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EXISTENCE OF SOLUTIONS FOR A CLASS OF DIFFERENTIAL INCLUSIONS WITH MEMORY

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1. INTRODUCTION

It is well known that differential equations with the continuous right-hand side in Banach spaces, even in the separable Hilbert space, may not have solutions. However, solutions always exist for differential inclusions with the continuous right-hand side which in a certain sense is substantially multi — valued. This is shown in the paper [1] by F.S. De Blasi and G. Piagiani. Namely they proved that the following differential inclusion in reflexive separable Banach spaces always has solutions

$$\begin{cases} \dot{x}(t) \in \Gamma\,(t, x\,(t)). \\ x\,(0) = x_0, \end{cases}$$

if Γ is a continuous map such that, for any t, x, the closed convex hull of $\Gamma(t, x)$ has nonempty interior.

In [3] or [4], Phan Van Chuong has proved that under an additional condition that $\Gamma\,(t, x)$ is convex (with nonempty interior) the inclusion

$$\begin{cases} \dot{x}\,(t) \in \operatorname{Extr}\Gamma\,(t, x\,(t)) \\ x\,(0) = x_0 \end{cases}$$

admits solutions, where Extr A denotes the set of extremal points of $A \subset X$.

The aim of this paper is to extend the just mentioned result of [3] to differential inclusions with memory. Our approach is also based on the well known Baire's category theorem which was initiated by A. Cellina and then developed in [1], [3] and [4].

2. STATEMENT OF THE MAIN RESULT

Throughout this paper, for fixed $T > 0$ and $h \geqslant 0$, I denotes the interval $[-h, T]$ in $\mathbf{R}$, U is a closed ball of a reflexive separable real Banach space X and $G: [0, T] \times U \to 2^X$ is a multifunction taking closed convex values with nonempty interior. We shall suppose that G is continuous with respect to the Hausdorff distance H associated with the norm $\|\cdot\|$ in X and that $G([0, T] \times U)$ is bounded.

We denote by Extr $G(t, x)$ the set of the extremal points of $G(t, x)$. Let φ^0 be a given function in $C_X([-h, 0])$ such that $\varphi^0(\theta) \in \text{int}\, U$ (the interior of U) for all $\theta \in [-h, 0]$ and let $r : [0, T] \to [0, h]$ be a continuous function. We set $t - r(t) = \alpha(t)$ and $x(\alpha(t)) = (T^r x)(t)$.

Consider the following differential inclusion with a retarded argument in the right-hand-side:

$$\begin{cases} \dot{x}(t) \in \text{Extr}\; G(t, T^r x)(t)), & \text{if } t \in [0, T], \\ \dot{x}(\theta) = \varphi^0(\theta), & \text{if } \theta \in [-h, 0] \end{cases} \tag{1}$$

We say that a function $x(.)$ defined on $[-h, T_0]$ $(0 < T_0 \leqslant T)$ is a solution of (1) on $[-h, T_0]$ if it is continuous on $[-h, T_0]$, absolutely continuous on $[0, T_0]$ and satisfies the inclusion (1) on $[-h, T_0]$.

The main result of the paper is stated as follows:

THEOREM 1. *Under the above hypotheses on X and G, the inclusion (1) admits at least one solution defined on an interval $[-h, T_0]$ with $0 < T_0 \leqslant T$.*

3. PROOF OF THEOREM 1

We shall use the following notations: int A (resp. $\overline{A}$): the interior (resp. the closure) of a set $A \subset X$ in the norm topology of X; $B(x,\delta)$: the ball of center $x \in X$ and of radius $\delta > 0$; $B = B(0,1)$; $B((t,x), \delta)$: the ball of center $(t,x) \in R \times X$ and of radius $\delta > 0$ where $R \times X$ is equipped with the norm $\|(t,x)\| = \max\{|t|, \|x\|\}$; $\langle.,.\rangle$: the bilinear form pairing between X and its topological X'; Gr(G): the graph of G, i. e., the set $\{(t, x;\, u) \in [0,T] \times U \times X : u \in G(t,x)\}$; $C_X[-h,T]$(resp. $L^1_X[-h,T]$): the space of continuous function (resp. The space of equivalence classes of integrable functions) from $[-h,T]$ into X ; $\chi_A(.)$: the characteristic function of a measurable set A in $\mathbf{R}$.

We shall also consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in G(t, (T^r x)(t)), & \text{if } t \in [0, T]. \\ x(\theta) = \varphi^0(\theta), & \text{if } \theta \in [-h, 0]. \end{cases} \tag{2}$$

It is easily seen that there exists $T_0 > 0$ such that the differential inclusion (2) admits a solution $x^1(.)$ on $[-h,T_0]$ with constant derivative $\dot{x}^1(.)$ and, moreover, $\dot{x}^1(t) \doteq a \in \text{int}\, G(t, (T^r x^1)(t))$ $(\forall t \in [0, T_0])$.

