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SUFFICIENT OPTIMALITY CONDITIONS FOR DISCRETE MINIMAX PROBLEMS IN THE PRESENCE OF CONSTRAINTS IN BANACH SPACES

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I. INTRODUCTION

Let us consider the following discrete minimax problem:

$$(\mathrm{I}) \begin{cases} \text{Minimize } \max\limits_{i \in [0\,:\,N]} f_i(x), \\ \text{Subject to } F(x) \in K \text{ and } x \in C, \end{cases}$$

where f_i is a functional defined on a real Banach space X ($i = 0,1, \ldots, N$), C is a non-empty closed convex subset of X, F is a map from X into a real Banach space Y and K is a closed convex cone in Y with vertex at the origin.

The discrete minimax problem (I) is a nonsmooth problem which is closely connected with the smooth one studied in [2], [3]. In the finite-dimensional case, necessary and sufficient optimality conditions for Problem (I) involving only the second constraint are given in [1]. In the case of infinite-dimensional spaces necessary conditions for Problem (I) with $N = 0$ are established in [2], and sufficient conditions are studied in [3].

This paper presents some results concerning sufficient optimality conditions for Problem (I). The paper consists of 4 sections. After the introduction, in Sections 2 and 3, using an approximation property of the feasible set, we obtain first and second-order sufficient optimality conditions for Problem (I). Section 4 is devoted to the discussion of first-order sufficient optimality conditions for the problem with inequality — type constraint involving a finite number of functionals. From results of the paper, give us in particular some known results including those in [3] for $N = 0$ and those of Dem'yanov, Malozemov in [1] for the case when X and Y are finite dimensional.

2. FIRST-ORDER SUFFICIENT OPTIMALITY CONDITIONS.

Throughout this paper we assume that the map F is continuously Fréchet differentiable at a feasible point $\bar{x}$.

Recall that a feasible point $\bar{x}$ is regular for Problem (I) (in the sense of Zowe and Kurcyusz [2]) if :

$$F'(\bar{x})C(\bar{x}) - K(F(\bar{x})) = Y, \tag{2.1}$$

where

$$C(\bar{x}) = \{\lambda\ (x - \bar{x}) \mid x \in C,\ \lambda \geqslant 0\},$$

$$K(F(\bar{x})) = \{k - \lambda F(\bar{x}) \mid k \in K,\ \lambda \geqslant 0\}.$$

The set of feasible points for Problem (I) is denoted by M.

It should be noted that the feasible set of Problem (I) coincides with the feasible set of the problem considered in [2], [3]. If $\bar{x}$ is a regular point of Problem (I), then by virtue of a result in [3], the feasible set M is approximated at $\bar{x}$ by the linearizing cone L of M at $\bar{x}$, i. e., there exists a map $\zeta : M \to L$ such that

$$\| \zeta(x) - (x - \bar{x}) \| = 0(\| x - \bar{x} \|) \quad \text{for} \quad x \in M, \tag{2.2}$$

where

$$0(\| x - \bar{x} \|) \,/\, \| x - \bar{x} \| \to 0 \qquad (\text{when } \| x - \bar{x} \| \to 0),$$

$$L = \{\ x \in C(\bar{x}) \mid F'(\bar{x})x \in K(F(\bar{x}))\ \}.$$

Throughout the forthcoming, φ will denote :

$$\varphi(x) = \max_{i \in [0:N]} f_i(x).$$

DEFINITION. The feasible point $\bar{x}$ is said to be a strict local minimum of Problem (I), if there exists a number $\delta > 0$ such that $\varphi(x) > \varphi(\bar{x})$ for every $x \in M$ satisfying $\| x - \bar{x} \| < \delta$, $x \neq \bar{x}$.

We are now in a position to formulate a first-order sufficient optimality condition for the discrete minimax Problem (I).

THEOREM 2.1. *Let $\bar{x}$ be a regular point of Problem (I). Suppose that the maps $f_0, \ldots, f_N$, F are Fréchet differentiable at $\bar{x}$. Assume, in addition, that there is a number $\beta > 0$ such that*

$$\max_{i \in R(\bar{x})} \langle f_i'(\bar{x}), x \rangle \geqslant \beta \| x \| \quad \textit{for all} \quad x \in L, \tag{2.3}$$

where

$$R(\bar{x}) = \{ i \in [0:N] \mid f_i(\bar{x}) = \max_{j \in [0:N]} f_j(\bar{x}) \}$$

Then, $\bar{x}$ is a strict local minimum of Problem (I).

