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ON SOME CHARACTERIZATIONS OF q-BESSEL POLYNOMIALS

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1. INTRODUCTION

The Bessel polynomials were introduced by Krall and Frink [10] in connection with the solution of the wave equation in spherical coordinates. They are the polynomial solution of the differential equation

$$x^2 y''(x) + (ax + b)\, y'(x) = n(n + a - 1)\, y(x), \tag{1.1}$$

where n is a positive integer and a and b are arbitrary parameters. These polynomials are orthogonal on the unit circle with respect to the weight function

$$\rho^{(x,\alpha)} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{T_{(\alpha)}}{T(\alpha+n-1)} \left(-\frac{2}{x}\right)^n \tag{1.2}$$

Several other authors including Agarwal [2], Al-Salam [3], Brafman [4], Burchnall [5], Carlitz [6], Dickinson [8], Grosswald [9], Rainville [11], and Toscano [12] have contributed to the study of the Bessel polynomials.

In 1965, Abdi [1] defined q-Bessel polynomials and discussed some of the important properties. He denoted this polynomial by $J(q;\, c,\, n;\, x)$ and defined it as

$$J(q;\, c,\, n;\, x) = \frac{(q^c)_n}{(q)_n}\, {}_2\Phi_0 \left[q^{-n},\, q^{c+n};\, x\right] \tag{1.3}$$

The aim of the present paper is to establish some characterizations for q-Bessel polynomials.

2. DEFINITIONS AND NOTATIONS

For $|q| < 1$, let

$$[\alpha] = \frac{1 - q^{\alpha}}{1 - q} \;, \tag{2.1}$$

where α may be a real or a complex number.

$$(q^{\alpha})_n = (1 - q^{\alpha})(1 - q^{\alpha+1}) \ldots (1 - q^{\alpha+n-1}); \; (q^{\alpha})_0 = 1 \tag{2.2}$$

$$ {}_r\Phi_s\left[\begin{matrix} q^{(a_r)} ; x \\ q^{(b_s)} ; q^{\lambda} \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(1-q^{a_1})_n (1-q^{a_2})_n \ldots (1-q^{a_r})_n \, q^{\frac{1}{2}\lambda n(n+1)} x^n}{(q)_n (1-q^{b_1})_n (1-q^{b_2})_n \ldots (1-q^{b_s})_n} \tag{2.3}$$

$$D_q f(x) = \frac{f(xq) - f(x)}{(q-1)}. \tag{2.4}$$

We shall adopt in this paper a somewhat different notation from that used by Abdi for q-Bessel polynomials.

In the notation of q-hypergeometric series, the q-Bessel polynomials are given by

$$Y^{(\alpha)}_{n,q}(x) = {}_2\Phi_0\,[q^{-n}, q^{n+\alpha+1}; x] \tag{2.5}$$

$$= \sum_{k=0} \frac{(q^{-n})_k (q^{n+\alpha+1})_k \, x^k}{(q)_k} \tag{2.6}$$

Thus

$$Y^{(\alpha)}_{n,q}(x) = \frac{(q)_n}{(q^{1+\alpha})_n} \; J(q\,;\,1+\alpha, n;\, x) \tag{2.7}$$

3. RECURRENCE RELATIONS

From formula (2.6) we see that

$$Y^{(\alpha+1)}_{n,q}(x) - Y^{(\alpha)}_{n,q}(x) = q^{n+\alpha+1}(1 - q^{-n})x\, Y^{(\alpha+2)}_{n-1,q}(x) \tag{3.1}$$

This suggests the difference formula

$$\Delta_\alpha Y^{(\alpha)}_{n,q}(x) = q^{n+\alpha+1}(1 - q^{-n}) \times Y^{(\alpha+2)}_{n-1,q}(x)\;; \tag{3.2}$$

where $\Delta_\alpha f(\alpha) = f(\alpha+1) - f(\alpha)$.

The q-derivative of the q-Bessel polynomials are themselves q-Bessel polynomials with the parameter increased by two. Indeed we find from formula (2. 6).

$$(1-q)\,D_q\,Y^{(\alpha)}_{n,q}(x) = x^{-1}(1-q^{-n})(1-q^{n+\alpha+1})\,Y^{(\alpha+2)}_{n-1,q}(x) \tag{3. 3}$$

which can also be written as

$$\frac{1}{1-q}\,D_q Y^{(\alpha)}_{n,q}(x) = [-n]\,[n+\alpha+1]\,x^{-1}\,Y^{(\alpha+2)}_{n-1,q}(x) \tag{3. 4}$$

From (3. 2) and (3. 3), we see that the q-Bessel polynomials satisfy the mixed equation.

$$\alpha\, Y^{(\alpha)}_{n,q}(x) = \frac{x\,q^{n+\alpha+1}}{[n+\alpha+1]}\,D_q\,Y^{(\alpha)}_{n,q}(x) \tag{3. 5}$$

