf{R}^*$, $\varphi \in (0\ \pi)$.

(ii) Index $C^*(V_4, \mathcal{F}_{4,1}) = \gamma_2 = (1,1)$ *in the KK-group:*

$$Ext(C_0(\mathbf{R}_+)\otimes\mathcal{K}, C_0(\mathbf{R}^* \times \mathbf{R})\otimes\mathcal{K}) \cong \mathbf{Z}^2$$

(iii) Index $C^*(V_4, \mathcal{F}_{4,2}) = \{\gamma_3, \gamma_4\}$ *with*

$$\gamma_3 = (1,1) \text{ in } Ext(C_0(\mathbf{R}^2)^* \times \mathbf{R}) \rtimes_\rho \mathbf{R}^2, C_0(\mathbf{R}^* \times \mathbf{R})\otimes\mathcal{K}) \cong \mathbf{Z}^2,$$

$$\gamma_4 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \text{ in } Ext(\mathbf{C}^4\otimes\mathcal{K}, C_0(\mathbf{R}^*\cup\mathbf{R}^*)\otimes\mathcal{K}) \cong Hom_{\mathbf{Z}}(\mathbf{Z}^4, \mathbf{Z}^4).$$

Proof of Proposition 3. The proof is straightforward by theorem 3.

Proofs of Theorems 4 and 5. Theorem 4 is obtained similarly to the case of the real diamond group $\mathbf{R}.H_3$ (see [13, Th. 1]).

Theorem 5 is an immediate consequence of Theorem 3 and [13, Th. 2], using the formal properties of the Thom-Connes isomorphism of [1] (see [11, Rem. 3.4.2 and Lem. 3.4.3]).

Concluding Remark. It should be noted that the results of Sections §2, §3, §4 are true for all indecomposable connected MD4-groups.

We emphasized in Section §2 that the similar problem for the decomposable *MD4*-groups can be directly reduced to the case of the *MD*-groups of dimension 3 or less. In particular, it is clear that:

Remark 2. Let G be a decomposable *MD4*-group and $\mathcal{G} = \mathbf{R}^n \times \widetilde{\mathcal{G}}$ be its Lie algebra ($1 \leqslant n \leqslant 4$). Then the dual spaces $\mathcal{G}^*$ of $\mathcal{G}$ can be given by $\mathcal{G}^* = \mathbf{R}^n \times \widetilde{\mathcal{G}}^*$, where $\widetilde{\mathcal{G}}^*$ is the dual space of $\mathcal{G}$. Let $F(\alpha, \beta) \in \mathcal{G}^* = \mathbf{R}^n \times \widetilde{\mathcal{G}}^*$. and $\widetilde{F}(\beta) \in \widetilde{\mathcal{G}}^*$. Denote by Ω_F and $\widetilde{\Omega}_{\widetilde{F}}$ the K-orbits including F and $\widetilde{F}$ in $\mathcal{G}^*$ and $\widetilde{\mathcal{G}}^*$ respectively, we get

$$\Omega_F = \{(\alpha)\} \times \widetilde{\Omega}_{\widetilde{F}} .$$

Furthermore, $\widetilde{\mathcal{G}}$ is an indecomposable MD-algebra of dimension $(4 - n)$.

Let us consider an example on the group $G = \mathbf{R} \times G_{3,2(-1)}$. It is a decomposable MD4-group corresponding to decomposable MD4-algebra $\mathcal{G} = \mathbf{R} \times G_{3,2(-1)}$. Recall that $G_{3,2(-1)}$ is simply connected Lie group corresponding to the 3-dimensional Lie algebra $\mathcal{G}_{3,2(-1)}$ with basis $\{X, Y, T\}$ satisfying the relations : $[T, X] = -X, [T, Y] = Y, [X, Y] = 0$ (see) [6]).

PROPOSITION 4. *Let* $F(u, \alpha, \beta, \delta) \in \mathcal{G}^* = \mathbf{R} \times \mathcal{G}^*_{3,2(-1)} \cong \mathbf{R} \times \mathbf{R}^3$ *and* $\widetilde{F}(\alpha, \beta, \delta) \in \mathcal{G}^*_{3,2(-1)} \cong \mathbf{R}^3$. *Denote by* Ω_F, $\widetilde{\Omega}_{\widetilde{F}}$ *the K-orbits including* F, $\widetilde{F}$ *in* $\mathcal{G}^*$, $\mathcal{G}^*_{3,2(-1)}$ *respectively. Then* $\Omega_F = \{u\} \times \Omega_{\widetilde{F}}$ *is given as follows*

(*i*) $\alpha = \beta = 0$ *then* $\Omega_F = \{F\}$. (*the 0-dimensional orbit*)

(*ii*) $\alpha \neq 0 = \beta$ *then* $\Omega_F = \{(u, x, 0, t), \alpha x > 0\}$: *a cordinate half-plane* (*the 2-dimensional orbit*)

(*iii*) $\alpha = 0 \neq \beta$ *then* $\Omega_F = \{(u, 0, y, t), \beta y > 0\}$: *a coordinate half-plane* (*the 2-dimensional orbit*)

(*iv*) $\alpha\beta \neq 0$ *then* $\Omega_F = \{(u, x, y, t), xy = \alpha\beta, \alpha x > 0, \beta y > 0\}$: *a vertical hyperbolic cylinder.* (*the 2-dimensional orbit*)

PROPOSITION 5. (*i*) *The family* $\widetilde{\mathcal{F}}$ *of all two-dimensional K-orbits of* $G_{3,2(-1)}$ *forms a measured foliation on* $\widetilde{V} = \cup\{\widetilde{\Omega}/\widetilde{\Omega} \in \widetilde{\mathcal{F}}\} \cong (\mathbf{R}^2)^* \times \mathbf{R}$. *The foliation* $(\widetilde{V}, \widetilde{\mathcal{F}})$ *can be identified with* $(V_2, \mathcal{F}_2)$ *given in* [13, Sect.2].