Proof. Take $x \in M, x \neq \bar{x}$. For each $i \in [0:N]$, by virtue of the differentiability of f_i, we have

$$f_i(x) = f_i(\bar{x}) + \langle f_i'(\bar{x}), x - \bar{x} \rangle + 0_i(\| x - \bar{x} \|), \tag{2.4}$$

where $0_i(\| x - \bar{x} \|) / \| x - \bar{x} \| \to 0$ (as $\| x - \bar{x} \| \to 0$).

We shall use the following inequality :

$$\max_{i \in [0:N]} \{ a_i + b_i \} \geqslant \max_{i \in [0:N]} a_i + \max_{i \in R} b_i, \tag{2.5}$$

where a_i, b_i are real numbers ($i = 0, \ldots, N$),

$$R = \{ i \in [0:N] \; a_i = \max_{j \in [0:N]} a_j \}.$$

It follows from (2.4), (2.5) that

$$\varphi(x) = \max_{i \in [0:N]} f_i(x) \geqslant \varphi(\bar{x}) + \max_{i \in R(\bar{x})} \{ < f_i'(\bar{x}), x - \bar{x} > + 0_i(\| x - \bar{x} \|) \}$$

$$\geqslant \varphi(\bar{x}) + \max_{i \in R(\bar{x})} \langle f_i'(\bar{x}), x - \bar{x} \rangle + \min_{i \in [0:N]} 0_i(\| x - \bar{x} \|). \tag{2.6}$$

From (2.6), we see that for $\varepsilon > 0$ there exists $\delta_1 > 0$ such that

$$\varphi(x) \geqslant \varphi(\bar{x}) + \max_{i \in R(\bar{x})} < f_i'(\bar{x}), x - \bar{x} > - \varepsilon \| x - \bar{x} \|, \tag{2.7}$$

for all $x \in B(\bar{x}, \delta)$, where $B(\bar{x}, \delta_1)$ stands for the closed ball around $\bar{x}$ with radius δ_1.

Since the feasible set M is approximated at $\bar{x}$ by L, for $x \in M$, $x - \bar{x}$ may be expressed as $x - \bar{x} = x_1 + x_2$ with $x_1 \in L$, $\| x_2 \| = 0(\| x - \bar{x} \|)$. Consequently,

$$\varphi(x) \geqslant \varphi(\bar{x}) + \max_{i \in R(\bar{x})} (< f_i'(\bar{x}), x_1 > - \| f_i'(\bar{x}) \| \| x_2 \|) - \varepsilon \| x - \bar{x} \|$$

$$\geqslant \varphi(\bar{x}) + \max_{i \in R(\bar{x})} < f_i'(\bar{x}), x_1 > - \max_{i \in R(\bar{x})} \| f_i'(\bar{x}) \| \| x_2 \| - \varepsilon \| x - \bar{x} \| \tag{2.8}$$

It follows from Assumption (2.3) that

$$\varphi(x) \geqslant \varphi(\bar{x}) + \beta \| x_1 \| - \max_{i \in R(\bar{x})} \| f_i'(\bar{x}) \| \| x_2 \| - \varepsilon \| x - \bar{x} \|. \tag{2.9}$$

Since $\| x_2 \| = 0 (\| x - \overline{x} \|)$, there is a number $\delta_2 > 0$ such that, for every $x \in B(\overline{x}, \delta_2) \cap M$,

$$\| x_2 \| \leqslant \varepsilon \| x - \overline{x} \| , \tag{2.10}$$

which implies

$$\| x_1 \| = \| x - \overline{x} - x_2 \| \geqslant (1 - \varepsilon) \| x - \overline{x} \| . \tag{2.11}$$

Taking $\delta = \min \{\delta_1 , \delta_2\}$ and substituting from (2.10), (2.11) in (2.9) we have

$$\varphi (x) \geqslant \varphi(\overline{x}) + [\beta(1 - \varepsilon) - \varepsilon A_1 - \varepsilon] . \| x - \overline{x} \| , \tag{2.12}$$

for every $x \in B(\overline{x}, \delta) \cap M$; here

$$A_1 = \max_{i \in R(\overline{x})} \| f'_i (\overline{x}) \| .$$

For $\varepsilon > 0$ small enough, $\beta (1 - \varepsilon) - \varepsilon A_1 - \varepsilon > 0$. Therefore, $\varphi (x) > \varphi (\overline{x})$ for all $x \in B(\overline{x}, \delta) \cap M$, $x \neq \overline{x}$.