Indeed, first let us set $x^1(s) = \varphi^o(s)$ $(\forall s \in [-h,0])$ and take $a \in \text{int}\, G(0, \varphi^o(-r(0)))$. Consider the following two cases:

i) $r(0) > 0$. There exists $T_o > 0$ such that $t - r(t) < 0$ and $a \in \text{int}\, G(t, \varphi^o(\alpha(t)))$ $(\forall t \in [0, T_o])$. For every $t \geqslant 0$ we set $x^1(t) = \varphi^o(0) + at$. Obviously, $x^1(0) = \varphi^o(0)$ and $\dot{x}^1(t) = a \in \text{int}\, G(t, (T^r x^1)(t))$ $(\forall t \in [0, T_o])$.

ii) $r(0) = 0$. We take $a \in \text{int}\, G(0, \varphi^o(0))$. Obviously, there exists $T_o > 0$ such that $a \in \text{int}\, G(t, \varphi^o(0) + a\alpha(t))$ $(\forall t \in [0, T_o])$. We set $x^1(t) = \varphi^o(0) + at$. It is easy to see that $\dot{x}^1(t) = a \in \text{int}\, G(t, (T^r x^1)(t))$ $(\forall t \in [0, T_o])$.

Further, let $S_{\ddot{G}}$ (resp. S_G) denote the set of all solutions of (1) (resp. (2)) on $[-h, T_o]$ and let $\overset{\circ}{S}_G$ be the set of all solutions x(.) of (2) on $[-h, T_o]$ with the following property: $\dot{x}(t)$ takes piecewise constant values on $[0,T_o]$ and $\dot{x}(t) \in \text{int}\, G(t, (T^r x)(t))$ $(\forall t \in [0, T_o])$. As was shown above, $x^1(\,.\,) \in \overset{\circ}{S}_G$, hence $\overset{\circ}{S}_G \neq \varnothing$.

By using the same arguments as in [1] it can be shown that S_G is closed in the Banach space $C_X[-h, T_o]$ and hence the closure $\overline{\overset{\circ}{S}}_G$ of $\overset{\circ}{S}_G$ in $C_X[-h, T_o]$ is a nonempty complete set, contained in S_G. In what follows, S_G and $\overline{\overset{\circ}{S}}_G$ will be endowed with the metric of $C_X[-h, T_o]$.

It is well known (see e.g. [3]) that there exists a function $\varphi : \text{Gr}(G) \to [0, +\infty)$ satisfying the following properties:

i) φ is upper bounded on $\text{Gr}(G)$ and upper semicontinuous on $[0, T] \times U \times X$, and for each $(t, x) \in [0, T] \times U$, $\varphi(t, x;.)$ is a concave function on X;

ii) $\varphi(t, x; u) = 0$ if and only if $u \in \text{Extr}\, G(t,x)$.

Consider the following functional on $C_X[-h,T_o] \times L^1_X[-h, T_o]$;

$$J[x(.), u(.)] = \int_0^{T_o} \varphi(t,(T^r x)(t); u(t))dt.$$

For each $\alpha > 0$ we set

$$S^\alpha = \{x(.) \in \overline{\overset{\circ}{S}}_G : J[x(.), \dot{x}(.)] < \alpha\}.$$

LEMMA 1. $$\bigcap_{P=1}^{\infty} S^{1/P} = \overline{\overset{\circ}{S}}_G \cap S_{\ddot{G}}.$$

The proof of this lemma is analogous to that of Lemma 3 in [3].

To prove the main result, it suffices, by Lemma 1, to show that $\bigcap_{P=1}^{\infty} S^{1/P} \neq \varnothing$.

We shall show that each set $S^{1/P}$ is open and dense in $\overline{\overset{\circ}{S}}_G$. Since $\overline{\overset{\circ}{S}}_G$ is a nonempty complete metric space, the conclusion of theorem then follows from the Baire's category theorem.

LEMMA 2. *For any $\alpha > 0$, S^α is open in $\overline{\overset{\circ}{S}}_G$.*

This lemma is proved in the same way as the proof of Lemma 4 in [3].

