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ON INEQUALITIES FOR DERIVATIVES OF MULTIVARIATE FUNCTIONS

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1. INTRODUCTION

Let R be the real field, N be the set of natural numbers and T be the interval $[-\pi,\pi]$. Denote by $L_2(T^n)$ the space of periodic functions $x(t)=x(t_1,t_2,...,t_n)$ with period 2π on each variable, for which the norm

$$\| x \|_e = \left(\int_{T^n} | x(t) |^2 dt\right)^{1/2}$$

is finite and

$$\int_T x(t)\, dt_j = 0 \ , \ j = 1, 2,..., n$$

For $x(t) \in L_2^0(T^n)$ and $\alpha \in R^n$, denote by $x^{(\alpha)}(t)$ the Weyl derivative of order α of the function $x(t)$ (see e. g. [1]).

For any $\alpha^j \in R^n$, $\gamma_j > 0$, $j = 1, 2,..., m$, let

$$\overline{\alpha} = \{\alpha^1 , \alpha^2 ,..., \alpha^m\}, \qquad \gamma = (\gamma_1 ; \gamma_2 ,..., \gamma_m) .$$

By $W_2^{\overline{\alpha}}(\gamma)$ we denote the class of functions $x(t) \in L_2^0(T^n)$ such that

$$\| x^{(\alpha^j)} \|_2^2 \leqslant \gamma_j , \ j = 1, 2, ... , m.$$

Given $\alpha^0 \in R^n$, consider the problem of inequality of Hardy-Littlewood—Polya's type for derivatives : Find the supremum

$$S(\gamma) = sup \left\{ \| x^{(\alpha^0)} \|_2^2 \mid x \in W_2^{\overline{\alpha}}(\gamma)\right\}. \tag{1}$$

Problem (1) was studied and solved for the case $n = 1$ by Din' Zung—Tihomirov [1]. They also solved an analogous problem for functions in R^n. It was shown in [1 —2] that problem (1) may be led to a linear programming problem. We refer the reader to [3] for a survey on the problem of inequalities for derivatives.

The aim of this paper is to use Din' Zung—Tihomirov's method for solving some special cases of problem (1). Moreover, we shall find sufficient and necessary conditions under which $S_{(\gamma)}$ is of the multiplicative form, i. e.

$$S_{(\gamma)} = \prod_{j=1}^{m} \gamma_j^{\lambda_j} .$$

2. NOTATIONS AND RESULTS

In this paper we denote by conv $\bar{\alpha}$ the convex hull of $\bar{\alpha}$ and by R_+ the set of all nonnegative real numbers.

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n$, and $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in R^n$, we write $\alpha \leqslant \beta$ if $\alpha_i \leqslant \beta_i$ for all $i = 1, 2, \ldots, n$.

We say that α is comparable with β if either $\alpha \leqslant \beta$ or $\beta \leqslant \alpha$ holds.

DEFINITION 1. We say that $\bar{\alpha}$, γ satisfy the condition (Q1) if for any $\varepsilon \in (0,1)$ there exists $K_\varepsilon \in N^n$ such that $\bigcap_{j=1}^{m} \left[\frac{\varepsilon\gamma_j}{K_\varepsilon^{2\alpha j}} , \frac{\gamma_j}{K_\varepsilon^{2\alpha^j}} \right]$ is nonempty.

DEFINITION 2. We say that $\bar{\alpha}$ satisfies the condition (Q2) if for any constants A_i, B_i $(i = 1, 2, \ldots, m-1)$, satisfying $0 < A_i < B_i$ $(i = 1, 2, \ldots, m-1)$, there exists $K \in N^n$ such that

$$A_i \leqslant K^{2(\alpha^i - \alpha^m)} \leqslant B_i, \; i = 1, 2, \ldots, m-1$$

The following statement is proved by Din' Zung-Tihomirov [1]:

$$S(\gamma) < +\infty \Leftrightarrow (\alpha^0 + R_+^n) \cap conv\ \bar{\alpha} \neq \emptyset. \tag{2}$$

This is equivalent to the fact that there exist nonnegative numbers λ_1, $\lambda_2, \ldots, \lambda_m$ satisfying

$$\sum_{i=1}^{m} \lambda_i = 1, \; \alpha^0 \leqslant \sum_{i=1}^{m} \lambda_i \alpha^i \tag{2'}$$

