ACTA MATHEMATICA VIETNAMICA

Vol. 15 No 1 (1990)

OPTIMAL STATE ESTIMATION FOR A STOCHASTIC DYNAMICAL SYSTEM FROM POINT PROCESS OBSERVATIONS

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This paper is devoted to the problem of optimal state estimation for a stochastic dynamical system of the form

$$dX_t = a(t, X_t)\, dt + \sum_{j=1}^{m} a_j(t, X_t)\, d\mu_t^j \qquad (*)$$

where $\mu_t = (\mu_t, ..., \mu_t)$ is a martingale, and the observation process is a point process Y_t of intensity h_t.

Some preliminaries on Innovation Method for Point Process Filtering are given in the two first sections. In Section 3, we recall some results on filtering for a semimartingale from point observations. In Section 3, we consider the filtering process corresponding to system (∗) and derive the filtering equations in the two important cases of Brownian motion and Poisson martingales.

1. PRELIMINARIES

Let (Ω, F, P) be a complete probability space equipped with a right continuous increasing family of σ-fields $F_t\ (t \geqslant 0)$ of F. The signal process will be an F_t-adapted stochastic process X_t. The observations will be given by a n-dimensional point process Y_t of the form

$$Y_t = \int_0^t h_s\, ds + M_t \qquad (1.1)$$

where M_t is an n-dimensional F_t-martingale such that:

a) for any t the future σ-field $\sigma(M_u - M_t : u \geqslant t)$ is independent of the past one $\sigma(\gamma_u, h_u : u \leqslant t)$

b) the n-dimensional process $h_t = (h_t^1, \ldots, h_t^n)$ contains informations about X and $E[\sum_{i=1}^{n} \int_0^t |h_s^i|^2 ds] < \infty$ for any $t \geqslant 0$.

Denote by F_t^Y the σ-field generated by Y_t :

$$F_t^Y = \sigma(Y_s, s \leqslant t).$$

The conditional expectation

$$E(X_t \mid F_t^Y) \tag{1. 2}$$

will be called the optimal state estimation (filtering) of X_t. It will be denoted by π. Thus

$$\pi(X_t) = E(X_t \mid F_t^Y)$$

2. INNOVATION FROM POINT PROCESS OBSERVATIONS

Let m_t be an n-dimensional process defined by

$$m_t = Y_t - \int_0^t \pi(h_s)\, ds \tag{2. 1}$$

THEOREM 1. *m_t is a point process F_t^Y— martingale and for any t, the future σ-field $\sigma(m_u - m_t : u \geqslant t)$ is independent of F_t^Y.*

Proof. We have, for any $t > s$

$$E(m_t - m_s \mid F_s^Y) = E\left(\int_0^t (h_u - \pi(h_u))\, du \mid F_s^Y\right) + E\left(M_t - M_s \mid F_s^Y\right).$$

Since $\quad E\left(h_u \mid F_s^Y\right) = E\left(\pi(h_u) \mid F_s^Y\right) (u \leqslant s)$

it follows that the first member of the right hand side is 0. The second member $E\left[M_t - M_s) \mid F_t^Y\right]$ is also 0 because of independence of the future $\sigma(M_t - M_s,$

$t \geqslant s$) on the past $F_t^Y = \sigma(Y_s, s \leqslant t)$. Therefore $E(m_t - m_s \mid F_s^Y) = 0$, so m_t is an F_t^Y — martingale and $\sigma(m_t - m_s, t \geqslant s)$ is independent of F_s^Y.

Remark. Since m_t is F_t^Y — measurable, it is obvious that the σ—field generated by m_t is included in F_t^Y : $\sigma(m_s ; s \leqslant t) \subseteq F_t^Y$. If $\sigma(m_s ; s \leqslant t) \equiv F_t^Y$ for any t, the process m_t is called the innovation.