The proof is complete.

We would like to point out two consequences of Theorem 2.1 for the class of smooth problems studied in [3] and for the class of finite-dimensional discrete minimax problems studied by Dem'yanov, Malozemov in [1]. First, let us consider the following problem:

$$\text{(II)} \begin{cases} \text{minimize } f_0(x) \\ \text{subject to } F(x) \in K, \\ \text{and } x \in C , \end{cases}$$

where f_o, F, K, C are as in Problem (I). Problem (II) is discussed in [3] by the author, and in [2] by Zowe and Kurcyusz.

COROLLARY 2. 1. Let $\overline{x}$ be a regular point of Problem (II). *Suppose that the maps f_o, F are Eréchet differentiable at $\overline{x} \in M$. Assume, in addition, that there is a number $\beta > 0$ such that*

$$< f'_o(\overline{x}), x > \geqslant \beta \; \| x \| \qquad \textit{for all } x \in L.$$

Then, $\overline{x}$ is a strict local minimum of Problem (II).

Consider the following discrete minimax problem:

$$\text{(III)} \begin{cases} \text{minimize} \quad \max\limits_{i \in [0 : N]} f_i(x), \\ \text{subject to } x \in C, \end{cases}$$

where f_i are functionals defined on a n-dimensional space $\mathbf{R}^n$, C is a non-empty closed convex subset of $\mathbf{R}^n$. This problem is discussed by Dem'yanov and Malozemov in [1].

COROLLARY 2.2. *Suppose that* $f_o, \ldots, f_N$ *are differentiable at* $\bar{x} \in C$. *Furthermore, assume that*

$$\min_{\substack{g \in C(\bar{x}) \\ \|g\|=1}} \max_{i \in R(\bar{x})} < \frac{\partial f_i(\bar{x})}{\partial x}, g > > 0, \tag{2.13}$$

where $\frac{\partial f_i(\bar{x})}{\partial x} = \left(\frac{\partial f_i(\bar{x})}{\partial x_1}, \ldots, \frac{\partial f_i(\bar{x})}{\partial x_n} \right)$ *(usual derivative). Then,* $\bar{x}$ *is a strict local solution of Problem (III).*

Proof. Since the set $\{ g \in \mathbf{R}^n / \| g \| = 1\}$ is compact, Condition (2. 13) is equivalent to

$$\max_{i \in R(\bar{x})} < \frac{\partial f_i(\bar{x})}{\partial x}, g > \geqslant \beta \| g \| \text{ for some } \beta > 0 \text{ and all } g \in C(\bar{x}).$$

Thus, all the hypotheses of Theorem 2.1 hold and therefore the corollary follows.

Corollarises 2.1 and 2.2 may be found in [3] and [1] resp.

3. SECOND-ORDER SUFFICIENT OPTIMALITY CONDITIONS

We now formulate a second-order sufficient optimality condition for Problem (I).

THEOREM 3.1. *Let* $\bar{x}$ *be a regular point of Problem (I). Suppose that the maps* $f_o, \ldots f_N$, F *are twice continuously Fréchet differentiable at* $\bar{x}$ *and*

a) there is a Lagrange multiplier $\wedge \in K*$ *such that for every* $x \in M$,

$$\max_{i \in (Rx)} \langle f'_i(\bar{x}), x - \bar{x} \rangle - \langle \wedge, F'(\bar{x})(x - \bar{x}) \rangle \geqslant 0, \tag{3.1}$$

$$\langle \wedge, F(\bar{x}) \rangle = 0 ; \tag{3.2}$$

b) there exists a number $\sigma > 0$ *such that for every* $g \in L$, $x \in M$,

$$\max_{i \in R_2(\bar{x}, x-\bar{x})} f''_i(\bar{x})(g, g) - \langle \wedge, F''(\bar{x})(g, g) \rangle \geqslant \sigma \| g \|^2, \tag{3.3}$$

where

$$R_2(\bar{x}, x - \bar{x}) = \{ i \in R(\bar{x}) \mid \langle f'_i(x), x - \bar{x} \rangle = \max_{j \in R(\bar{x})} \langle f'_i(\bar{x}), x - \bar{x} \rangle \}.$$

Then, $\bar{x}$ is a strict local solution of Problem (I).