In the case $\alpha^0 \in \operatorname{conv} \bar{\alpha}$ we have $\alpha^0 = \sum_{i=1}^{m} \lambda_i \alpha^i$ where

$$\lambda_i \geqslant 0,\ i = 1, 2, \dots, m,\ \sum_{i=1}^{m} \lambda_i = 1 \tag{3}$$

THEOREM 1. *Let $\alpha^0 \in \operatorname{conv} \bar{\alpha}$ and suppose that $\bar{\alpha}$ and γ satisfy Condition (Q1). Then*

$$S(\gamma) = \prod_{j=1}^{m} \gamma_j^{\lambda_j}$$

where λ_j, $j = 1, 2, \dots, m$ satisfy (3).

THEOREM 2. *Assume that $\alpha^0 \in \operatorname{conv} \bar{\alpha}$ and $\bar{\alpha}$ satisfies Condition (Q2). Then the conclusion of Theorem 1 is true.*

Now we consider the case when $\alpha^0, \alpha^1, \dots, \alpha^m$ are on a straight line. In other words, assume that there exist real numbers v_j, $j = 0, 1, 2, \dots m$ such that

$$\alpha^j - \alpha^1 = v_j(\alpha^2 - \alpha^1),\ j = 0, 1, 2, \dots, m.$$

We denote by Q the set of all real numbers of the form $K^{2(\alpha^2 - \alpha^1)}$ where $K \in N^n$. Elements of Q are called ‹Q-integers›.

Let $\alpha^0 \in [\alpha^i, \alpha^j]$ and let λ (i, j) and v (i, j) be nonnegative numbers such that

$$\lambda(i, j) + v(i, j) = 1,\ \alpha^0 = \lambda(i, j)\, \alpha^i + v(i, j)\, \alpha^i.$$

If α^i, α^j are comparable for any i, j, $1 \leqslant i, j \leqslant m$, then we may assume without loss of generality that $\alpha^1 < \alpha^2 < \dots < \alpha^m$ (cf. [2]). Hence, for any $A \geqslant 1$, there exists the largest ' Q-integer' $\leqslant A$, which is denoted by $[A]_Q$ and called the' Q-integer' part of A''.

Let $1 \leqslant i \leqslant j \leqslant m$ and define $S(i, j)$ as follows

a) If $\alpha^0 \notin [\alpha^i, \alpha^j]$, then $S(i, j) = \infty$.

b) If $\alpha^0 \in [\alpha^i, \alpha^j]$, we consider the following two cases:

Case 1. α^i is comparable with α^j

i) If $\gamma_j \leqslant \gamma_i$ then $S(i, j) = \gamma_j$.

ii) If $\gamma_j \quad \gamma_i$ then $S(i, j) = \gamma_i p + \gamma_j p'$, where $p = p(i, j)$, $p' = p'(i, j)$ satisfy the conditions

$$q^{v_i} p + q^{v_j} p' = q^{v_o}$$
$$\tilde{q}^{v_i} p + \tilde{q}^{v_j} p' = \tilde{q}^{v_o}$$

with $q = \left[\left(\frac{\gamma_j}{\gamma_i}\right)^{\frac{1}{v_j - v_i}}\right]_Q$

and $\tilde{q}$ being the smallest' Q-integer' $\geqslant q$.

Case 2. α^i is not comparable with α^j. In this case we put

$$S(i,\ j) = \gamma_i^{\lambda(i,\ j)}\ \gamma_j^{\nu(i,\ j)}$$

Let

$$G(i) = \begin{cases} \gamma i & \text{if } \alpha^i \geqslant \alpha^o \\ +\infty & \text{otherwise} \end{cases}$$

Then we have the following

THEOREM 3. *Let α^o, α^1 .... α^m be on a straight line in R^n. Then*

$$S(\Upsilon) = \min\left(\min_i G(i),\ \min_{i<j} S(i,j)\right)$$

3. PROOFS OF THEOREMS

Using the Parseval identity we see that for any $\mathbf{x}(t) \in L_2^o(T^n)$, there exists a sequence $\{y_K\}$ ($y_K \geqslant 0$, $K \in N^n$) satisfying the equality for any $\alpha \in R^n$

$$\| x^{(\alpha)} \|_2^2 = \sum_{K \in N^n} K^{2\alpha} y_K \tag{4}$$

Therefore, Problem (1) can be led to the following linear programming problem: find

$$S(\gamma) = \sup \sum_{K \in N^n} K^{2\alpha^0} y_K \tag{5}$$

subject to

$$y_K \geqslant 0\ (K \in N^n);\ \sum_{K \in N^n} K^{2\alpha^i} y_K \leqslant \Upsilon_j (j = 1,2,\dots,m) \tag{6}$$

