ACTA MATHEMATICA VIETNAMICA

Vol. 15 No 1 (1990)

ON THE PROBABILISTIC HAUSDORFF DISTANCE AND FIXED POINT THEOREMS FOR MULTIVALUED CONTRACTIONS

DO HONG TAN

As is known, the Hausdorff distance plays an important role in fixed point theory for multivalued mappings. The aim of this note is to establish a relationship between the probabilistic Hausdorff distance and the usual one, and some fixed point theorems for multivalued contractions in Menger spaces.

1. ON THE PROBABILISTIC HAUSDORFF DISTANCE

Let $(X, \mathcal{F}, \min)$ be a Menger space, i.e. X is a set, $\mathcal{F}$ is a family of distribution functions F_{xy} for each pair $x, y \in X$, which satisfy the following conditions:

$E_{xy}(t) = 1$ for all $t > 0$ if and only if $x = y$,

$F_{xy}(o) = 0$,

$F_{xy} = F_{yx}$

$F_{xy}(t + s) \geqslant \min \{F_{xz}(t), F_{zy}(s)\}$

for all $x, y \in X$ and $t, s \in R$.

When X is a metric space, $F_{xy}(t)$ for $x, y \in X$, $t \in R$ (the real line) can be interpreted as the probability that the distance between x and y is less that t. The topology in X, called the (ε, λ)-topology, is generated by the family of neighbourhoods of $x \in X$

$$N_x(\varepsilon, \lambda) = \{y \in X : F_{xy}(\varepsilon) > 1-\lambda\}, (\varepsilon > 0,\ 0 < \lambda < 1).$$

It is known that a Menger space is a uniform space with the family of pseudometrics

$$d_\lambda(x,y) = \sup\{t : F_{xy}(t) \leqslant 1-\lambda\},\ (0<\lambda<1)\,. \tag{1}$$

A set $A \subset X$ is said to be probabilistic bounded if

$$\operatorname*{Sup}_{t\in R}\ \inf_{x,y\in A} F_{xy}(t) = 1.$$

The class of nonempty probabilistic bounded and closed subsets of X is denoted by CB(X). For $A, B \in CB(X)$ the probabilistic Hausdorff distance between A and B is defined as follows (see, for instance, [1]):

$$H_{AB}(t) = \sup_{s<t} \min\{\inf_{x\in A}\ \sup_{y\in B} F_{xy}(s),\ \inf_{y\in B}\ \sup_{x\in A} F_{xy}(s)\}.$$

It is known that $(CB(X), \mathcal{H}, \min)$ where $\mathcal{H} = \{H_{AB} : A, B \in CB(X)\}$ is a Menger space too, i. e. H_{AB} possesses all properties analogous to that of F_{xy} listed in the beginning of this section (see [3, p. 300]).

For a Menger space $(X, \mathcal{F}, \min)$ we define a family of pseudometrics d_λ by (1) and then let

$$D_\lambda(A,B) = \max\{\sup_{x\in A}\ \inf_{y\in B} d_\lambda(x,y),\ \sup_{y\in B}\ \inf_{x\in A} d_\lambda(x,y)\} \tag{2}$$

be the λ-Hausdorff distance between A and B in CB (X). In [2] Hadzic has proved that

$$D_\lambda(A,B) \leqslant \sup\{t : H_{AB}(t) \leqslant 1-\lambda\}$$

and hence $H_{AB}(D_\lambda(A,B)) \leqslant 1-\lambda$.

The aim of this section is to answer the question: When have we the equality

$$D_\lambda(A,B) = \sup\{t : H_{AB}(t) \leqslant 1-\lambda\}? \tag{3}$$

Moreover, we shall show that in this case we have also the inverse formula

$$H_{AB}(t) = 1 - \sup\{\lambda \in (0,1) : D_\lambda(A,B) \geqslant t\}, \tag{4}$$

where we agree that the supremum over any empty set is equal to zero.

Before giving the answer we need the following

LEMMA 1.1. *For fixed $y \in X$ and $t \in R$, the function $F_{xy}(t) : X \to [0,1]$ is lower semicontinuous.*

Proof. Fixing $x_0 \in X$ we have to show that there exists a neighbourhood U of x_0 such that $F_{x_0y}(t) - \varepsilon \leqslant F_{xy}(t)$ for all $x \in U$, where ε is an arbitrarily given positive number. Without loss of generality we may assume that $F_{x_0y}(t) > 0$ and $\varepsilon < F_{x_0y}(t)$. Denote $a = F_{x_0y}(t) - \varepsilon$ and $b = 1 - a$. Since $F_{x_0y}(t) > 1 - b$ by the left-continuity of F_{x_0y} (as a distribution function) there exists an $s < t$ such that $F_{x_0y}(s) > 1 - b$. Putting $U = \{x \in X : F_{x_0x}(t-s) > 1-b\}$ we get

$$F_{xy}(t) \geqslant \min\{F_{xx_0}(t-s), F_{x_0y}(s)\} > 1 - b = F_{x_0y}(t) - \varepsilon$$

for every $x \in U$. The lemma is proved.