Proof. Take $x \in M$, $x \neq \bar{x}$. As f_i is twice differentiable one has

$$f_i(x) = f_i(\bar{x}) + \langle f_i'(\bar{x}), x - \bar{x} \rangle + \frac{1}{2} f_i''(\bar{x})(x - \bar{x}, x - \bar{x}) + 0_i(\| x - \bar{x} \|^2), \quad (3.4)$$

where $0_i(\| x - \bar{x} \|^2) / \| x - \bar{x} \|^2 \to 0$ (when $\| x - \bar{x} \| \to 0$).

Since $\langle \Lambda, F(x) \rangle \geqslant 0$, it follows from (3.4) and Assumption (3.2) that, for each $i \in [0 : N]$,

$$f_i(x) \geqslant f_i(x) - \langle \Lambda, F(x) \rangle = f_i(\bar{x}) + \langle f_i'(\bar{x}), x - \bar{x} \rangle +$$

$$+ \frac{1}{2} f_i''(\bar{x})(x - \bar{x}, x - \bar{x}) - \langle \Lambda, F'(\bar{x})(x - \bar{x}) \rangle -$$

$$- \frac{1}{2} \langle \Lambda, F''(\bar{x})(x - \bar{x}, x - \bar{x}) \rangle + \tilde{0}_i(\| x - \bar{x} \|^2), \quad (3.5)$$

where $\tilde{O}_i(\| x - \bar{x} \|^2) / \| x - \bar{x} \|^2 \to 0$ (when $\| x - \bar{x} \| \to 0$).

Using (2.5) and (3.5), we have

$$\varphi(x) = \max_{i \in [0 : N]} f_i(x) \geqslant \varphi(\bar{x}) + \max_{i \in R(\bar{x})} \{ \langle f_i'(\bar{x}), x - \bar{x} \rangle + \frac{1}{2} f_i''(\bar{x})(x - \bar{x}, x - \bar{x}) +$$

$$+ \tilde{0}_i(\| x - \bar{x} \|^2) \} - \langle \Lambda, F'(\bar{x})(x - \bar{x}) \rangle - \frac{1}{2} \langle \Lambda, F''(\bar{x})(x - \bar{x}, x - \bar{x}) \rangle,$$

which yields

$$\varphi(x) \geqslant \varphi(\bar{x}) + \max_{i \in R(\bar{x})} \langle f_i'(\bar{x}), x - \bar{x} \rangle + \frac{1}{2} \max_{i \in R_2(\bar{x}, x - \bar{x})} f_i''(\bar{x})(x - \bar{x}, x - \bar{x}) +$$

$$+ \min_{i \in [0 : N)} \tilde{0}_i(\| x - \bar{x} \|^2) - \langle \Lambda, F'(\bar{x})(x - \bar{x}) \rangle - \frac{1}{2} \langle \Lambda, F''(\bar{x})(x - \bar{x}, x - \bar{x}) \rangle. \quad (3.6)$$

It follows from (3.6) and Assumption (3.1) that for $\varepsilon > 0$ there exists $\delta_1 > 0$ such that, for every $x \in B(\bar{x}, \delta_1) \cap M$,

$$\varphi(x) \geqslant \varphi(\bar{x}) + \frac{1}{2} \{ \max_{i \in R_2(\bar{x}, x - \bar{x})} f_i''(\bar{x})(x - \bar{x}, x - \bar{x}) - \langle \Lambda, F''(\bar{x})(x - \bar{x}, x - \bar{x}) \rangle -$$

$$\varepsilon \| x - \bar{x} \|^2 \}. \quad (3.7)$$

Arguing as in the proof of Theorem 2.1, for $x \in M$, we have $x - \bar{x} = x_1 + x_2$ with $x_1 \in L$, $\| x_2 \| = 0(\| x - \bar{x} \|)$. Hence, there is a number $\delta_2 > 0$ such that, for very $x \in B(\bar{x}, \delta_2) \cap M$, the following relations hold

$$\| x_2 \| \leqslant \varepsilon \| x - \bar{x} \|, \quad \| x_1 \| \geqslant (1 - \varepsilon) \| x - \bar{x} \|. \quad (3.8)$$

