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ON LYAPUNOV EXPONENTS AND CENTRAL EXPONENTS OF LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS UNDER RANDOM PERTURBATIONS

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In this paper we shall be concerned with an asymptotic behavior of Lyapunov exponens and central exponents of a linear system of differential equations with almost periodic coefficients under small random perturbations. It is proved that under small nondegenerate perturbations by a white noise, with probability 1 the Lyapunov exponents and the central exponents of a perturbed system coincide with auxiliary numbers of the given system which tend to corresponding numbers of the probability spectrum of the initial system.

We consider a linear system of differential equations

$$\dot{x} = A(t)x, \tag{1}$$

where $t \in R$, $x \in R^n$, $A(t)$ is an almost periodic matrix-valued function (see [1]). To system (1) we associate its random perturbations

$$\dot{y} = (A(t) + \sigma C(t, \omega))y, \tag{2}$$

where $y \in R^n$, $\sigma \in R^+_*$, the elements of matrix $C(t, \omega)$ are independent white noises, ω belongs to the probability space (Ω, P). Let us denote by $X(t, \tau)$ and $Y_\sigma(t, \tau, \omega)$ the Cauchy matrices of systems (1) and (2), respectively. We shall consider Lyapunov exponents λ_k, central exponents Ω_k and Θ_k of the system (1) ($k = 1, \ldots, n$) (see [2,3,4]) defined as follows:

$$\lambda_k = \min_{R^{n-k+1} \subset R^n} \max_{\zeta \in R_*^{n-k+1}} \overline{\lim_{t \to +\infty}} \frac{1}{t} \ln | X(t, 0)\zeta |, \tag{3}$$

$$\Omega_k = \inf_{R^{n-k+1} \subset R^n} \inf_{T \in R^+} \overline{\lim_{m \to +\infty}} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \| X((i+1)T, iT)\Big|_{X(iT,0)R^{n-k+1}} \|, \tag{4}$$

$$\Theta_k = \sup_{R^k \subset R^n} \sup_{T \in R_+^1} \overline{\lim_{m \to +\infty}} \frac{1}{mT} \sum_{i=0}^{m-1} \ln \| X(iT, (i+1)T)\Big|_{X((i+1)T,0)R^k} \|^{-1} \tag{5}$$

where R^d is a dimentional linear subspace in R^n, $R^d_* = R^d \setminus \{0\}$ and $X|_{R^d}$ is the restriction of the operator X to R^d. It is easy to show that $\Theta_k \leqslant \lambda_k \leqslant \Omega_k$. If in (3)–(5) we replace $X(t, \tau)$ by $Y_\sigma(t,\tau;\ \omega)$ we shall get Lyapunov exponents $\lambda_k(\sigma,\omega)$ and central exponents $\Omega_k(\sigma,\omega)$, $\Theta_k(\sigma,\omega)$ of the system (2) ($k = 1,.., n$).

We now introduce the following auxiliary functions of systems (1) — (2)

$$\nu_k\ (\sigma, T) = \overline{\lim_{m \to +\infty}} \frac{1}{mT} \sum_{i=0}^{m-1} E \ln d_k\ (Y_\sigma((i+1)\ T,\ iT,\ \omega)), \tag{6}$$

where $E\,\xi\,(\omega)$ is the expectation of the random variable $\xi\,(\omega)$ and $d_1\,(X) \geqslant \ldots \geqslant d_n(X)$ are the singular numbers of the nondegenerate $n \times n$ matrix X, i. e they are the positive square roots of the eigenvalues of th matrix X^*X. The following lemma is proved in [4].

LEMMA. *For all* $\sigma \in (0,1)$ *and* $k \in \{1, \ldots, n\}$ *there exists he following limit*

$$\nu_k\ (\sigma) = \lim_{\substack{T \to +\infty \\ T\, \in\, \mathbf{N}}} \nu_k\ (\sigma, T)$$

V.M. Millionshchikov introduced a notion of a probability spectrum of a linear system of differential equations with uniformly continuous coefficients and showed that the probability spectrum of an almost periodic system consists of no more than n elements $\nu_1 \geqslant \cdots \geqslant \nu_n$ and that almost all systems in the space of shifts of argument t of the given system have Lyapunov spectrum consisting of $\nu_1 \geqslant \ldots \geqslant \nu_n$ (see [5]). In [6] he proved a theorem on stochastic stability of the probability spectrum of a linear system with uniformly continuous coefficients, but his proof contains some flaws which are shown in [7,8]. In [7] he infers that this theorem still holds if the given system is almost periodic, but the proof of that result is not clear. In this paprer we formulate and prove a more powerful theorem.