THEOREM 2. *(Bremaud*, cf. [1]). *Integral Representation*

THEOREM. *Let R_t be a F_t^Y — martingale. Then there exists a F_t^Y — predictable vector process $K_t = (K_t, \dots, K_t)$ such that for all $t \geqslant 0$*

$$\sum_{i=1}^{N} \int_0^t K_s^i \, \pi(h_s^i) \, ds < \infty \qquad P\text{—a.s.} \tag{2.2}$$

and such that R_t has the following representation

$$R_t = R_0 + \sum_{i=1}^{n} \int_0^t K_s^i \, dm_s^i \, ,$$

or using Kunita's notation [2] :

$$R_t = R_0 + \int_0^t (K_s, dm_s) \, . \tag{2.3}$$

3. NON-LINEAR FILTERING FOR A SEMIMARTINGALE FROM POINT PROCESS OBSERVATIONS

In this Section the signal process is supposed to be an one-dimensional semimartingale of the form

$$X_t = X_0 + \int_0^t H_s \, d_s + Z_t \tag{3.1}$$

where Z_t is an F_t-martingale, H_t is a bounded F_t-progressive process and $E(\sup_{s \leqslant t} |X_s|) < \infty$.

The observations are still given by a one dimensional point process of intensity h_t :

$$Y_t = \int_0^t h_s \, d_s + M_t \tag{3.2}$$

where M_t is an F_t -martingale of O-mean and $h_t = h(X_t)$ is a positive bounded F_t -progressive process.

Denote again by m_t the corresponding innovation:

$$m_t = Y_t - \int_0^t \pi(h_s)\, ds \qquad (3.3)$$

which is a F_t^Y -martingale of point process. Then, the filtering process $\pi(X_t)$ is determined by

THEOREM 3 (*see* [4])

$$\pi(X_t) = \pi(X_0) + \int_0^t \pi(H_s) ds + \int_0^t K_s \,.\, dm_s \qquad (3.4)$$

where

$$K_s = \pi^{-1}(h_s)[\pi(X_{s-}) h_s - \pi(X_{s-}) h_s + \pi(u_s)] \qquad (3.5)$$

$u_s = \frac{d}{ds}\langle Z, M\rangle_s$ *and* $\langle\ ,\ \rangle$ *stands for quadratic variation of two processes.*

Remark. The filtering $\pi(X_t)$ is an F_t^Y-semimartingale since $R_t \equiv \int_0^t K_s\, dm_s$ is an F_t^Y-martingale.

Indeed, R_t is F_t^Y-adapted and in view of (3.4):

$$E(R_t - R_s \big| F_s^Y) = E[\pi(X_t) - \pi(X_0) - \int_0^t \pi(H_u)\, du \Big| F_s^Y]$$

$$= E[X_t - X_0 - \int_0^t H_u du \Big| F_s^Y]$$

$$= E[Z_t - Z_s - F_s^Y] = E[E(Z_t - Z_s \mid E_s(\mid F_s^Y] = 0.$$

4. FILTERING FOR DYNAMICAL SYSTEM FROM POINT OBSERVATIONS

Suppose that the n-dimensional signal process $X_t = (X_t^1, \dots, X_t^n$ satisfies the following equation for a dynamical system:

$$dX_t = a_0(t, X_t) dt + \sum_{j=1}^{m} a_j(t, X_t)\, d\mu_t^j, \qquad (4.1)$$

where: a) The components X_t^i $(i = 1,..., n)$ of X_t have no common jumps.

b) μ_t^j $(j = 1,..., m)$ are independent F_t-martingales. In particular, $\mu_t^j = W_t^j$ $(j = 1,..., m)$ are independent F_t-Brownian motion; μ_t^j may be also F_t-standard Poisson martingales, i. e. $\mu_t^j = N_t^j - t$ $(j = 1,..., m)$ where the N_t^j are independent standard Poisson processes. Note that in the two latter particular cases we have

$$\langle \mu^j, \mu^l \rangle_t = t\delta_{jl} \tag{4. 2}$$

c) The vector coefficients $a_o(t, x) \in R^n$, $a_j(t, x) \in R^n$ $(j = 1, ..., m)$ are continuously differentiable in t, twice continuously differentiable in x and their first derivatives are bounded.

d) The integrals $\int_o^t a_j(t, X_s)\, d\mu_s^j$ are F_t — martingales. In the case of Brownian martingales $\mu_t^j = W_t^j$, this is obvious for Itô integrals $\int_o^t a_j(t, X_s)\, dW_s^j$.

e) The integrals $\int_o^t a_j^i(s, X_s) \frac{\partial}{\partial x^i} f(X_s)\, d\mu_s^j$

$(i = 1,..., n;\ j = 1,..., m)$ are F_t-martingales and this is also the case for Brownian martingales $\mu_t^j = W_t^j$.

$$X_t = X_o + \int_o^t a_o(s, X_s)\, ds + \sum_{j=1}^{m} \int_o^t a_j(s, X_s)\, d\mu_s^j \tag{4.3}$$

Assume now that $f : R^n \to R$ is a function of class C^2 such that its first and second derivatives are bounded.