(*ii*) $C^*(\widetilde{V}, \widetilde{\mathcal{F}})$ *can be included into the following canonical extension :*

$$(\widetilde{Y}): 0 \to C_0(\mathbf{R}^* \cup \mathbf{R}^*) \otimes \mathcal{K} \to C^*(\widetilde{V}, \widetilde{\mathcal{F}}) \to \mathbf{C}^4 \otimes \mathcal{K} \to 0$$

with its topological invariant in KK-theory given by

$$\text{Index } C^*(\tilde{V}, \tilde{\mathcal{F}}) \doteq \tilde{Y} = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix}$$

in $Ext(\mathbf{C}^4 \otimes \mathcal{K}, C_0(\mathbf{R}^* \cup \mathbf{R}^*) \otimes \mathcal{K}) \cong Hom_z(\mathbf{Z}^4, \mathbf{Z}^4)$.

The Proposition 4 is obtained by the same woy as in [12, Th.1]. The Proposition 5 is an immediate consequence of [13, Th.1 and 2]. It follows from Proposition 6 that:

COROLLARY 1. *(i) The family $\mathcal{F} = \mathbf{R} \times \tilde{\mathcal{F}}$ of all two-dimensional K-orbits Ω of $G = \mathbf{R} \times G_{3,2(-1)}$ forms a measured foliation on $V = \bigcup\{\Omega/\Omega \in \mathcal{F}\} \cong \mathbf{R} \times \tilde{V} \cong \cong \mathbf{R} \times (\mathbf{R}^{2*}) \times \mathbf{R}$.*

(ii) $C^*(V, \mathcal{F}) \cong C_0(\mathbf{R}) \otimes C^*(\tilde{V}, \tilde{\mathcal{F}})$ can be included into the following canonical extension:

$$(\gamma): 0 \to C_0(\mathbf{R}) \otimes C_0(\mathbf{R}^* \cup \mathbf{R}^*) \otimes \mathcal{K} \to C^*(V, \mathcal{F}) \to C_0(\mathbf{R}) \otimes \mathbf{C}^4 \otimes \mathcal{K} \to 0$$

with its topological invariant in KK-theory given by

$$\text{Index } C^*(V, \mathcal{F}) \doteq \gamma = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \text{ in the KK-group}$$

$$Ext(C_0(\mathbf{R}) \otimes \mathbf{C}^4 \otimes \mathcal{K}, \ C_0(\mathbf{R}) \otimes C_0(\mathbf{R}^* \cup \mathbf{R}^*) \otimes \mathcal{K}) \cong Hom_z(\mathbf{Z}^4, \mathbf{Z}^4).$$

REFERENCES

[1] A. Connes, *An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of* $\mathbf{R}$, Adv. in Math. 39 (1981), 31 – 55.

[2] A. Connes, *A survey of foliations and operator algebras.* Proc. Symp. Pure Math., 38(1982), 521 – 628, Part I.

[3] D.N. Diep, *Structure of the group C^*-algebra of the group of affine transformations of the straight line.* Funkt. Anal. Prilozen., 9(1975), n$^{\underline{o}}$ 1, 63 – 64 (in Russian).

[4] G.G. Kasparov, *Operator K-functor and extension of C^*-algebras,* Izv. Akad. Nauk SSSR, Ser. Mat., 44(1980), 571 – 636 (in Russian).

[5] A.A. Kirillov, *Elements of the theory of representations,* Springer-Verlag, Berlin-Heidenberg-New York, 1976.

[6] J. Rosenberg, *The C^*-algebras of some real and p-adic solvable groups,* Pacific J. Math., 65 (1976), 175 – 192.

[7] J. Rosenberg, *Homological invariants of extension of C^* -algebras*, Proc. Symp. Pure Math., 38 (1982), AMS, Providence R.I., 1982, 35 − 75.

[8] M. Saito, *Sur certains groupes de Lie resolubles*, Sci. Papers of the College of General Education, Univ. of Tokyo, 7 (1957), 1-11, 157-168.

[9] V.M. Son, H.H. Viet, *Sur la structure des C^*-algebres d'une classe de groupes de Lie*, J. Operator Theory, 11 (1984), 77-90.

[10] J. Taylor, *Banach algebras and topology*, in *« Algebra in Analysis »*, (J. Williamson Ed.), Academic Press, New York 1975.

[11] A.M Torpe, *K−theory for the leaf space of foliations by Reeb components*, J. Func. Anal., 61(1985), 15−71.

[12] L.A. Vu, *The foliation formed by the K−orbits of maximal dimension of the real diamond group*, J. Math. Vietnam, XV(1987), n^0 3, 7-10 in Vietnamese).

[13] L.A. Vu, *On the structure of the C^*−algebra of the foliation formed by the K−orbits of maximal dimension of the real diamond group* (to appear in J. Operator Theory).

[14] L. A. Vu, *On the foliations formed by the generic K−orbits of a class of solvable Lie groups* (to appear).

Received September 13, 1988

INSTITUTE OF MATHEMATICS, P.O. BOX 631, BO HO, 10.000 HANOI VIETNAM.