This implies

$$\| x_2 \| \leqslant \frac{\varepsilon}{1-\varepsilon} \| x_1 \| . \tag{3.9}$$

Moreover,

$$\max_{i \in R_2(\bar{x}, x-\bar{x})} f_i''(\bar{x})(x-\bar{x}, x-\bar{x}) = \max_{i \in R_2(\bar{x}, x-\bar{x})} \{ f_i''(\bar{x})(x_1, x_1) + 2 f_i''(\bar{x})(x_1, x_2) + f_i''(\bar{x})(x_2, x_2) \}$$

$$\geqslant \max_{i \in R_2(\bar{x}, x-\bar{x})} \{ f_i''(\bar{x})(x_1, x_1) - 2 \| f_i''(\bar{x}) \| \| x_1 \| \| x_2 \| - \| f_i''(\bar{x}) \| . \| x_2 \|^2 \}$$

$$\geqslant \max_{i \in R_2(\bar{x}, x-\bar{x})} f_i''(\bar{x})(x_1, x_1) - 2 \max_{i \in R(\bar{x})} \| f_i''(\bar{x}) \| \| x_1 \| \| x_2 \| - \max_{i \in R(\bar{x})} \| f_i''(\bar{x}) \| \| x_2 \|^2 . \tag{3.10}$$

Hence, in view of Assumption b) and (3.10), one has

$$\varphi(x) \geqslant \varphi(\bar{x}) + \frac{\sigma}{2} \| x_1 \|^2 - \max_{i \in R(\bar{x})} \| f_i''(\bar{x}) \| \| x_1 \| \| x_2 \| -$$

$$- \frac{1}{2} \max_{i \in R(\bar{x})} \| f_i''(\bar{x}) \| \| x_2 \|^2 - \frac{\varepsilon}{2} \| x - \bar{x} \|^2 . \tag{3.11}$$

Setting $\delta = \min \{ \delta_1, \delta_2 \}$ and substituting (3.8), (3.9) in (3.11), we have

$$\varphi(x) \geqslant \varphi(\bar{x}) + \frac{1}{2} \{ \sigma(1-\varepsilon)^2 - 2\varepsilon(1-\varepsilon) A_2 - \varepsilon^2 A_2 - \varepsilon \} \| x - \bar{x} \|^2 , \tag{3.12}$$

for all $x \in B(\bar{x}, \delta) \cap M$, here $A_2 = \max_{i \in R(\bar{x})} \| f_i''(\bar{x}) \|$.

Consequently, for $\varepsilon > 0$ small enough, $\varphi(x) > \varphi(\bar{x})$ for all $x \in B(\bar{x}, \delta) \cap M$, $x \neq \bar{x}$. This completes the proof.

Applying Theorem 3.1 to Problem (II), we obtain a second-order sufficient optimality condition for this problem (which may be found in [3]).

COROLLARY 3.1. *Let $\bar{x}$ be a regular point of Problem (II).*
Suppose that f_0, F are twice continuously Fréchet differentiable at $\bar{x}$ and

a) there exists a Lagrange multiplier $\Lambda \in K^*$ such that
$\mathcal{L}'_x(\bar{x}, \Lambda) \in (C(\bar{x}))^*$, where $\mathcal{L}(x, \Lambda) = f_0(x) - \langle \Lambda, F(x) \rangle$,
$\langle \Lambda, F(\bar{x}) \rangle = 0$;

b) there is a number $\sigma > 0$ such that

$$\mathcal{L}''_x(\bar{x}, \Lambda)(\xi, \xi) \geqslant \sigma \|\xi\|^2 \quad \text{for all } \xi \in L.$$

Then, $\bar{x}$ is a strict local solution of Problem (II).

We close this section with an application to Problem (III).

COROLLARY 3.2. Suppose that $f_0, \ldots, f_N$ are twice continuously differentiable at $\bar{x} \in C$. Furthermore, assume that

$$a) \quad \min_{\substack{g \in C(\bar{x}) \\ \|g\| = 1}} \max_{i \in R(\bar{x})} \left\langle \frac{\partial f_i(\bar{x})}{\partial x}, g \right\rangle \geqslant 0, \tag{3.13}$$

$$b) \quad \min_{\substack{g \in C(\bar{x}) \\ \|g\| = 1}} \max_{i \in R_2(\bar{x}, g)} \left\langle \frac{\partial^2 f_i(\bar{x})}{\partial x^2} g, g \right\rangle > 0, \tag{3.14}$$

$$\text{where} \quad \frac{\partial^2 f_i(\bar{x})}{\partial x^2} = \begin{pmatrix} \frac{\partial^2 f_i(\bar{x})}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f_i(\bar{x})}{\partial x_1 \partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 f_i(\bar{x})}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f_i(\bar{x})}{\partial x_n \partial x_n} \end{pmatrix}$$

Then, $\bar{x}$ is a strict local solution of Problem (III).