We wish to calculate the filtering by f of X_t, i. e.

$$\pi_f(X_t) = \pi(f(X_t)) = E[f(X_t) \mid F_t^Y] \tag{4.4}$$

from a point process observation of the form (1.1):

$$Y_t = \int_o^t h_s\, ds + M_t.$$

The Itô's formula is then written as follows

$$f(X_t)-f(X_0)=\sum_{i=1}^{n}\int_0^t D_i f(X_s)\,dX_s^i+$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_0^t D_i D_k f(X_s)\,d\langle (X^c)^i (X^c)^k\rangle_s$$

$$+\sum_{o\leqslant s\leqslant t}[f(X_s)-f(X_{s-})-\sum_{i=1}^{n}D_i f(X_{s-})\,\Delta X_s^i] \tag{4.5}$$

where D_i denotes the derivative with respect to the ith variable, X^c is the continuous part of X and as usual, $\langle , \rangle$ stands for quadratic variation of two processes.

Since

$$X_t^i=X_0^i+\int_0^t a_0^i(s,X_s)\,ds+\sum_{j=1}^{m}\int_0^t a_j^i\,(s,\;X_s)\;d\mu_s^j, \tag{4.6}$$

we have

$$\int_0^t D_i f(X_s)\,dX^i=\int_0^t a_0^i(s,X_s)\;\frac{\partial}{\partial x^i}\,f(X_s)\,ds+$$

$$+\sum_{j=1}^{m}\int_0^t a_j^i\,(s,X_s)\,\frac{\partial}{\partial x^i}\,f(X_s)\;d\mu_s^j,$$

$$\sum_{i=1}^{n}\int_0^t D_i\,f(X_s)\,dX_s^i=\int_0^t A_0(s)\,f\,(X_s)\,ds+\sum_{j=1}^{m}\int_0^t A_j\,(s)\,f\,(X_s)\,d\mu_s^j, \tag{4.7}$$

where

$$A_j(s)\,f(x)=\sum_{i=1}^{n}a_j^i\,(s,x)\frac{\partial}{\partial x^i}f(x). \tag{4.8}$$

$$(j=0,1,2,\dots,m).$$

Because X_t^i and X_t^k have no common jumps and μ_t^j $(j=1,\dots,m)$ are independent, it is easy to see that $\langle\mu^j,\mu^l\rangle_t=0$ if $j\neq l$ and

$$\langle (X^c)^i,(X^c)^k\rangle_t=\sum_{j=1}^{m}\int_0^t a_j^i\,(s,X_s)\,a_j^k\,(s,X_s)\;d\;\langle\mu^j,\mu^j\rangle_s. \tag{4.9}$$

(In both cases of standard Brownian and Poissonian martingales $\langle \mu^j, \mu^j \rangle_t = t$).

It follows that

$$F_t = \frac{1}{2} \sum_i \sum_k \int_o^t D_i D_k f(X_s) \, d \langle (X^c)^i, (X^c)^k \rangle_s =$$

$$= \int_o^t \frac{1}{2} \sum_{j=1}^m \sum_{i=1}^n \sum_{k=1}^n a_j^i(s, X_s) \, a_j^k(s, X_s) \frac{\partial^2}{\partial x^i \partial x^k} f(X_s) \, d \langle \mu^i, \mu^j \rangle_s \quad . \tag{4.10}$$

and that F_t is of bounded variation.

Denote by G_t the third term in the right hand side of (4.5). Obviously G_t is of bounded variation. Then (4.5) can be rewritten as

$$f(X_t) = f(X_o) + [F_t + G_t + \int_o^t A_o(s) f(X_s) \, ds] + \sum_{j=1}^m \int_o^t A_j(s) f(X_s) \, d\mu_s^j. \tag{4.11}$$

Taking account of hypothesis d) and of the above mentioned remarks, we see from (4.11) that $f(X_t)$ is again a semimartingale. Applying then Theorem 3 to $x_t = f(X_t)$ yields an equation for the filtering process $\pi(f(X_t))$. Let us consider the two important cases where $\mu_t = W_t$ and μ_t = Poisson martingale. Then $\langle \mu^j \rangle_t = t$.