THEOREM. *For all* $\varepsilon > 0$ *and* $k \in \{1,\ldots, n\}$ *there exists* $\delta > 0$ *such that, for each* $\sigma \in (0, \delta)$,

$$|\, \nu_k\ (\sigma) - \nu_k\ \,| < \varepsilon,$$

where $\nu_1 \geqslant \ldots \geqslant \nu_n$ *are the numbers of the probability spectrum of the almost periodic system* (1). *Furthermore, the following equalities are satisfied with probability 1*

$$\Omega_k\ (\sigma, \omega) = \lambda_k\ (\sigma, \omega) = \Theta_k\ (\sigma, \omega) = \nu_k\ (\sigma).$$

Proof. We take a system $\tilde{A}(t)$ from the space of shifts of argument t of system (1) such that $\tilde{A}(t)$ is absolutely regular and its Lyapunov spectrum consists of $v_1 \geqslant \dots \geqslant v_n$ (the existence of such a system follows from [5]). The matrix $\tilde{A}$ (t) has the form

$$\tilde{A}(t) = \lim_{n \to +\infty} A(t + t_n), \tag{7}$$

where the limit is uniform on segments. It follows from Bochner's Theorem [1] that from the sequence $\{A(t + t_n)\}$ $(n \in N)$ we may extract a subsequence convergent uniformly on R. For simplicity we assume that the sequence $\{A(t + t_n)\}$ $(n \in N)$ itself has this property. To every system

$$\dot{x} = A(t + t_n)\,x$$

we associate the following perturbed system

$$\dot{Z} = (A(t + t_n) + \sigma\, C(t, \omega))\, Z$$

and the auxiliary function $v_k(\sigma, T; t_n)$ defined by formula (6).

Denote by $\tilde{v}_k(\sigma, T)$ the auxiliary functions of systems

$$\dot{x} = \tilde{A}(t)x,$$

$$\dot{y} = (\tilde{A}(t) + \sigma\, C(t, \omega))\, y.$$

Since the sequence $\{A(t + t_n)\}$ $(n \in N)$ uniformly converges to $\tilde{A}(t)$ we get the following equality for all $\sigma \in (0,1)$, $T \in R^+$ and $k \in \{1,\dots,n\}$

$$\lim_{n \to +\infty} v_k(\sigma, T; t_n) = \tilde{v}_k(\sigma, T). \tag{8}$$

Using the results in [4] we may easily prove that for all $n \in N$ and $\varepsilon \in (0, 1)$ the following inequality holds

$$|v_k(\sigma, T; t_n) - v_k(\sigma, T)| \leqslant c\sqrt{\varepsilon} - \frac{1}{T} \ln \frac{\delta\, \sigma^{n(n^2+2)}}{2},$$

where the positive constant c does not depend on ε, σ, T, k, t_n while the constant δ depends on ε but not on σ, T, k, t_n. Consequently, by (8) we have

$$|v_k(\sigma, T) - \tilde{v}_k(\sigma, T)| \leqslant c\sqrt{\varepsilon} - \frac{1}{T} \ln \frac{\delta\, \sigma^{n(n^2+2)}}{2}$$

Therefore, by the above lemma for each $\varepsilon \in (0, 1)$ we have

$$|v_k(\sigma) - \tilde{v}_k(\sigma)| \leqslant c\sqrt{\varepsilon}.$$

Since ε is arbitrarily chosen, we get

$$v_k(\sigma) = \tilde{v}_k(\sigma). \tag{9}$$

Proof. We take a system $\tilde{A}(t)$ from the space of shifts of argument t of system (1) such that $\tilde{A}(t)$ is absolutely regular and its Lyapunov spectrum consists of $v_1 \geqslant \ldots \geqslant v_n$ (the existence of such a system follows from [5]). The matrix $\tilde{A}(t)$ has the form

$$\tilde{A}(t) = \lim_{n \to +\infty} A(t + t_n), \tag{7}$$

where the limit is uniform on segments. It follows from Bochner's Theorem [1] that from the sequence $\{A(t + t_n)\}$ $(n \in N)$ we may extract a subsequence convergent uniformly on R. For simplicity we assume that the sequence $\{A(t + t_n)\}$ $(n \in N)$ itself has this property. To every system

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and the auxiliary function $v_k(\sigma, T; t_n)$ defined by formula (6).