LEMMA 3. *Let $\alpha > 0$. $I_1 = [t', t'']$, $I_1 \subset [0, T_0]$, $x(.) \in \overset{\circ}{S}_G$ with $\dot{x}(t) = a$ (constant) for every $t \in I_1$. Then there exist $\delta^* > 0$, $c_1 > 0$ such that for every $\delta \in (0, \delta^*]$, and $t_0 \in I_1$ satisfying $[t_0, t_0 + c_1\delta] \subset I_1$, every absolutely continuous function $y(.)$ on $[0, t_0]$ with $y(\theta) = \varphi^0(\theta)$ ($\forall \theta \in [-h, 0]$ such that*

a) $\dot{y}(.)$ is piecewise constant on $[0, t_0]$ and $\dot{y}(t) \in \operatorname{int} G(t, (T^r y)(t))$ for every $t \in [0, t_0]$,

b) $y(t_0) = x(t_0)$,

c) $\| y(t) - x(t) \| < \delta$ *for every* $t \in [0, t_0]$,

d) $\left(\int_0^{t_0} \varphi(t, (T^r y)(t); \dot{y}(t))\, dt < \alpha \frac{t_0}{T_0}\right.$,

can be extended to an absolutely continuous function onto $[-h, \tilde{t}_0]$ with $\tilde{t}_0 = t_0 + \delta c_1$ such that all properties a), b), c), d) with $\tilde{t}_0$ in place of t_0 remain valid.

Proof. Let us take

$$\beta \in \left(0, \min \left\{ 1, \frac{\alpha}{T_0(1 + c)} \right\} \right) \tag{3}$$

where $c = \max \{1, \sup [\| v \| : v \in G([0, T] \times U)],$
$\operatorname{Sup} [\varphi(t, x; u) : (t, x; u) \in \operatorname{Gr}(G)]\}$.

It follows from the boundedness of $G([0, T] \times U)$ and the upper boundedness of φ that $1 \leqslant c < +\infty$.

According to Krein — Milmann's convexity theorem, for every $s \in I_1$ there exist $\xi_s > 0$, $\lambda_i^s > 0$ and $b_i^s \in \operatorname{Extr} G(s, (T^r x)(s))$ $(i = 1, 2, \ldots, n_s)$ such that

$$B(a, 2\xi_s) \subset G(s, (T^r x)(s)), \quad \sum_{i=1}^{n_s} \lambda_i^s = 1$$

and

$$\| a - \sum_{i=1}^{n_s} \lambda_i^s b_i^s \| \leqslant \xi_s \beta. \tag{4}$$

Since $\varphi(s, (T^r x)(s); b_i^s) = 0$ $(i = 1, 2, \ldots n_s)$ there exists $\gamma_s \in (0,1)$ such that $\varphi(s, (T^r x)(s); c_i^s) < \beta/4$, where $c_i^s = (1 - \gamma_s)\, b_i^s + \gamma_s \alpha$ $(i = 1, 2, \ldots n_s)$.

Further, let $\delta_s > 0$ be such that

$$B(a, \xi_s) \subset G(x, z),$$
$$B(c_i^s\, \gamma_s \xi_s) \subset G(x, z), \tag{5}$$
$$\varphi(t, z;\ c_i^s) < \beta/4$$

for every $(t, z) \in B((s, x(\alpha(s))), \delta_s)$ $(i = 1, 2, \ldots, n_s)$.

Let $\{(s_i - \delta s_i/4;\ s_i + \delta s_i/4\}_{i=1}^k$ be a finite subcovering of the open covering $\{(s - \delta s/4, s + \delta s/4)\}_{s \in I_1}$ of I_1. Set

$$\delta_0 = \min_{1 \leqslant i \leqslant k} \{\delta s_i/4\}. \tag{6}$$

For every $\delta_0 > 0$, there exists $\delta_{00} > 0$ such that for every $t_0 \in [0, T_0]$ and for every function $y(.)$ on $[-h, t_0]$ with $y(\theta) = \varphi^0(\theta)$ $(\forall\ \theta \in [-h, t_0])$, $\| \dot{y}(t) \| \leqslant c$ $(\forall\ t \in [0, t_0])$ we have, for all $t, s \leqslant t_0$,

$$| t - s | < \delta_{00} \Rightarrow \| (T^r y)(t) - (T^r y)(s) \| < \delta_0.$$

Indeed, given $\delta_0 > 0$, let δ_0'' be chosen so that for any $t_1, t_2 \in [-h, 0]$, the inequality $|t_1 - t_2| < \delta_0''$ implies $\| \varphi^0(t_1) - \varphi^0(t_2) \| < \delta_0/4$. Set $\delta_0' = \min \{\delta_0'', \delta_0/4c\}$. It is clear that for any $t_1, t_2 \in [-h, 0]$ satisfying $|t_1 - t_2| < \delta_0'$ we have $\| y(t_1) - y(t_2) \| < \delta_0/2$. Take now $\delta_{00} > 0$ such that $|\alpha(t) - \alpha(s)| < \delta_0'$ and $\| T^r y)(t) - (T^r y)(s) \| < \delta_0/2 < \delta$ whenever $t, s < t_0$ and $|t - s| < \delta_{00}$.