Set

$$L(s) f(x) = \sum_{i=1}^n a_o^i(s, x) \frac{\partial}{\partial x_i} f(x) +$$

$$+ \frac{1}{2} \sum_{j=1}^m \sum_{i=1}^n \sum_{k=1}^n a_j^i(s, x) \, a_j^k(s, x) \frac{\partial^2}{\partial x^i \partial x^k} f(x)$$

$$= A_o(s) f(x) + \frac{1}{2} \sum_{j=1}^m A_j(s)^2 f(x) \, . \tag{4.12}$$

(4.11) becomes then:

$$f(X_t) = f(X_o) + G_t + \int_o^t A_o(s) f(X_s) \, ds +$$

$$+ \int_o^t L(s) f(X_s) \, ds + \sum_{j=1}^m \int_o^t A_j(s) f(X_s) \, d\mu_s^j \tag{4.13}$$

Denote by Q_t the martingale component in (4.13)

$$Q_t = \sum_{j=1}^m \int_o^t A_j(s) f(X_s) \, d\mu_s^i. \tag{4.14}$$

Let μ_s^j be a process defined by

$$\mu_s^j = \frac{d}{ds} \langle \mu^j , M^l \rangle_s ; \tag{4.15}$$

where M_t is the martingale in the point observation (1.1). Consequently,

$$\langle Q, M^l \rangle_t = \int_0^t \sum_{j=1}^m u_s^{jl} A_j(s) f(X_{s-})\, ds,$$

hence

$$\frac{d}{ds} \langle Q, M^l \rangle_s = \sum_{j=1}^m u_s^{jl} A_j(s) f(X_{s-}), \tag{4.16}$$

and

$$\frac{d}{ds} \langle Q, M \rangle_s = \left(\frac{d}{ds} \langle Q, M^1 \rangle_s, \ldots, \frac{d}{ds} \langle Q, M^n \rangle_s \right). \tag{4.17}$$

Denote the vector (4. 17) by

$$D_s f(X_s) = \left(D_s^1 f(X_s), \ldots, D_s f(X_s) \right). \tag{4.18}$$

Then the filtering process $\pi(f(X_t))$ is obtained directly by Theorem 3 and we thus have;

THEOREM 4. *Under the assumptions at the beginning of Section 4, the filtering process* $\pi(f(X_t))$ *for dynamical system*

$$dX_t = a_0(t, X_t)dt + \sum_{j=1}^m a_j(t, X_t)\, d\mu_t^j$$

from a point observation $Y_t = \int_0^t h_s\, ds + M_t$, *is defined as follows:*

a) if μ_t^j *are standard Poisson martingales:*

$$\pi(f(X_t)) = \pi(f(X_0)) + \pi(G_t) + \int_0^t \pi(A_0(s) f(X_s))\, ds + \int_0^t \pi^{-1}(h_s)\,[\pi(f(X_{s-}) h_s) - \pi(f(X_{s-}))\pi(h_s) + \pi(D_s f(X_s))]\, dm_s \tag{4.19}$$

where m_t *is the innovation from the observation* Y_t *:*

$$m_t = Y_t - \int_0^t \pi(h_s) ds \tag{4.20}$$

and

$$\pi(G_t) = \sum_{0 \leqslant s \leqslant t} [\pi(f(X_s)) - \pi(f(X_{s-})) - \pi \sum_{i=1}^n D_i f(X_{s-}) \Delta X_s^i] \tag{4.21}$$

b) If μ_t^j *are standard Brownian martingale, the second term in the right hand side of (4.19) is omitted i.e.*

$$\pi(f(X_t)) = \pi(f(X_0)) + \int_0^t \pi(A_0(s)f(X_s))ds +$$

$$+ \int_0^t \pi^{-1}(h_s)[\pi(f(X_{s-})h_s) - \pi(fX_{s-})n(h_s) +$$

$$+ \pi(D_s f(X_s))]dm_s \qquad (4.22)$$

where m_t *is the same as (4.20).*

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Received October 16, 1988

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