Denote by $\tilde{v}_k(\sigma, T)$ the auxiliary functions of systems

$$\dot{x} = \tilde{A}(t)x,$$

$$\dot{y} = (\tilde{A}(t) + \sigma\, C(t, \omega))\, y.$$

Since the sequence $\{A(t + t_n)\}$ $(n \in N)$ uniformly converges to $\tilde{A}(t)$ we get the following equality for all $\sigma \in (0,1)$, $T \in R^+$ and $k \in \{1, \ldots, n\}$

$$\lim_{n \to +\infty} v_k(\sigma, T; t_n) = \tilde{v}_k(\sigma, T). \tag{8}$$

Using the results in [4] we may easily prove that for all $n \in N$ and $\varepsilon \in (0, 1)$ the following inequality holds

$$|v_k(\sigma, T; t_n) - v_k(\sigma, T)| \leqslant c\sqrt{\varepsilon} - \frac{1}{T} \ln \frac{\delta\, \sigma^{n(n^2+2)}}{2},$$

where the positive constant c does not depend on $\varepsilon, \sigma, T, k, t_n$ while the constant δ depends on ε but not on σ, T, k, t_n. Consequently, by (8) we have

$$|v_k(\sigma, T) - \tilde{v}_k(\sigma, T)| \leqslant c\sqrt{\varepsilon} - \frac{1}{T} \ln \frac{\delta\, \sigma^{n(n^2+2)}}{2}$$

Therefore, by the above lemma for each $\varepsilon \in (0, 1)$ we have

$$|v_k(\sigma) - \tilde{v}_k(\sigma)| \leqslant c\sqrt{\varepsilon}.$$

Since ε is arbitrarily chosen, we get

$$v_k(\sigma) = \tilde{v}_k(\sigma). \tag{9}$$

It follows from a theorem in [4] and the stochastic stability of Lyapunov exponents of absolutely regular system $A(t)$ (see [6, 7]) that

$$\lim_{\delta \to 0} \tilde{\nu}_k(\delta) = \nu_k.$$

Consequently, taking account of equality (9), we get the first assertion of the theorem. The second one is proved in [4]. The proof is complete.

COROLLARY. *In order that the least Lyapunov exponent of an almost periodic system be stable, a sufficient and necessary condition is that this exponent coincide with the lower central exponent of the given system.*

Proof. It is proved in [9] that for the almost periodic system (1) we have $\nu_n = \omega_0$, where

$$\omega_0 = \underline{\lim}_{t-\tau \to \infty} \frac{1}{t-\tau} \ln \| X(t, 0) X(0, \tau) \|$$

is the lower special exponent of system (1). It is proved in [10] that $\omega = \omega_0$, where

$$\omega = \underline{\lim}_{T \to +\infty} \underline{\lim}_{s \to +\infty} \frac{1}{sT} \sum_{i=0}^{s-1} \ln \| X(iT, (i+1)T \|^{-1}$$

is the lower central exponent of system (1). The corollary now follows from the equalities $\omega_0 = \omega = \nu_n$ and the above theorem.

REFERENCES

1. S. Bochner, Beitrage zur Theorie der pastperiodische Funktionen, I Teil. *Funktionen einer Variablen*, Math. Ann., 96 (1927), 119 — 147.
2. A. M Lyapunov, *Collected works*, Vol. 2, Izd. Akad. Nauk SSSR, Moscow-Leningrad (1956) (in Russian).
3. V. M. Millionshchikov, *Typical properties of conditional exponential stability I*, Differensial'nye Uravneniya, 19 (1983), № 8, 1344 – 1356; English transl. in Differential equation 19 (1983).
4. Nguyen Dinh Cong, *On central exponents of linear systems with coefficients perturbed by a white noise*, Institute of Mathematics, Hanoi, Preprint, № 19 (1988) (in Russian).
5. V. M. Millionshchikov, *A stability criterion for the probability spectrum of differential equations with recurrent coefficients and a criterion for almost reducibility of systems with almost periodic coefficients*, Mat. Sb. 78 (120) (1969), 179 — 201; English transl. in Math. USSR Sb. 7 (1969).
6. V. M. Millionshchikov. *On the theory of the Lyapunov characteristic exponents*, Mat. Zametki 7 (1970), 501 – 513; English transl. in Math. Notes 7 (1970).

7. V.M Milionshchikov. *The stochastic stability of Lyapunov characteristic exponents*, Differensial'nye Uravneniya 11(1975), 581—583 (in Russian).
8. Nguyen Dinh Cong, *A criterion for stochastic stablity of the greatest Lyapunov exponent*, Mat. Zametki 43(1988), N^{0}1, 82—97 (in Russian).
9. V.M. Millionshchikov, *Metric theory of linear systems of differential equations*, Mat. Sb. 73(119) (1968), 163—173; English transl. in Math. USSR Sb. 6(1968).
10. B.F.Bylov, *On upper stability of the greatest characteristic exponent of a linear system of differential equations with almost periodic coefficients*, Mat. Sb.48 (90) (1959), N^{0}1, 118—128 (in Russian).

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