PROPOSITION 1.2. *If* $(X, \mathcal{F}, min)$ *is a Menger space then we have* (1) *and* (2) *for compact subsets* A, B *of* X.

Proof. Let us fix two compact subsets A,B and $\lambda \in (0,1)$. Using the techniques of Hadzic in [3] and ourselves in [7,8], we shall first prove that

$$\sup \{ t : H_{AB}(t) \leqslant 1 - \lambda\} = \sup \{ t : G(t) \leqslant 1 - \lambda\}. \tag{5}$$

where $G(t) = \min \{ \inf_{x\in A} \sup_{y\in B} F_{xy}(t), \inf_{y\in B} \sup_{x\in A} F_{xy}(t)\}$.

For this we denote $M_1 = \{t : G(t) \leqslant 1 - \lambda\}$.

$M_2 = \{ t : \sup_{s<t} G(s) \leqslant 1 - \lambda \}$ and we need only to show that $\sup M_1 = \sup M_2$. Since $G(t)$ is a nondecreasing function we have $\sup_{s<t} G(s) \leqslant G(t)$, hence $M_1 \subset M_2$ and consequently, $\sup M_1 \leqslant \sup M_2$. Now let $t \in M_2$. For $s < t$ we have $G(s) \leqslant 1 - \lambda$ which implies that $s \in M_1$. From this $t \leqslant \sup M_1$, so $\sup M_2 \leqslant \sup M_1$. This implies $\sup M_1 = \sup M_2$, as desired.

Next, denoting $P(t) = \inf_{x\in A} \sup_{y\in B} F_{xy}(t)$, $Q(t) = \inf_{y\in B} \sup_{x\in A} F_{xy}(t)$ we shall show that

$$\sup \{ t : \min \{ P(t), Q(t) \} \leqslant 1 - \lambda\} = \max \{ \sup \{ t : P(t) \leqslant 1 - \lambda \}, \sup \{ t : Q(t) \leqslant 1 - \lambda\}\} \tag{6}$$

Obviously,

$$\{ t : P(t) \leqslant 1 - \lambda \} \subset \{ t : \min \{ P(t), Q(t)\} \leqslant 1 - \lambda\},$$
$$\{ t : Q(t) \leqslant 1 - \lambda \} \subset \{ t : \min \{ P(t), Q(t) \} \leqslant 1 - \lambda \}.$$

Hence we have

$$\max \{ \sup \{ t : P(t) \leqslant 1 - \lambda \}, \sup \{ t : Q(t) \leqslant 1-\lambda \}\} \leqslant \leqslant \sup \{ t : \min \{ P(t), Q(t) \} \leqslant 1 - \lambda \}.$$

To prove (6) we assume the contrary that there exists a real number c such that

$$\max \{ \sup \{t : p(t) \leqslant 1 - \lambda\}, \sup \{t : Q(t) \leqslant 1 - \lambda \}\} < c < \sup \{t : \min \{P(t), Q(t)\} \leqslant 1 - \lambda\}$$

Since $\sup \{t : P(t) \leqslant 1 - \lambda \} < c$ and $\sup \{t : Q(t) \leqslant 1 - \lambda\} < c$ we get $P(c) > 1 - \lambda$, $Q(c) > 1 - \lambda$ and hence $\min\{P(c), Q(c)\} > 1 - \lambda$. But since $c < \sup \{t : \min \{P(t), Q(t)\} \leqslant 1 - \lambda\}$ and P, Q are nondecreasing we obtain a contradiction:

$\min \{P(c), Q(c)\} \leqslant 1 - \lambda$. Thus, (6) is proved.