The following recurrence relation can be verified directly.

$$Y^{(\alpha)}_{n+1,q}(x) - Y^{(\alpha)}_{n,q}(x) = (q^{n+\alpha+1} - q^{-n-1})\,x\,Y^{(\alpha+1)}_{n,q}(x) \tag{3. 6}$$

This can also be written as

$$\nabla_n Y^{(\alpha)}_{n,q}(x) = (q^{n+\alpha+1} - q^{-n-1})\,x\,Y^{(\alpha+1)}_{n,q}(x) \tag{3. 7}$$

4. CHARACTERIZATIONS

In this section we obtain some characterizations of the q-Bessel polynomials similar to those for (i) the Jacobi polynomials obtained by Al-Salam [3].

We prove here the following:

THEOREM 1. *Given a sequence* $\{f^{(\alpha)}_{n,q}(x)\}$ *of* q*-polynomials in* x *where* *deg* $f^{(\alpha)}_{n,q}(x) = n$, *and* α *is a parameter, such that*

$$(1-q)\,D_q\,f^{(\alpha)}_{n,q}(x) = x\,(1-q^{-n})\,(1-q^{n+\alpha+1})\,f^{(\alpha+2)}_{n-1,q}(x) \text{ and } f^{(\alpha)}_{n,q}(0) = 1 \tag{4.1}$$

Then $f^{(\alpha)}_{n,q}(x) = Y^{(\alpha)}_{n,q}(x)$.

Proof. Assume

$$f^{(\alpha)}_{n,q}(x) = \sum_{k=0}^{n} A_{k,q}(\alpha, n)\,x^k$$

Now by (4.1), we have

$$A_{k,q}(\alpha, n) = \frac{(1-q^{-n})(1-q^{n+\alpha+1})}{(1-q^k)}\,A_{k-1,q}(\alpha+2, n-1)\,.$$

Since $f_{n,q}^{(\alpha)}(0)=1$ so $A_{o,q}(\alpha,n)=1$. Consequently, we obtain

$$A_{k,q}(\alpha,n)=\frac{(q^{-n})_k\,(q^{n+\alpha+1})_k}{(q)_k}$$

which proves the theorem.

Another characterization is suggested by (3.2). Indeed we have

THEOREM 2. *Given a sequence of q-functions* $\{f_{n,q}^{(\alpha)}(x)\}$ *such that*

$$\Delta_\alpha f_{n,q}^{(\alpha)}(x)=q^{n+\alpha+1}(1-q^{-n})\times f_{n-1,q}^{(\alpha+2)}(x) \tag{4.2}$$

$$f_{n,q}^{(\alpha)}(0)=1\,,\ f_{o,q}^{(\alpha)}(x)=1 \tag{4.3}$$

Then $$f_{n,q}^{(\alpha)}(x)=Y_{n,q}^{(\alpha)}(x)$$

Proof. From (4.2) it is evident that $f_{n,q}^{(\alpha)}(x)$ is a q-polynomial in α of degree n. Hence we can write

$$f_{n,q}^{(\alpha)}(x)=\sum_{r=o}^{n}A_r(n,x)\,\frac{(q^{n+\alpha+1})_r}{(q)_r}$$

Hence (4.2) implies

$$A_r(n,x)=(q^{-n})_r\,x^r$$

This proves the theorem.

Equation (3.5) implies the following

THEOREM 3. *If the sequence* $f_{n,q}^{(\alpha)}(x)$, *where* $f_{n,q}^{(\alpha)}(x)$ *is a polynomial of degree n in x, and* α *is a parameter, satisfies*

$$\Delta_\alpha f_{n,q}^{(\alpha)}(x)=\frac{x\,q^{n+\alpha+1}}{[n+\alpha+1]}D_q f_{n,q}^{(\alpha)}(x) \tag{4.4}$$

such that $f_{n,q}^{(o)}(x)=Y_{n,q}^{(o)}(x)$, *then* $f_{n,q}^{(\alpha)}(x)=Y_{n,q}^{(\alpha)}(x)$.

The proof is similar to that of Theorems 1 and 2.

Finally we give the theorem suggested by formula (3.7).

THEOREM 4. *Given a sequence of q-functions* $\left\{f_{n,q}^{(\alpha)}(x)\right\}$ *such that*

$$\Delta_n f_{n,q}^{(\alpha)}(x)=(q^{n+\alpha+1}-q^{-n-1})\,x\,f_{n,q}^{(\alpha+1)}(x$$

and $f_{o,q}^{(\alpha)}(x)=1$ for all x and α

Then $f_{n,q}^{(\alpha)}(x)=Y_{n,q}^{(\alpha)}(x)$.

The proof of this theorem follows by induction on n.

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