In order to prove the above stated theorems we need

LEMMA 1. *If α^0 satisfies condition (2') then $S(\gamma) \leqslant \prod_{j=1}^{m} \gamma_j^{\lambda_j}$.*

Proof. Using (4) and Holder's inequality we have

$$\| x^{(\alpha^0)} \|_2^2 = \sum_{K \in N^n} K^{2\alpha^0} y_K \leqslant \sum_{K \in N^n} \prod_{j=1}^{m} (K^{2\alpha^j} y_K)^{\lambda j} \leqslant$$

$$\prod_{j=1}^{m} \left(\sum_{K \in N^n} K^{2\alpha^j} y_K\right)^{\lambda_j} = \prod_{j=1}^{m} \| x^{(\alpha^j)} \|_2^{2\lambda_j} \leqslant \prod_{j=1}^{m} \gamma_j^{\lambda_j}$$

The lemma is thus proved.

Remark 1. From Theorem 2 and Lemma 1 it follows that, in general, there does not exist any constant $C \in (0,1)$, not depending on γ and satisfying

$$S(\gamma) = C \prod_{j=1}^{m} \gamma_j^{\lambda_j} \quad \text{for all } \gamma .$$

Also, if (2') holds and if there exist $C, v_1, v_2, \ldots, v_m$, not depending on γ and satisfying the just written equality, then $C = 1$, $v_j = \lambda_j$, $j = 1, 2, \ldots, m$

and $\alpha^o = \sum_{j=1}^{m} \lambda_j \alpha^j$.

Proof of Theorem 1. For any $\varepsilon \in (0,1)$ there are $K_\varepsilon \in N^n$, $p_\varepsilon > 0$ such that

$$\frac{\varepsilon\gamma_j}{K_\varepsilon^{2\alpha^j}} \leqslant p_\varepsilon \leqslant \frac{\gamma_j}{K_\varepsilon^{2\alpha^j}}, \; j = 1, 2, \ldots, m ,$$

which shows that

$$\varepsilon\gamma_j \leqslant K_\varepsilon^{2\alpha^j} p_\varepsilon \leqslant \gamma_j, \; j = 1, 2, \ldots, m . \qquad (7)$$

Now let us introduce the sequence y_K, $K \in N^n$ by setting

$$y_K = \begin{cases} p_\varepsilon & \text{if } K = K_\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Then by (7) it is easy to verify that the sequence y_K, $K \in N^n$ satisfies (6). Hence

$$S(\gamma) \geqslant \sum_{K \in N^n} K^{2\alpha^o} y_K = K_\varepsilon^{2\alpha^o} = \prod_{j=1}^{m} \left(K_\varepsilon^{2\alpha^j} p_\varepsilon\right)^{\lambda_j} \geqslant \prod_{j=1}^{m} (\varepsilon\gamma_j)^{\lambda_j}$$

Letting $\varepsilon \to 1$, we obtain

$$S(\gamma) \geqslant \prod_{j=1}^{m} \gamma_j^{\lambda_j} ,$$

which together with Lemma 1 proves the theorem.

Remark 2. Using well-known results of linear programming theory and Holder's inequality we can prove that if (3) holds with $\lambda_i > 0$, $i = 1, 2, \ldots, m$ and

$$S(\gamma) = \prod_{j=1}^{m} \gamma_j^{\lambda_j}$$

then $\bar{\alpha}$ and γ satisfy condition (Q1).

Proof of Theorem 2. We shall show that if $\bar{\alpha}$ satisfies (Q2) then $\bar{\alpha}$ and γ satisfy (Q1) for all γ.