Proof. In R^n, Condition (3.14) is equivalent to

$$\max_{i \in R_2(\bar{x}, g)} \left\langle \frac{\partial^2 f_i(\bar{x})}{\partial x^2} g, g \right\rangle \geqslant \sigma \|g\|^2 \text{ for some } \sigma > 0 \text{ and every } g \in C(\bar{x}).$$

Thus, all the assumptions of Theorem 3.1 hold and the corollary follows.

4. CASE $C = X$ AND $K = \{0\} \times R^m_-$

Let us consider the case $K = \{0\} \times R^m_-$, $C = X$, where $\{0\} \subset Y_1$ (Banach space), R^m_- is the non-positive orthant of R^m. The problem we are concerned with can be formulated as follows:

$$
\text{(IV)} \quad \begin{cases} \max\limits_{i \in [0:N]} f_i(x) \to \min, \\ F(x) = 0, \\ h_i(x) \leqslant 0 \qquad (i = 1, \dots, m), \end{cases}
$$

where F is a map from X into Y_1; f_i, h_j $(i = 0, \dots, N; \; j = 1, \dots, m)$ are functionals defined on X.

Denote by M_1 the feasible set of Problem (IV). Consider a point $x \in M_1$ and the set

$$
I = \{ i \in [1 : m] \; / \; h_i(\bar{x}) = 0 \},
$$

and the linearizing cone L_1 of M_l at $\bar{x}$:

$$
L_1 = \{ x \in X \, / < h_i'(\bar{x}), x \rangle \leqslant 0 \qquad (i \in I), \qquad F'(\bar{x})x = 0 \}.
$$

In this section we assume that the maps F, f_i, h_j $(i \in [0 : N]$, $j \in I)$ are Fréchet differentiable at $\bar{x}$.

It follows fom Theorem 4. 1 in [3] that if $F'(\bar{x}) = Y_1$, the feasible set M_1 may be approximated at $\bar{x}$ by L_1.

By an argument analogous to that used in the proof of Theorem 2. 1 we obtain the following first-order sufficient optimality condition of Problem (IV).

THEOREM 4.1. *Assume that $F'(\bar{x})X = Y_1$ and there is a number $\beta > 0$ such that*

$$
\max_{i \in R(\bar{x})} \langle f_i'(\bar{x}), x \rangle \geqslant \beta \| x \| \qquad \text{for all } x \in L_l . \tag{4.1}
$$

Then, $\bar{x}$ is a strict local minimum of Problem (IV).

Consider the following smooth Problem :

$$
\text{(V)} \quad \begin{cases} f_o(x) \to \min, \\ F(x) = 0, \\ h_i(x) \leqslant 0 \qquad (i = , \dots 1, m) \end{cases}
$$

where f_o, F, h_i $(i \in [1 : m])$ are as in Problem (IV).

From Theorem 4.1, taking $N = 0$, we obtain as an immediate consequence:

COROLLARY 4. 1. *Assume that $F'(\bar{x})X = Y_1$ and there is a number $\beta > 0$ such that*

$$
\langle f_0'(\bar{x}), x \rangle \geqslant \beta \| x \| \; \text{for all } x \in L_1.
$$

Then, $\bar{x}$ is a strict local minimum of Problem (V).

A modified first-order sufficient condition for Problem (IV) can be formulated as follows

THEOREM 4.2. *Assume that* $F'(\bar{x})X = Y_1$. *Suppose, furthermore, that there exist Lagrange multipliers* $y^* \in Y^*$, $\lambda_i \geqslant 0$, $\lambda_j > 0$ $(i \in I_1, j \in I \setminus I_1, I_1$ *is a non-empty subset of* I), *and a number* $\beta > 0$ *such that*

$$a)\ \max_{i \in (R\bar{x})} \langle f'_i(\bar{x}), x - \bar{x} \rangle + \langle y^*, F'(\bar{x})(x - \bar{x}) \rangle +$$

$$+ \sum_{i \in I} \lambda_i \langle h'_i(\bar{x}), x - \bar{x} \rangle \geqslant 0 \tag{4.2}$$

for all $x \in M_1$;

$$b)\ \max_{i \in R\bar{x})} \langle f'_i(\bar{x}), g \rangle \geqslant \beta \| g \| \text{ for all } g \in L_2, \text{ where} \tag{4.3}$$

$$L_2 = \{ x \in X \mid \langle h'_i(\bar{x}), x \rangle \leqslant 0, \langle h'_j(\bar{x}), x \rangle = 0 \quad (i \in I_1, j \in I \setminus I_1), F'(\bar{x})x = 0 \}.$$

Then, $\bar{x}$ *is a strict local minimum of Problem (IV).*

Remark. It is interesting to note that $L_2 \subset L_1$. Thus, Condition (4.3) is weaker than (4.1), and hence, in Theorem 4.2, Condition (4.2) must be added. Theorem 4.2 contains Theorem 4.4 in [3] as a special case.