For each $x \in A$ we have $\sup_{y\in B} F_{xy}(t) \geqslant P(t)$

and hence

$$\sup_{x\in A} \sup \{t : \sup_{y\in B} F_{xy}(t) \leqslant 1 - \lambda\} \leqslant \sup \{t : P(t) \leqslant 1 - \lambda\}.$$

We shall prove that if A is compact then it must be an equality. Suppose the contrary, that

$$\sup_{x\in A} \sup \{t : \sup_{y\in B} F_{xy}(t) \leqslant 1 - \lambda\} < s < \sup \{t : \inf_{x\in A} \sup_{y\in B} F_{xy}(t) \leqslant 1 - \lambda\}.$$

The first inequality shows that for each $x \in A$ there exists $y \in B$ such that $F_{xy}(s) > 1 - \lambda$, while the second inequality shows that

$$\inf_{x\in A} \sup_{y\in B} F_{xy}(s) < 1 - \lambda.$$

For fixed s and y, the function $F_{xy}(s)$ is lower semicontinuous in x by the above lemma, so is the function $\sup_{y\in B} F_{xy}(s)$. As A is compact, the last function attains its minimum at some point $x_0 \in A$. Thus we have $F_{x_0y}(s) \leqslant 1 - \lambda$ for all $y \in B$, a contradiction. So we obtain the equality

$$\sup_{x\in A} \sup \{ t : \sup_{y\in B} F_{xy}(t) \leqslant 1 - \lambda \} = \sup \{ t : p(t) \leqslant 1 - \lambda \}. \tag{7}$$

Analogously, because B is compact, we have

$$\sup_{y\in B} \sup \{ t : \sup_{x\in A} F_{xy}(t) \leqslant 1 - \lambda \} = \sup \{ t : Q(t) \leqslant 1 - \lambda \}. \tag{8}$$

To complete the proof of (3) we shall prove that for each $x \in A$

$$\sup \{t : \sup_{y\in B} F_{xy}(t) \leqslant 1 - \lambda\} = \inf_{y\in B} \sup \{t : F_{xy}(t) \leqslant 1 - \lambda\}. \tag{9}$$

Since for every $x \in A$ and $y \in B$ we have $F_{xy} \leqslant \sup_{y\in B} F_{xy}$, it follows that

$$\sup \{t : \sup_{y\in B} F_{xy}(t) \leqslant 1 - \lambda\} \leqslant \sup \{t : F_{xy}(t) \leqslant 1 - \lambda\}$$

and hence

$$\sup \{t : \sup_{y\in B} F_{xy}(t) \leqslant 1 - \lambda\} \leqslant \inf_{y\in B} \sup \{t : F_{xy}(t) \leqslant 1 - \lambda\}.$$

To prove the equality (9) we assume the contrary that for some $x_0 \in A$ we have

$$\sup \{t : \sup_{y\in B} F_{x_0y}(t) \leqslant 1 - \lambda\} < s < \inf_{y\in B} \sup \{t : F_{x_0y}(t) \leqslant 1 - \lambda\}.$$

The first inequality shows that $\sup_{y\in B} F_{x_0y}(s) > 1 - \lambda$ while the second inequality shows that

$$\sup \{t : F_{x_0y}(t) \leqslant 1 - \lambda\} > s \qquad \text{for all } y \in B$$

which implies $F_{x_0y}(s) \leqslant 1 - \lambda$ for all $y \in B$, a contradiction, Thus, (9) is proved.

Similarly, we get

$$\sup \{t : \sup_{x\in A} F_{xy}(t) \leqslant 1 - \lambda\} = \inf_{x\in A} \sup \{t : F_{xy}(t) \leqslant 1 - \lambda \} \tag{10}$$

Combining (5), (6), (7), (8), (9), (10) and (1), (2) we obtain (3).

To prove (4), fixing A,B and t we denote

$$a = 1 - \sup \{\lambda \in (0,1) . D_\lambda(A, B) \geqslant t\}$$

and prove that $H_{AB}(t) = a$,

If the set $\{\lambda : D_\lambda (A,B) \geqslant t\}$ is empty then $a = 1$. On the other hand, this shows that $t > D_\lambda(A, B)$ for all $\lambda \in (0,1)$ and from (3), $H_{AB}(t) > 1-\lambda$ for all λ, hence $H_{AB}(t) = 1 = a$.

We now consider the case where the mentioned set is nonempty. First we prove $a \leqslant H_{AB}(t)$. Assume the contrary, that

$$1 - \sup \{\lambda : D_\lambda(A, B) \geqslant t\} > H_{AB}(t).$$

Then by (3),

$$1 - H_{AB}(t) > \sup \{\lambda : \sup \{ s : H_{AB}(s) \leqslant 1 - \lambda\} \geqslant t\}$$

It follow that $\sup \{s : H_{AB}(s) \leqslant H_{AB}(t)\} < t$

which is obviously a contradiction.