Indeed, for any $\varepsilon \in (0, 1)$ there exists $K = K_\varepsilon \in N^n$ such that

$$\frac{\varepsilon\gamma_j}{\gamma_m} \leqslant K^{2(\alpha^j - \alpha^m)} \leqslant \frac{\gamma_j}{\gamma_m};\ j = 1, 2,\dots, m.$$

Putting

$$x_\varepsilon = \frac{\gamma_m}{K^{2\alpha^m}}$$

we have

$$\frac{\varepsilon\gamma_j}{\gamma_m} \leqslant K^{2(\alpha^j - \alpha^m)} \cdot K^{2\alpha^m}\ x_\varepsilon \cdot \frac{1}{\gamma_m} \leqslant \frac{\gamma_j}{\gamma_m},\ j = 1, 2,\dots, m.$$

Therefore,

$$\frac{\varepsilon\gamma_j}{K^{2\alpha^j}} \leqslant x_\varepsilon \leqslant \frac{\gamma_j}{K^{2\alpha^j}},\ j = 1, 2,\dots, m,$$

which means that $\bar{\alpha}$ and γ satisfy condition (Q1). To complete the proof, it remains to apply Theorem 1.

Now we consider some examples, where either condition (Q1) or (Q2) is satisfied.

Example 1. Assume that $m = 1$. Then $\bar{\alpha}$ and γ always satisfy condition (Q1).

Example 2. Suppose that there exist $\lambda > 0$ and $K \in N^n$ such that

$$\gamma_j = \lambda K^{2\alpha^j},\ j = 1, 2,\dots, m.$$

Then $\bar{\alpha}$ and γ satisfy Condition (Q1). Hence, in this case if $\alpha^0 \in \operatorname{conv}\bar{\alpha}$ then

$$S(\gamma) = \prod_{j=1}^{m} \gamma_j^{\lambda_j},$$

where λ_j, $j = 1, 2,\dots, m$ are defined by (3).

Example 3. Assume that $m = 2$ and n is an arbitrary positive integer. We consider the following two cases:

Case 1. α^1 and α^2 are not comparable and $\alpha^0 \in [\alpha^1, \alpha^2]$. Then it can be verified that $\bar{\alpha}$ satisfies (Q2). Consequently,

$$S(\gamma) = \gamma_1^{\lambda(1,2)} \gamma_2^{\nu(1,2)}.$$

Case 2. α^1 and α^2 are comparable and there exists $K \in N^n$ such that

$$K^{2(\alpha^2 - \alpha^1)} = \frac{\gamma_2}{\gamma_1}.$$

Then it is easy to verify that $\bar{\alpha}$ and γ satisfy condition (Q1).

Example 4. Assume that $n = 2(m-1)$. We choose $\alpha^1, \alpha^2, \ldots, \alpha^m$ such that

$$\alpha^j = \alpha^m + (0, 0, \ldots, 0, x_{2j-1}, 0, \ldots, 0)\,;\ j = 1, 2, \ldots, m-1,$$

where α^m is an arbitrary point in R^n and $x_1, x_2, \ldots, x_n$ are real numbers satisfying

$$x_{2j-1}\, x_{2j} < 0 \text{ for } j = 1, 2, \ldots, m-1.$$

Then using the result of the case 1 of example 3 we can check that $\bar{\alpha}$ satisfies (Q2).

Example 5. Assume that $n = m = 3$. For any $\alpha^3 \in R^3$, we put $\alpha^1 = \alpha^3 + (\alpha_1, \alpha_2, \alpha_3)$, $\alpha^2 = \alpha^3 + (\beta_1, \beta_2, \beta_3)$ where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are real numbers such that

$$\alpha_1 \geqslant 0,\ \alpha_3 \geqslant 0,\ \alpha_1 + \alpha_3 > 0,\ \alpha_2 < 0,\ \beta_2 > 0 \text{ and}$$
$$(\alpha_1 \beta_2 - \alpha_2 \beta_1)(\alpha_3 \beta_2 + \alpha_2 \beta_3) < 0.$$

We can prove without difficulty that $\bar{\alpha}$ satisfies (Q2).

Conversely, if $\bar{\alpha}$ satisfies (Q2) then we can reduce our problem to the case, where all conditions of Example 5 holds.

Example 6. Assume that $m = 3$ and $n \geqslant 3$. Let $\alpha' = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \in R^n$, $\beta' = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in R^n$, where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ satisfy all conditions in Example 5. For any $\alpha^3 \in R^n$ we take $\alpha^1 = \alpha^3 + \alpha'$, $\alpha^2 = \alpha^3 + \beta'$. Then $\bar{\alpha}$ satisfies Condition (Q2).