Further, set $\delta^* = \min \{\delta_0, \delta_{00}\}$ and $c_1 = \dfrac{(1+\beta)}{6c}$. It is clear that $\delta^* \leqslant \min\limits_{1 \leqslant i \leqslant k} \{\delta s_i / 4\}$. Let $\delta \in (0, \delta^*]$ and $l = c_1 \delta$. Then $l = \dfrac{(1+\beta)\,\delta}{6c} < \delta \leqslant \delta s_i / 4$ $(i = 1, 2, \ldots, k)$. Hence, there exists $j \in \{1, 2, \ldots, k\}$ such that $[t_0, t_0 + l] \subset [s_j - \delta_j / 2, s_j + \delta_j / 2]$, where $\delta_j := \delta s_j$. For simplicity we shall write $\bar{s}$ and $\bar{\delta}$ instead of s_j and δ_j, respectively. Setting $t_i = t_{i-1} + \lambda_i^{\bar{s}}\, \bar{\delta} / 6c$ $(i = 1, 2, \ldots, n_{\bar{s}} + 1)$ with $\lambda_{n_{\bar{s}}+1}^{\bar{s}} = \beta$ and $\Delta_i = [t_{i-1}, t_i]$, we see easily that $t_{\bar{s}+1} = t_0 + l$ and, hence, $[t_0, t_0 + l] = \bigcup_{i=1}^{n_{\bar{s}}+1} \Delta_i$.

For every t_o in $[0, T_o]$ such that $[t_o, t_o + l] \subset I_1$ and every absolutely continuous function $y(.)$ on $[0, t_o]$ satisfying a), b), c), d), set

$$c^{\bar{s}}_{n_{\bar{s}}+1} = a + \frac{\left(a - \sum_{i=1}^{n_{\bar{s}}} \lambda^{\bar{s}}_i c^{\bar{s}}_i\right)}{\beta} \text{ and}$$

$$y(t) = X(t_o) + \int_{t_o}^{t} u(\tau)\, d\tau \quad \text{for every } t \in [t_o, t_o + l], \tag{7}$$

where

$$u(t) = \sum_{i=1}^{n_{\bar{s}}+1} c^{\bar{s}}_i \chi_{\Delta_i}(t) \tag{8}$$

We have

$$\int_{t_0}^{t_0+l} u(\tau)\, d\tau = \int_{t_0}^{t_0+l} \dot{x}(\tau)\, d\tau. \tag{9}$$

Hence, $\| y(t) - x(t) \| < \delta$ for every $t \in [-h, t_o + l]$. Indeed, for $t \leqslant t_o$ the inequality is obvious. If $t \in [t_o, t_o + l]$ we have

$$\| y(t) - x(t) \| = \| \int_{t_0}^{t} (u(\tau) - \dot{x}(\tau))\, d\tau \| = \| \int_{t}^{t_0+l} (u(\tau) - \dot{x}(\tau))\, d\tau \| \leqslant l/2 .$$

$2c < \delta$.

Thus we have c) on $[-h, \tilde{t}_o]$.

It follows from (6) that

$$\| (T^r y)(t) - (T^r x)(\bar{s}) \| \leqslant \| (T^r y)(t) - (T^r y)(\bar{s}) \| +$$

$$\| (T^r y)(\bar{s}) - (T^r x)(\bar{s}) \| \leqslant \delta_o + \delta < 2\delta_o < \bar{\delta} \text{ . Thus}$$

$$B(c^{\bar{s}}_i, \gamma_{\bar{s}} \xi_{\bar{s}}) \subset G(t, T^r y)(t)),$$

$$B(a, \xi_{\bar{s}}) \subset G(t, (T^r y)(t)),$$

hence a) holds on $[-h, \tilde{t}_o]$.

Further, b) follows from (9) and c).

Finally, we have

$$\int_0^{\tilde{t}_o} \varphi(t, (T^r y); \dot{y}(t))\, dt = \int_0^{t_o} \varphi(t, (T^r y)(t); \dot{y}(t))\, dt +$$

$$\int_{t_o}^{t_o+l} \varphi(t, (T^r y)(t); \dot{y}(t))\, dt < \frac{\alpha t_o}{T_o} + \frac{\beta l}{4} < \frac{\alpha(t_o + \delta c_1)}{T_o} = \frac{\alpha \tilde{t}_o}{T_o}$$

Thus, d) also holds on $[-h, \tilde{t}_o]$ and the proof of the lemma hereby it complete.

It follows immediately from Lemma 3 that for any $\alpha > 0$, S^{α} is dense in $\overline{\tilde{S}}_G$. As stated above this completes the proof of the Theorem 1.

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