To prove that $a = H_{AB}(t)$ we assume the contrary that $a < b < H_{AB}(t)$. Then we have

$$1 - b < \sup \{ \lambda : \sup \{s : H_{AB}(s) \leqslant 1 - \lambda\} \geqslant t\}.$$

From this $\sup \{s : H_{AB}(s) \leqslant b\} \geqslant t$. Hence $H_{AB}(t) \leqslant b < H_{AB}(t)$, a contradiction. Thus, (4) is proved and the proof of the proposition is complete.

In the sequel by $C(X)$ we denote the class of all nonempty compact subsets of X.

COROLLARY 1.3. *If $(X, \mathcal{F}, \min)$ is a Menger space then $(C(X), \mathcal{H}, \min)$ is a uniform space with the family of pseudometrics defined by (3).*

2. SOME FIXED POINT THEOREMS

Up to now there is no analogue of the Nadler fixed point theorem for multivalued contractions in Menger spaces. In fact, in 1983 Hadzic [2] partically solved this problem for compact Menger spaces using the corresponding results in uniform spaces. On the other hand, in 1981 Mishra [5] established some fixed point theorems for generalized multivalued contractions in uniform spaces. It is natural to expect an analogue of the Nadler's theorem by combining the mentioned Mishra's theorem with the corollary presented at the end of Section 1. Unfortunately, there is an error in the proofs of Mishra: the sequence of iterates constructed there in depends in general on the index of each pseudometric. By this reason in the present section we must still restrict ourselves to the case of compact Menger spaces. Our first theorem generalizes the mentioned result of Hadzic in [2] while the second one is an analogous result for probabilistic generalized multivalued contractions in the sense of Kirk [4]. They are based upon the following result of Pai and Veeramani [6].

THEOREM [6]. *Let* $(X, d_i, i \in I)$ *be a compact uniform space and let* $T: X \to C(X)$ *satisfy the following conditions:*

a) *For every* $i \in I$ *and* $\varepsilon > 0$ *there exist* $j \in I$ *and* $\delta > 0$ *such that* $d_j(x,y) < \delta$ *implies* $D_i(Tx, Ty) < \varepsilon$,

b) *For each* $\varepsilon > 0$ *and* $i \in I$ *the set* $\{x \in X: d_i(x, Tx) < \varepsilon\}$ *is nonempty. Then* T *has a fixed point.*

Before stating the main results we need the following

LEMMA. 2.1. *Let* $(X, d_i . i \in I)$ *be a compact uniform space and let* $T: X \to C(X)$ *satisfy the following condition: for every* $i \in I$ *and* $x, y \in X$ *such that* $d_i(x, y) > 0$ *we have*

$$D_i(Tx, Ty) < d_i(x, y). \tag{11}$$

Then T **has a fixed point.**

Proof. Fixing $i \in I$ we denote $p_i(x) = d_i(x, Tx)$. It is known that (11) implies the continuity of T as a multivalued mapping and this in turn implies the continuity of p_i. From the compactness of X it follows that there exists an $x_i \in X$ such that $p_i(x_i) = \min\{p_i(x): x \in X\} = a \geq 0$. If $a > 0$ then there is an $y_i \in Tx_i$ such that $d_i(x_i, y_i) = d_i(x_i, Tx_i) = a > 0$. Then by (11) we have

$$D_i(Tx_i, Ty_i) < d_i(x_i, y_i) = a \leq d_i(y_i, Ty_i).$$

But this contradicts the fact that $d_i(y_i, Ty_i) \leq D_i(Tx_i, Ty_i)$ because $y_i \in Tx_i$. Thus, $a = 0$ and Condition b) of the previous theorem is satisfied. Condition a) is derived immediately from (11). The lemma now follows from applying the above theorem.

THEOREM 2.2. *Let* $(X, \mathcal{F}, \min)$ *be a compact Menger space and let* $T: X \to C(X)$ *satisfy the following condition: for each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that*

$$H_{TxTy}(\varepsilon) \geq F_{xy}(\varepsilon + \delta) \tag{12}$$

for all $x, y \in X$. *Then* T *has a fixed point.*

Proof. First we prove that from (12) we get

$$D_\lambda(Tx, Ty) < \varepsilon \qquad \text{if } 0 < d_\lambda(x, y) < \varepsilon + \delta. \tag{13}$$

Fix $\lambda \in (0,1)$ and suppose that $d_\lambda(x, y) < \varepsilon + \delta$. From (1) it follows that $F_{xy}(\varepsilon + \delta) > 1 - \lambda$ and hence $H_{TxTy}(\varepsilon) > 1 - \lambda$ by (12). From this and (3) we obtain $D_\lambda(Tx, Ty) < \varepsilon$.