Remark 3: Assume $\tilde{\alpha}^1, \tilde{\alpha}^2, \ldots, \tilde{\alpha}^m$ satisfy Condition (Q2) and let

$$\alpha^i = \Sigma_i (\tilde{\alpha}^i - \tilde{\alpha}^m),\ i = 1, 2, \ldots, m-1,$$

where $\varepsilon_i \neq 0$, $i = 1, 2, \ldots, (m-1)$. Then for any $\alpha^m \in R^n$, $\bar{\alpha} = \{\alpha^1, \alpha^2, \ldots, \alpha^m\}$ satisfies **Condition** (Q2) **too.**

Proof of Theorem 3.

In the case when $\alpha^0, \alpha^1, \ldots, \alpha^m$ are on a straight line, the problem (5)–(6) can be led to the following equivalent problem: find

$$S(\Upsilon) = \sup \sum_{q \in Q} q^{\nu_0} y_q \tag{8}$$

subject to $y_q \geqslant 0$

$$\text{for all } q \in Q, \ \sum_{q \in Q} q^{\nu_i} y_q \leqslant \Upsilon_i, \ i = 1, 2, \dots, m. \tag{9}$$

Applying the method used by Din' Zung [2] in the case $L_2^o(T^1)$ to our case by replacing ' positive integers ' by ' Q-integers ', we can prove that there exist i, j such that the value $S(\gamma)$ of problem (8)—(9) coincides with

$$\tilde{S}(i, j) = \sup \sum_{q \in Q} q^{\nu_0} y_q,$$

where $y_q \geqslant 0$, $q \in Q$; $\sum_{q \in Q} q^{\nu_i} y_q \leqslant \gamma_i$; $\sum_{q \in Q} q^{\nu_j} y_q \leqslant \gamma_j$.

In other words,

$$S(\gamma) = \min \{ \tilde{S}(i, j) \mid 1 \leqslant i < j \leqslant m \}$$

For computation of the value $S(i, j)$ we consider the two following cases.

Case 1: $\alpha^i \leqslant \alpha^j$. It is easy to see that $\alpha^0, \alpha^1, \dots, \alpha^m$ are pairwise comparable. Then without any loss of generality we may assume that $\alpha^1 < \alpha^2 < \dots < \alpha^m$. The method for computing $S(i, j)$ is similar to that of Din' Zung [2] by replacing, positive integers ' by ' Q-integers ' and ' integer part of x ' by ' Q-integer part of x '. We obtain

$S(\gamma) = \min(\min \{ \gamma_r \mid \alpha^r \geqslant \alpha^0 \}, \min \{ \Upsilon_i p + \gamma_j p' \mid \alpha^i < \alpha^0 < \alpha^j, \gamma_i \leqslant \gamma_j).$

where $p = p(i, j)$, $p' = p'(i, j)$ satisfy the following conditions

$$q^{\nu_i} p + q^{\nu_j} p' = q^{\nu_0}$$
$$\tilde{q}^{\nu_i} p + \tilde{q}^{\nu_j} p' = \tilde{q}^{\nu_0},$$

where

$$q = \left[\left(\frac{\gamma_j}{\gamma_i} \right)^{\frac{1}{\nu_j - \nu_i}} \right]_Q \text{ and } \tilde{q} \text{ is the smallest } Q\text{-integer} \geqslant q.$$

Case 2. α^i is not comparable with α^j. From the criterion (2) and the result in Example 3 it follows that

$$\tilde{S}(i, j) = \infty \text{ if } \alpha^0 \notin [\alpha^i, \alpha^j],$$
$$\tilde{S}(i, j) = \gamma_i^{\lambda(i, j)} \gamma_j^{\nu(i, j)} \text{ if } \alpha^0 \in [\alpha^i, \alpha^j],$$

where $\lambda(i, j)$, $\nu(i, j)$ satisfy the following conditions

$$\lambda(i, j) \geqslant 0, \nu(i, j) \geqslant 0, \lambda(i, j) + \nu(i, j) = 1 \text{ and}$$
$$\alpha^0 = \lambda(i, j)\, \alpha^i + \nu(i, j)\, \alpha^j .$$

The theorem is thus proved.

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