Proof of Theorem 4.2. Let x be a feasible point of Problem (IV), $x \neq \bar{x}$. In virtue of the approximation property of M_1 and Hoffman's lemma (see, [4]), x may be expressed as the sum $x = \bar{x} + x'_1 + x''_1 + x_2$, with $x_1 = x'_1 + x''_1 \in L_1$,

$x'_1 \in L_2$, $\| x_2 \| = 0(\| x - \bar{x} \|)$, and x'_1 satisfying

$$\| x''_1 \| \leqslant C_1 \{ \sum_{i \in I \setminus I_1} | \langle h'_i(\bar{x}), x_1 \rangle | + \sum_{i \in I_1} \langle h'_i(\bar{x}), x_1 \rangle_+ \}, \ (C_1 > 0) \tag{4.4}$$

where

$$\langle h'_i(\bar{x}), x_1 \rangle_+ = \begin{cases} \langle h'_i(\bar{x}), x \rangle, & \text{if } \langle h'_i(\bar{x}), x \rangle \geqslant 0 \\ 0 & , \text{otherwise.} \end{cases}$$

Since $x_1 \in L_1$, it follows readily from (4.4) that

$$\| x''_1 \| \leqslant C_1 \{ - \sum_{i \in I \setminus I_1} \langle h'_i(\bar{x}), x_1 \rangle \}. \tag{4.5}$$

Observe that, for $\varepsilon > 0$, there exists $\delta_1 > 0$ such that for every $x \in B(\bar{x}, \delta_1) \cap M_1$,

$$\| x_2 \| \leqslant \varepsilon \| x - \bar{x} \|, \ \| x_1 \| \geqslant (1 - \varepsilon) \| x - \bar{x} \| . \tag{4.6}$$

For each $i \in [0:N]$, we have

$$f_i(x) = f_i(\bar{x}) + \langle f_i'(\bar{x}), x - \bar{x} \rangle + 0_i(\| x - \bar{x} \|)$$

One can choose a number $C_2 > 0$ such that

$$C_1^{-1} C_2 \min_{i \in I \setminus I_1} \lambda_i - | I | \max_{i \in I} (\lambda_i \| h_i'(\bar{x}) \|) - 1 > 0, \quad (4.7)$$

where $| I |$ is the number of elements of I.

Consider two cases

a) $\| x_1 \| > C_2 \varepsilon \| x - \bar{x} \|$.

From (4.5), we see that

$$C_2 \| x - \bar{x} \| < \| x_1'' \| \leqslant C_1 \{ - \sum_{i \in I \setminus I_1} \langle h_i'(\bar{x}), x_1'' \rangle \}. \quad (4.8)$$

In view of Assumption a) it follows from Inequality (2.5) that, for $\varepsilon > 0$, there is a number $\delta_2 > 0$ ($\delta_2 \leqslant \delta_1$) such that for every $x \in B(\bar{x}, \delta_2) \cap M_1$,

$$\varphi(x) = \max_{i \in [0:N]} f_i(x) \geqslant \varphi(\bar{x}) + \max_{i \in R(\bar{x})} \langle f_i'(\bar{x}), x - \bar{x} \rangle + \min_{i \in [0:N]} 0_i(\| x - \bar{x} \|)$$

$$\geqslant \varphi(\bar{x}) + \max_{i \in R(\bar{x})} \langle f_i'(\bar{x}), x - \bar{x} \rangle + \langle y^*, F(x) \rangle +$$

$$+ \sum_{i \in I} \lambda_i \langle h_i'(\bar{x}), x - \bar{x} \rangle - \sum_{i \in I} \lambda_i \langle h_i'(\bar{x}), x - \bar{x} \rangle - \varepsilon \| x - \bar{x} \|$$

$$\geqslant \varphi(\bar{x}) - \sum_{i \in I} \lambda_i \langle h_i'(\bar{x}), x - \bar{x} \rangle - \varepsilon \| x - \bar{x} \|. \quad (4.9)$$