We now show that (13) is equivalent to the following

$$D_\lambda(Tx, Ty) < \varepsilon \text{ if } \varepsilon \leq d_\lambda(x, y) < \varepsilon + \delta. \tag{14}$$

Indeed, obviously (13) implies (14). To prove the inverse implication we note by putting $d_\lambda(x, y) = \varepsilon$ in (14) that

$$D_\lambda(Tx, Ty) < d_\lambda(x, y) \text{ if } d_\lambda(x, y) > 0 \tag{15}$$

Now, if $\varepsilon \leqslant d_\lambda(x, y) < \varepsilon + \delta$ then we have $D_\lambda(Tx, Ty) < \varepsilon$ by (14). If $0 < d_\lambda(x, y) < \varepsilon$ then by (15) we have $D_\lambda(Tx, Ty) < \varepsilon$ again. Thus (13) always holds.

Because T satisfies (15), Theorem 1 follows by applying the above lemma.

COROLLARY 2.3. *Let $(X, \mathfrak{F}, \min)$ be a compact Menger space and let $T: X \to C(X)$ satisfy the following condition: there exists a function $k: R_+ \to R_+$ (the set of all non-negative numbers) which is upper semicontinuous from the right with $k(t) < t$ for all $t > 0$ and*

$$H_{TxTy}(k(t)) \geqslant F_{xy}(t) \tag{16}$$

for all $x, y \in X$, $t \in R$. Then T has a fixed point.

Proof. Let $\varepsilon > 0$ be given. Since $k(\varepsilon) < \varepsilon$ and k is upper semicontinuous from the right, there exists $\delta > 0$ such that $k(\varepsilon + \delta) < \varepsilon$. Since H_{TxTy} is nondecreasing as a distribution function, from (16) we get.

$$F_{xy}(\varepsilon + \delta) \leqslant H_{TxTy}(k(\varepsilon + \delta)) \leqslant H_{TxTy}(\varepsilon),$$

i. e. (12) holds. The corollary follows from Theorem 1.

COROLLARY 2. 4. (Hadzic [2]). *Let $(X, \mathfrak{F}, \min)$ be a compact Menger space and let $T: X \to C(X)$ satisfy the condition that there exists a $k < 1$ such that*

$$H_{TxTy}(kt) \geqslant F_{xy}(t) \tag{17}$$

for all $x, y \in X$ and $t \in R$. Then T has a fixed point.

Proof. It suffices to put $k(t) = kt$ in Corollary l.

THEOREM 2.5. *Let $(X, \mathfrak{F}, \min)$ be a compact Menger space and let $T: X \to C(X)$ satisfy the following condition : for each $x \in X$ there exists a $k(x) < 1$ such that*

$$H_{TxTy}(k(x)t) \geqslant F_{xy}(t) \tag{18}$$

for all $y \in X$ and $t \in R$. Then T has a fixed point.

Proof. Without loss of generality we may assume that $k(x) \neq 0$ for all $x \in X$. We shall prove that T satisfies the following inequality

$$D_\lambda(T_x, T_y) \leqslant k(x)\, d_\lambda(x, y) \tag{19}$$

for all $x, y \in X$ and $\lambda \in (0,1)$. Suppose the contrary that there exist x, y and λ such that $D_\lambda(Tx, Ty) > k(x) d_\lambda(x, y)$. Putting $t = D_\lambda(Tx, Ty)/k(x)$ we have $D_\lambda(Tx, Ty) = k(x)t$ and $d_\lambda(x, y) < t$. From (1) we get $F_{xy}(t) > 1 - \lambda$ and hence $H_{TxTy}(k(x)t) > 1 - \lambda$ by (18). From this

$$H_{TxTy}(D_\lambda(Tx, Ty)) > 1 - \lambda.$$

But from (3) we obtain $H_{TxTy}(D_\lambda(Tx, Ty)) \leqslant 1 - \lambda$. This contradiction shows that (19) holds and hence T satisfies (11). The proof is complete by applying the above lemma.

REMARK 2.6. Either Condition (12) or (18) implies Condition (17) (hence Theorem 2 serves another generalization of the Hadzic's theorem (Corollary 2)), but none of them implies the other (see an example in [9] for the metric case).

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Received April 29, 1989

INSTITUTE OF MATHEMATICS, P.O. BOX 631 BOHO, HANOI, VIETNAM