Substituting (4.6), (4.7), (4.8) in (4.9) yields that, for every $x \in B(\bar{x}, \delta_2) \cap M_1$, $x \neq \bar{x}$,

$$\varphi(x) \geqslant \varphi(\bar{x}) - \sum_{i \in I \setminus I_1} \lambda_i \langle h_i'(\bar{x}), x_1 \rangle - \sum_{i \in I} \lambda_i \langle h_i'(\bar{x}), x_2 \rangle - \varepsilon \| x - \bar{x} \|$$

$$\geqslant \varphi(\bar{x}) - \sum_{i \in I \setminus I_1} \lambda_i \langle h_i'(\bar{x}), x_1'' \rangle - | I | \max_{i \in I} (\lambda_i \| h_i'(\bar{x}) \|) \| x_2 \| - \varepsilon \| x - \bar{x} \|$$

$$\geqslant \varphi(\bar{x}) + C_1^{-1} C_2 \varepsilon (\min_{i \in I \setminus I_1} \lambda_i) \| x - \bar{x} \| - \varepsilon | I | \max_{i \in I} (\lambda_i \| h_i'(\bar{x}) \|) \| x - \bar{x} \| - \varepsilon \| x - \bar{x} \|$$

$$> \varphi(\bar{x}).$$

b) $\| x_1'' \| \leqslant C_2 \varepsilon \| x - \bar{x} \|$.

For each $i \in [O:N]$, one has

$$f_i(x) = f_i(\overline{x}) + \langle f'_i(\overline{x}), x'_1\rangle + \langle f'_i(\overline{x}), x''_i\rangle + \langle f'_i(\overline{x}), x_2\rangle + 0_i(\|x-\overline{x}\|)$$

Using Inequality (2.5) we see that

$$\varphi(x) = \max_{i\in[0:N]} f_i(x) \geqslant \varphi(\overline{x}) + \max_{i\in R(x)} (\langle f'_i(\overline{x}), x'_1\rangle - \|f'_i(\overline{x})\| \; \|x''_1\| -$$

$$- \|f'_i(\overline{x})\| \; \|x_2\|) + \min_{i\in[0:N]} 0_i(\|x-\overline{x}\|)$$

$$\geqslant \varphi(\overline{x}) + \max_{i\in R(\overline{x})} \langle f'_i(\overline{x}), x'_1\rangle - \max_{i\in R(\overline{x})} \|f'_i(\overline{x})\| \; \|x''_1\| - \max_{i\in R(\overline{x})} \|f'_i(\overline{x})\| \; \|x_2\|$$

$$+ \min_{i\in[0:N]} 0_i(\|x-\overline{x}\|). \tag{4.10}$$

Observe that, by (4.6)

$$\|x'_1\| \geqslant \|x_1\| - \|x''_1\| \geqslant (1-\varepsilon-C_2\varepsilon)\|x-\overline{x}\| \tag{4.11}$$

From (4.6), (4.10) and (4.11) there is a number $\delta_3 > 0$ ($\delta_3 \leqslant \delta_1$) such that

$\varphi(x) \geqslant \varphi(\overline{x}) + \beta(1-\varepsilon-C_2\varepsilon)\;\|x-\overline{x}\| - C_2A_1\varepsilon\,\|x-\overline{x}\| - A_1\varepsilon\|x-\overline{x}\| - \varepsilon\,\|x-\overline{x}\|$,
for all $x \in B(\overline{x}, \delta_3) \cap M_1$
where

$$A_1 = \max_{i\in R(\overline{x})} \|f'_i(\overline{x})\|.$$

Therefore, $\varphi(x) > \varphi(\overline{x})$ for sufficiently small $\varepsilon > 0$ and for all $x \in B(\overline{x}, \delta_3) \cap M_1$, $x \neq \overline{x}$.
The proof is complete.

COROLLARY 3.2. *Suppose that $F'(\overline{x})X = Y$ and there exist Lagrange multipliers $y^* \in Y^*$, $\lambda_i \geqslant 0$, $\lambda_j > 0$ ($i \in I_1$, $j \in I\setminus I_1$, I_1—a non-empty subset of I, and a number $\beta > 0$ such that*

a) $\mathcal{L}'_x(\overline{x}, \lambda_1, \dots, y^*) = 0$, where

$$\mathcal{L}(x, \lambda_1, \dots, y^*) = f_0(x) + \sum_{i\in I} \lambda_i h_i(x) + \langle y^*, F(x)\rangle,$$

b) $\langle f'_0(\overline{x}), x\rangle \geqslant \beta\|x\|$ for all $x \in L_2$.

Then, $\overline{x}$ is a strict local minimum of Problem (V).

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